A Theory of Simple Current Extensions of Vertex Operator Algebras and Applications to the Moonshine Vertex Operator Algebra

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Chapter 1

Introduction

The origin of the theory of vertex operator algebras comes from Borcherds' wonderful insight [Bo1] that the moonshine module $V^\natural$ which Frenkel, Lepowsky and Muerman constructed in [FLM] to solve the moonshine conjecture on the Monster sporadic finite simple group $\mathbb{M}$ has a natural structure of a vertex algebra which he devised. After the positive solution of the moonshine conjecture by Borcherds [Bo2], a movement to study the Monster as the full automorphism group of the moonshine vertex operator algebra was established.

Let us explain the FLM construction of the moonshine vertex operator algebra briefly. The moonshine vertex operator algebra $V^\natural$ is constructed as the $\mathbb{Z}_2$-twisted orbifold construction of the lattice VOA V_Λ associated to the Leech lattice Λ. Roughly speaking, we may understand it as follows. We can define an involution θ on V_Λ which comes from the (-1)-isometry on Λ. Then we take the θ-invariants V_Λ^+ of V_Λ, which is again a simple vertex operator algebra. It is shown in [FLM] that there is a θ-twisted representation V_Λ^T of V_Λ. The involution θ also acts on V_Λ^T. We take a fixed point submodule $(V_\Lambda^T)^+$ of V_Λ^T. Then we define the moonshine vertex operator algebra by $V^\natural := V_\Lambda^+ \oplus (V_\Lambda^T)^+$. In [FLM], Frenkel et. al. showed that $V^\natural$ has a structure of a simple vertex operator algebra with the q-character $J(\tau)$ and that the automorphism group $\mathrm{Aut}(V^\natural)$ is isomorphic to the Monster by using Griess' result in [G] (see also [ATLAS] [C] [Ti]). By the construction, $V^\natural$ has a natural $\mathbb{Z}_2$-symmetry, which is known to be the 2B-involution of the Monster. It is shown in [H3] that both V_Λ and $V^\natural$ are $\mathbb{Z}_2$-graded simple current extensions of the $\mathbb{Z}_2$-orbifold V_Λ^+ of V_Λ, and in this view point, the proof of existence of a structure of a vertex operator algebra on $V^\natural$ was simplified by Huang [H3].

After [FLM], Miyamoto showed quite another construction of $V^\natural$. In [DMZ], Dong, Mason and Zhu discovered an important fact that there is an orthogonal decomposition of the Virasoro vector of $V^\natural$ into 48 conformal vectors with central charge 1/2. This implies an existence of the unitary Virasoro frame $L(1/2,0)^{\otimes 48}$ in $V^\natural$ and we can study $V^\natural$ as a module for the frame $L(1/2,0)^{\otimes 48}$, where $L(c,h)$ denotes the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h. Motivated by this fact, Miyamoto builded a theory of code vertex operator algebras in [M2] [M3] [M4] and succeeded to reconstruct the moonshine vertex operator algebra started from the Virasoro frame $L(1/2,0)^{\otimes 48}$ in [M5]. In his construction, we have to perform simple current extensions in two steps. First, we construct a code vertex operator algebra as a simple current extension of $L(1/2,0)^{\otimes 48}$, and then we construct the moonshine vertex operator algebra as a simple current extension of the code VOA.

The Miyamoto's construction has an advantage that we can explicitly define many 2A-involutions of the Monster by using the representation theory of the unitary Virasoro vertex operator algebra $L(1/2,0)$. In [M1], Miyamoto showed that the $\mathbb{Z}_2$-symmetry of the fusion algebra for $L(1/2,0)$ gives rise to an involutive automorphism of any vertex operator algebra which contains $L(1/2,0)$ as a subalgebra. This involution is often called the *Miyamoto involution*. It is shown in [C] and [M1] that the Miyamoto involutions on the moonshine vertex operator algebra belong to the 2A-conjugacy of the Monster and that there is a one-to-one correspondence between the set of conformal vectors in $V^\natural$ with central charge 1/2 and the set of 2A-involutions of the Monster. After [M1], a method to define automorphisms of vertex operator algebras by using symmetries in the fusion algebras for the Virasoro vertex operator algebras and their simple current extensions was developed by many mathematicians (cf. [KMY] [M7] [M8] [LLY] [SY]). In particular, C.H. Lam, H. Yamada and the author has recently obtained an important achievement on the Miyamoto involutions in [LYY]. In [LYY], they also use some simple current extensions of vertex operator algebras to define automorphisms.

Now, we come to see that the simple current extensions of vertex operator algebras have an incredible significance to construct new vertex operator algebras and study their automorphisms. By this reason, the theory of simple current extensions is one of my central subject. In this article the author would like to make a comprehensive compilation on the theory of simple current extensions so far as I have ever obtained.

Let us review the definition of simple current extensions. Let V^0 be a simple vertex operator algebra and D an abelian group. Assume that a direct sum $V_D=\oplus_{\alpha\in D}V^\alpha$ of inequivalent irreducible V^0-modules $\{V^\alpha \mid \alpha\in D\}$ indexed by D forms a simple vertex operator algebra. If the fusion rule $V^\alpha\times V^\beta=V^{\alpha+\beta}$ for V^0-modules is satisfied, then the extension V_D is called a D-graded simple current extension of V^0. One of the main purpose of this article is to determine the representation theory of V_D. To complete it, we have to use the theory of fusion products. The theory of fusion products has been developed by Huang [H1]-[H4] and Huang-Lepowsky [HL1]-[HL4] and it provides us a powerful tool to study vertex operator algebras. The main tool we use in this paper is the associativity for the fusion products. By Huang's results, intertwining operators among simple current modules have a nice property so that in the representation theory of a simple current extensions with an abelian symmetry we can find certain twisted algebras associated to pairs of the abelian group and its orbit spaces. The twisted algebras can be considered as a deformation or a generalization of group rings and play a powerful role in our theory. Using the twisted algebras, we can show that every module for a simple current extension of a rational C_2-cofinite VOA of CFT-type is completely reducible. Furthermore, we can parameterize irreducible modules for extensions by irreducible representations of the twisted algebras (Theorem 4.4.7).

We also develop a method of induced modules for simple current extensions. We show that every irreducible module of a rational C_2-cofinite vertex operator algebra of CFT-type can be lifted to be a twisted module for a simple current extension with an abelian symmetry (Theorem 4.5.3). This result concerns the following famous conjecture: "For a simple rational vertex operator algebra V and its finite automorphism group G, the G-invariants V^G, called the G-orbifold of V, is also a simple rational vertex operator algebra. Moreover, every irreducible module for the G-orbifold V^G is contained in a g-twisted V-module for some $g \in G$." Our result is the converse of this conjecture in a sense. Actually, we prove the following. Let V^0 be a simple rational VOA of CFT-type and D a finite abelian group. Then a D-graded simple current extension $V_D = \oplus_{\alpha \in D} V^\alpha$ is σ-regular for all $\sigma \in D^*$. Here the term "σ-regular" means that every weak σ-twisted module is completely reducible (cf. [Y2]). Moreover, for an irreducible V^0-module W, we can attach the group representation $\chi : D \to \mathbb{Z}/n\mathbb{Z}$ such that the powers of z in a V^0-intertwining operator of type $V^\alpha \times (V^\beta \boxtimes_{V^0} W) \to V^{\alpha+\beta} \boxtimes_{V^0} W$ are contained in $\chi(\alpha) + \mathbb{Z}$ for all $\alpha, \beta \in D$. Finally we prove that W can be lifted to be an irreducible $\hat{\chi}$-twisted V_D-module, where $\hat{\chi}$ is an element in D^* defined as $\hat{\chi}(\alpha) = e^{-2\pi\sqrt{-1}\chi(\alpha)}$ for $\alpha \in D$.

In Miyamoto's reconstruction of the moonshine vertex operator algebra, sometimes we have to extend a D-graded simple current extension V_D of V^0 to an E-graded simple current extension V_E where E contains D as a subgroup. In this paper we present a method to construct the above extension in a general fashion in Theorem 4.6.1. We also present a lifting property of intertwining operators in Theorem 4.4.9 which describes fusion rules for V_D-modules in terms of that of V^0-modules. By this theorem, we can compute many fusion rules for the extensions.

Another main purpose of this article is to give applications of the theory of simple current extensions to the moonshine vertex operator algebra and the Monster. In Section 6 we recall FLM's construction of the moonshine module and introduce Huang's idea of the existence of a vertex operator algebra structure on the moonshine module in which he used some results on simple current extensions. In Section 7 we recall the Miyamoto's construction of the moonshine vertex operator algebra and we present a brief refinement of his construction based on our results on simple current extensions. It is worth to remark that by our refinement we can construct a vertex operator algebra which contains a Virasoro frame $L(1/2,0)^{\otimes n}$ for arbitrary length n as long as Hypothesis I in Section 7 is satisfied, while Miyamoto's original construction has a restriction on length n to be divisible by 8.

In Section 8 we give some new results on the moonshine vertex operator algebra. Let $e \in V^\natural$ be a conformal vector with central charge $1/2$ and $\tau_e \in \mathrm{Aut}(V^\natural) \simeq \mathbb{M}$ the corresponding Miyamoto involution. Then it is known that $C_{\mathrm{Aut}(V^\natural)}(\tau_e)$ is isomorphic to the 2-fold cover $\langle \tau_e \rangle \cdot \mathbb{B}$ of the baby monster sporadic finite simple group $\mathbb{B}$ (cf. [ATLAS]). On the other hand, since the subalgebra $\mathrm{Vir}(e)$ generated by e is isomorphic to the unitary Virasoro vertex operator algebra $L(1/2,0)$, which has exactly three irreducible module $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$, we have the following decomposition of $V^\natural$:

$$V^\natural = \bigoplus_{h=0,1/2,1/16} L(1/2,h) \otimes \mathrm{Hom}_{\mathrm{Vir}(e)}(L(1/2,h), V^\natural).$$

The space $T_e^\natural(h) := \mathrm{Hom}_{\mathrm{Vir}(e)}(L(1/2,h), V^\natural)$, $h \in \{0, 1/2, 1/16\}$, coincides with the space of highest weight vectors for $\mathrm{Vir}(e)$ with highest weight h, and by the commutant construction, $T_e^\natural(0)$ is a subalgebra of $V^\natural$ which commutes with $\mathrm{Vir}(e)$ and $T_e^\natural(h)$, $h = 0, 1/2, 1/16$, are module for $T_e^\natural(0)$. Therefore, the group $C_{\mathrm{Aut}(V^\natural)}(\tau_e)$ naturally acts on the spaces $T_e^\natural(h)$. On the other hand, it is well-known that $L(1/2,0) \oplus L(1/2,1/2)$ forms a simple vertex operator superalgebra. Therefore, it is natural for us to expect that the space $V\!B := T_e^\natural(0) \oplus T_e^\natural(1/2)$ also forms a simple vertex operator superalgebra. The study of $V\!B$ was first begun by Höhn in [Hö1] and he proved in [Hö2] that the automorphism group of $V\!B$ is exactly isomorphic to $2 \times \mathbb{B}$. In Section 8 we give a quite different proof of existence of a structure of a vertex operator superalgebra on $V\!B$ and the isomorphism $\mathrm{Aut}(V\!B) \simeq 2 \times \mathbb{B}$ by showing that $V\!B$ is a $\mathbb{Z}_2$-graded simple current extension of $T_e^\natural(0)$. Moreover, we prove that $V\!B_T := T_e^\natural(1/16)$ is an irreducible $\mathbb{Z}_2$-twisted $V\!B$-module and that the irreducible $T_e^\natural(0)$-modules are exactly given by $T_e^\natural(0)$, $T_e^\natural(1/2)$ and $T_e^\natural(1/16)$. This result gives an interesting consequence that the 2A-twisted orbifold construction applied to $V^\natural$ yields $V^\natural$ itself again, whereas the 2B-twisted orbifold construction applied to $V^\natural$ yields V_Λ as in [FLM].

In Section 8 we also review the 3A-algebra for the Monster in [SY]. This algebra was first studied by Miyamoto [M8] and Sakuma and the author studied it deeply in [SY]. After [SY], Lam, Yamada and the author greatly generalized Miyamoto's idea in [M8] and made a comprehensive development of the study of the McKay's observations on the 2A-conjugacy class of the Monster in [LYY]. In [LYY], many algebras related to the Monster were founded and studied. The 3A-algebra is a one of them and its fusion algebra has a natural $\mathbb{Z}_3$-symmetry which is known to define the 3A-triality of the Monster. Since the 3A-algebra is a $\mathbb{Z}_3$-graded simple current extension of the unitary Virasoro vertex operator algebras, we can study it by using a theory of simple current extensions we have developed. It is an interesting result that some coefficients of fusion rules for modules for the 3A-algebra is 6, whereas there are few examples of fusion rules whose coefficients are greater than 1.

Chapter 2

Basic Definitions

In this section we recall some basic definitions on vertex operator algebras. Sometimes it is convenient to consider not only vertex operator algebras but also vertex operator superalgebras, the superalgebra version of vertex operator algebra, together. So we also include the notion of vertex operator superalgebras. We shall work in an algebraic setting over the complex number field $\mathbb{C}$, however, all the results are valid over any algebraic field of characteristic 0. The set of non-negative integers will be denoted by $\mathbb{N}$, i.e., $\mathbb{N} = \{0, 1, 2, \dots\}$. The symbols z, z_0, $z_1, \dots$ will designate commuting formal variables.

2.1 Formal calculus

Here we discuss formal power series and fix the necessary notations. All the results are either elementary or in [FLM] and [FHL].

For a vector space V, we set

$$\begin{aligned}
V[z] &:= \{\textstyle\sum_{n\in\mathbb{N}} v_n z^n \mid v_n \in V,\ \text{all but finitely many } v_n = 0\},\\
V[z,z^{-1}] &:= \{\textstyle\sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V,\ \text{all but finitely many } v_n = 0\},\\
V[[z]] &:= \{\textstyle\sum_{n\in\mathbb{N}} v_n z^n \mid v_n \in V\},\\
V((z)) &:= \{\textstyle\sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V,\ v_n = 0 \text{ for sufficiently small } n\},\\
V[[z,z^{-1}] &:= \{\textstyle\sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V\},\\
V\{z\} &:= \{\textstyle\sum_{r\in\mathbb{C}} v_r z^r \mid v_r \in V\},\\
V\{\{z\}\} &:= \{\textstyle\sum_{r\in\mathbb{C}} v_r z^r \mid v_r \in V,\ {}^\forall r \in \mathbb{C}\ {}^\exists N \in \mathbb{N} \text{ s.t. } v_{r-i} = 0 \text{ for } {}^\forall i \geq N\}.
\end{aligned}$$

and we shall also use analogous notation for several variables.

For $r \in \mathbb{C}$, we define the binomial expansion by

$$(z_1 + z_2)^r := \sum_{i\in\mathbb{N}} \binom{r}{i} z_1^{r-i} z_2^i, \quad \text{where } \binom{r}{i} := \frac{r\cdot(r-1)\cdots(r-i+1)}{i!}.$$

Note that $(z_1 + z_2)^r \neq (z_2 + z_1)^r$ unless $r \in \mathbb{N}$.

For $f(z) \in V[z]$, we define its formal exponential series by

$$e^{f(z)} := \sum_{n=0}^{\infty} \frac{1}{n!} f(z)^n.$$

Then we have the following formal version of Taylor's theorem:

$$e^{z_0 \frac{d}{dz}} \cdot z^r = (z + z_0)^r \quad \text{for } r \in \mathbb{C}. \tag{2.1.1}$$

We introduce a basic generating function, the *formal δ-function at $z = 1$*:

$$\delta(z) := \sum_{n \in \mathbb{Z}} z^n.$$

The fundamental property of the δ-function is $z^n \delta(z) = \delta(z)$ for $n \in \mathbb{Z}$. We usually use the δ-function in the following way:

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^r = z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-r}, \quad \text{where } r \in \mathbb{C},$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) = z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right).$$

The following formal residue notation will be useful:

$$\mathrm{Res}_z \sum_{n \in \mathbb{Z}} v_n z^n := v_{-1}.$$

Formal residue enjoys some property of contour integration. We have

(i) Integration by parts. For $f(z) \in \mathbb{C}((z))$ and $g(z) \in V((z))$,

$$\mathrm{Res}_z f(z) \frac{d}{dz} g(z) = -\mathrm{Res}_z \frac{d}{dz} f(z) \cdot g(z).$$

(ii) Formula for change of variable. For $g(z) = \sum_{n \geq N} v_n z^n \in V((z))$ and $f(z) = \sum_{n \geq 1} a_n z^n \in \mathbb{C}[[z]]$ with $a_1 \neq 0$, we have the following formula for this change of variable:

$$\mathrm{Res}_z g(z) = \mathrm{Res}_{z_1} g(f(z_1)) \frac{d}{dz_1} f(z_1).$$

2.2 Vertex operator algebras

Definition 2.2.1. A *vertex superalgebra* is a quadruple $(V, Y(\cdot, z), \mathbb{1}, \partial)$ where $V = V^0 \oplus V^1$ is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space, $Y(\cdot, z)$ is a linear map called *vertex operator map* from $V \otimes_{\mathbb{C}} V$ to $V((z))$, where z is a formal variable, $\mathbb{1}$ is a specified element of V called *vacuum* *vector* and ∂ is a parity preserving endomorphism of V such that if we set $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$ for $a \in V$, which is called the *vertex operator* of a, then the following conditions hold:

(i) $Y(\mathbb{1}, z) = \mathrm{id}_V$;

(ii) $Y(a,z)\mathbb{1} \in V[[z]]$ and $Y(a,0)\mathbb{1} = a_{(-1)}\mathbb{1} = a$ for any $a \in V$;

(iii) $[\partial, Y(a,z)] = Y(\partial a, z) = \dfrac{d}{dz}Y(a,z)$ for any $a \in V$;

(iv) For $a \in V^i$ and $b \in V^j$, $Y(a,z)b \in V^{i+j}((z))$, where $i,j \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$;

(v) The following Jacobi identity holds for any $\mathbb{Z}_2$-homogeneous $a,b,v \in V$:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(a,z_1)Y(b,z_2)v - (-1)^{\varepsilon(a,b)}z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)Y(b,z_2)Y(a,z_1)v$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(a,z_0)b,z_2)v,$$

where $\varepsilon : V^0 \sqcup V^1 \to \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is a parity function such that $\varepsilon(a,b) = 1+\mathbb{Z}$ if and only if $a,b \in V^1$.

We usually denote $(V, Y(\cdot,z), \mathbb{1}, \partial)$ simply by V. In the case of $V^1 = 0$, V is called a *vertex algebra*. If a vertex superalgebra V has a $\frac{1}{2}\mathbb{Z}$-graded decomposition $V = \oplus_{n\in\frac{1}{2}\mathbb{Z}}V_n$ such that $V^0 = \oplus_{n\in\mathbb{Z}}V^0\cap V_n$, $V^1 = \oplus_{n\in\mathbb{Z}}V^1\cap V_{n+\frac{1}{2}}$ and $a_{(n)}V_s \subset V_{m+s-n-1}$ for any $a \in V_m$, then V is said to be a *graded vertex superalgebra*. For a graded vertex superalgebra V, we define the weight of a homogeneous element $a \in V_m$ by $\mathrm{wt}(a) := m$. This completes the definition.

Remark 2.2.2. For a vertex superalgebra $(V, Y(\cdot,z), \mathbb{1}, \partial)$, the underlying vector space V is sometimes called the *Fock space* of the structure.

The following are consequences of axioms (cf. [FLM] [FHL] [Li1]).

1° Skew-symmetry: $Y(a,z)b = (-1)^{\varepsilon(a,b)}e^{z\partial}Y(b,-z)a$. In particular, $Y(a,z)\mathbb{1} = e^{z\partial}a$.

2° Commutativity: Let N be an integer such that $a_{(n)}b = 0$ for $n \geq N$. Then

$$(z_1-z_2)^N Y(a,z_1)Y(b,z_2) = (-1)^{\varepsilon(a,b)}(z_1-z_2)^N Y(b,z_2)Y(a,z_1) \quad \text{on } V. \tag{2.2.1}$$

3° Associativity: Let N be a positive integer such that $Y(a,z)v$ involves only positive powers of z. Then

$$(z_2+z_0)^N Y(Y(a,z_0)b,z_2)v = (z_0+z_2)^N Y(a,z_0+z_2)Y(b,z_2)v. \tag{2.2.2}$$

Among the above conditions, the most important one is the commutativity as which characterizes a concept of vertex superalgebra. Here we present the following fact.

Proposition 2.2.3. *([Li1, Proposition 2.2.4]) In the definition of vertex superalgebra, the Jacobi identity can be equivalently replaced by the commutativity* (2.2.1).

Proof: Here we give a proof of the above proposition at the classical level as in Remark 2.2.5 of [Li1]. Let A be any algebra with a right identity ${}_A1$ and denote by $\mathrm{ad}(a)$ the left multiplication by an element $a \in A$. Suppose that the commutativity $\mathrm{ad}(a)\mathrm{ad}(b) = \mathrm{ad}(b)\mathrm{ad}(a)$ holds for any $a, b \in A$. Then $a(bc) = b(ac)$ for any $a, b, c \in A$. Setting $c = {}_A1$, we obtain the commutativity $ab = ba$. Furthermore, we obtain the associativity $a(bc) = a(cb) = c(ab) = (ab)c$ for any $a, b, c \in A$. Therefore A is a commutative associative algebra. The proof of Proposition 2.2.4 of [Li1] is exactly an analogue of the argument above. ∎

Later, we will present another approach to the definition of vertex superalgebras using a theory of local systems [Li1].

Definition 2.2.4. A *vertex operator superalgebra* $(V, Y(\cdot, z), \mathbb{1}, \omega)$ is a graded vertex superalgebra $(V, Y(\cdot, z), \mathbb{1}, \partial)$ with an additional element $\omega \in V$ called the *Virasoro vector* of V such that

(vi) V admits a representation of the Virasoro algebra:

$$[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0}\frac{m^3 - m}{12}c$$

for $m, n \in \mathbb{Z}$, where we set $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ and $c \in \mathbb{C}$ is called the *central charge* of ω;

(vii) $L(-1) = \partial$, i.e., $Y(L(-1)a, z) = [L(-1), Y(a, z)] = \frac{d}{dz}Y(a, z)$ for any $a \in V$;

(viii) The $\frac{1}{2}\mathbb{Z}$-graded decomposition $V = \oplus_{n\in\frac{1}{2}\mathbb{Z}}V_n$ coincides with $L(0)$-eigenspace decomposition, that is, $L(0)|_{V_n} = n \cdot \mathrm{id}_{V_n}$, and also we have $\dim_{\mathbb{C}} V_n < \infty$ and $V_n = 0$ for sufficiently small n. This completes the definition.

A vertex operator superalgebra $V = V^0 \oplus V^1$ with $V^1 = 0$ is called a *vertex operator algebra* and we call it a VOA for short. Similarly, a vertex operator superalgebra is shortly referred to as a super VOA or an SVOA.

In this article, we mainly treat VOAs of CFT-type:

Definition 2.2.5. An SVOA V is said to be *of CFT-type** if it has a grade decomposition $V = \oplus_{n\in\frac{1}{2}\mathbb{Z}}V_n$ such that $V_0 = \mathbb{C}\mathbb{1}$ and $V_n = 0$ for $n < 0$. A VOA of CFT-type is an SVOA of CFT-type with odd part component $V^1 = 0$.

* CFT=Conformal Field Theory, as this kind of condition is always assumed in the conformal field theory.

2.3 Modules for vertex operator algebras

Definition 2.3.1. Let $(V, Y_V(\cdot, z), \mathbb{1}, \partial_V)$ be a vertex superalgebra. A V-module is a triple $(M, Y_M(\cdot, z), \partial_M)$ where $M = M^0 \oplus M^1$ is a $\mathbb{Z}_2$-grades vector space and $Y_M(\cdot, z)$ is a linear map $V \otimes_{\mathbb{C}} M \to M((z))$ and ∂_M is a $\mathbb{Z}_2$-homogeneous endomorphism of M such that if we set $Y_M(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, where $a_{(n)} \in \mathrm{End}(M)$, then the following conditions hold:

(i) $Y_M(\mathbb{1}, z) = \mathrm{id}_M$;

(ii) For $a \in V^i$ and $v \in M^j$, $Y_M(a, z)v \in M^{i+j}((z))$, where $i, j \in \mathbb{Z}_2$;

(iii) $Y_M(\partial_V a, z) = [\partial_M, Y_M(a, z)] = \frac{d}{dz} Y_M(a, z)$;

(iv) The following Jacobi identity holds for any $\mathbb{Z}_2$-homogeneous $a, b \in V$ and $v \in M$:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2)v - (-1)^{\varepsilon(a,b)} Y_M(b, z_2)Y_M(a, z_1)v$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y_V(a, z_0)b, z_2)v.$$

We often denote $(M, Y_M(\cdot, z), \partial_M)$ simply by M. This completes the definition.

Definition 2.3.2. Let $(V, Y_V(\cdot, z), \mathbb{1}, \omega)$ be a vertex operator superalgebra. A V-*module* is a module $(M, Y_M(\cdot, z), \partial_M)$ of a vertex superalgebra $(V, Y_V(\cdot, z), \mathbb{1}, L(-1))$ such that $\partial_M = L(-1) = \mathrm{Res}_z Y_M(\omega, z)$. A V-module M is said to be $\frac{1}{2}\mathbb{N}$-*graded* if there is a $\frac{1}{2}\mathbb{N}$-grading $M = \oplus_{n\in\frac{1}{2}\mathbb{Z}} M(n)$ such that $M^0 = \oplus_{n\in\mathbb{N}} M^0 \cap M(n)$, $M^1 = \oplus_{n\in\mathbb{N}} M^1 \cap M(n + \frac{1}{2})$ and $a_{(n)}M(s) \subset M(s + \mathrm{wt}(a) - n - 1)$ for any $a \in V$. A $\frac{1}{2}\mathbb{N}$-graded V-module M is said to be a *strong module* if an operator $L(0) = \mathrm{Res}_z zY(\omega, z)$ acts on M semisimply and each eigenspace is of finite dimension. For a strong V-module M, we often use M_s to denote the eigen subspace $\{v \in M \mid L(0)v = sv\}$ for $s \in \mathbb{C}$. This completes the definition.

Remark 2.3.3. In the definition of modules, we do not have to assume the $L(-1)$-derivation property $Y_M(L(-1)a, z) = [L(-1), Y_M(a, z)] = \partial_z Y_M(a, z)$ if V is an SVOA, as will follow from the Jacobi identity on M.

Remark 2.3.4. It will be explained later that to give a module M for an SVOA V is equivalent to give a vertex superalgebra homomorphism from V to a local system on M. If this homomorphism preserves SVOA gradings, then a module M becomes a $\frac{1}{2}\mathbb{N}$-graded module. Namely, even if we consider a vertex operator algebra, its representation theory is studied in the level of vertex algebras.

For an $\mathbb{N}$-graded V-module $M = \oplus_{n\geq 0} M(n)$ with $M(0) \neq 0$, the subspace $M(0)$ is often called the *top level* of M, and for a strong V-module $M = \oplus_{n\geq 0} M_{n+h}$ with top level

M_h, we call its weight h *top weight* of M. For a strong module M with top weight h, we often consider its *q-dimension* or *q-character* given as follows:

$$\mathrm{ch}_M(q) := \mathrm{tr}_M q^{L(0)} = \sum_{n=0}^{\infty} \dim M_{n+h} q^{n+h}.$$

The following are consequences of definition (cf. [FLM] [FHL] [Li1]).

1° Commutativity: Let $a, b \in V$ and let N be an integer such that $z^n Y_V(a,z)b$ involves only positive powers of z. Then

$$(z_1 - z_2)^N Y_M(a, z_1) Y_M(b, z_2) = (-1)^{\varepsilon(a,b)} Y_M(b, z_2) Y_M(a, z_1) \quad \text{on } M. \tag{2.3.1}$$

2° Associativity: Let $a, b \in V$ and $v \in M$. Then for sufficiently N we have

$$(z_0 + z_2)^N Y_M(a, z_0 + z_2) Y_M(b, z_2) v = (z_2 + z_0)^N Y_M(Y_V(a, z_0)b, z_2) v. \tag{2.3.2}$$

It is well-known that the above two conditions are equivalent to the Jacobi identity.

Proposition 2.3.5. *([Li1, Proposition 2.3.3]) Let V be a vertex superalgebra. Then the Jacobi identity on a module is equivalently replaced by the commutativity* (2.3.1)*, the associativity* (2.2.2) *and the $L(-1)$-derivation property.*

2.4 Intertwining operators and fusion rules

Definition 2.4.1. Let V be a vertex operator algebra and let M^1, M^2 and M^3 be strong V-modules with $L(0)$-weight decompositions $M^i = \oplus_{n\in\mathbb{N}} M^i_{n+h_i}$, $i = 1, 2, 3$, where $h_i \in \mathbb{C}$ and $M^i_{n+h} = \{v \in M^i \mid L(0)v = (n+h_i)v\}$. A *$V$-intertwining operator* of type $M^1 \times M^2 \to M^3$ is a linear map $I(\cdot, z) : M^1 \otimes M^2 \to M^3\{\{z\}\}$ satisfying the following conditions:

(i) For any $u^1 \in M^1$ and $u^2 \in M^2$, $I(u^1, z)u^2 \in M^3((z))z^{-h_1-h_2+h_3}$;

(ii) $L(-1)$-derivation: $I(L(-1)u^1, z) = \partial_z I(u^1, z)$;

(iii) The following Jacobi identity holds for any $a \in V$, $u^1 \in M^1$ and $u^2 \in M^2$:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M^3}(a, z_1) I(u^1, z_2)u^2 - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) I(u^1, z_2) Y_{M^2}(a, z_1)u^2$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y_{M^1}(a, z_0)u^1, z_2)u^2.$$

The space of V-intertwining operators of type $M^1 \times M^2 \to M^3$ is denoted by $\binom{M^3}{M^1\ M^2}_V$ or simply by $\binom{M^3}{M^1\ M^2}$, and its dimension $\dim_{\mathbb{C}} \binom{M^3}{M^1\ M^2}$ is called the *fusion rule*. This completes the definition.

Remark 2.4.2. Let $I(\cdot, z)$ be as above and let $u \in M^1$. If we write $I(u, z) = \sum_{r\in\mathbb{C}} u_{(r)} z^{-r-1}$ with $u_{(r)} \in \mathrm{Hom}(M^2, M^3)$, then it follows from definition that each coefficient operator is homogeneous such that $u_{(r)} M^2_s \subset M^3_{s+\mathrm{wt}(u)-r-1}$.

The name "V-intertwining operator" comes from the following fact.

Lemma 2.4.3. *There is a canonical isomorphism between* $\binom{M^2}{V\ M^1}$ *and* $\mathrm{Hom}_V(M^1, M^2)$. *Therefore, if both* M^1 *and* M^2 *are irreducible, then* $\binom{M^2}{V\ M^1} = 0$ *unless* $M^1 \simeq M^2$.

Proof: Let $I(\cdot, z) \in \binom{M^2}{V\ M^1}$. Then by the $L(-1)$-derivation we have $\partial_z I(\mathbb{1}, z) = I(L(-1)\mathbb{1}, z) = 0$. Thus $I(\mathbb{1}, z)$ is in $\mathrm{Hom}_{\mathbb{C}}(M^1, M^2)$. Moreover, since $[a_{(n)}, I(\mathbb{1}, z)] = \sum_{i\geq 0} z^{n-i} I(a_{(i)}\mathbb{1}, z) = 0$ for any $a \in V$, $I(\mathbb{1}, z)$ defines an element in $\mathrm{Hom}_V(M^1, M^2)$. If $I(\mathbb{1}, z) = 0$, then

$$I(a, z) = I(a_{(-1)}\mathbb{1}, z) = \mathrm{Res}_w\{(w-z)^{-1} Y_{M^2}(a, w) I(\mathbb{1}, z) - (-z+w)^{-1} I(\mathbb{1}, z) Y_{M^1}(a, w)\}$$

implies $I(\cdot, z) = 0$. Thus we obtain a linear injection $\binom{M^2}{V\ M^1} \ni I(\cdot, z) \mapsto I(\mathbb{1}, z) \in \mathrm{Hom}_V(M^1, M^2)$. Conversely, for $\psi \in \mathrm{Hom}_V(M^1, M^2)$, define $J_\psi(\cdot, z) : V \times M^1 \to M^2((z))$ by $J_\psi(a, z) = Y_{M^2}(a, z)\psi = \psi Y_{M^1}(a, z)$. Then one can easily check that $J_\psi(a, z) \in \binom{M^2}{V\ M^1}$ and $J_\psi(\mathbb{1}, z) = \psi$. Thus $\binom{M^2}{V\ M^1}$ is isomorphic to $\mathrm{Hom}_V(M^1, M^2)$. ∎

Clearly, for every V-module M, the module vertex operator $Y_M(\cdot, z)$ is a V-intertwining operator of type $V \times M \to M$. Since the definition of intertwining operators are a generalization of that of module vertex operator maps, intertwining operators also enjoy the following properties.

Proposition 2.4.4. *Let* $I(\cdot, z)$ *be a* V*-intertwining operator of type* $M^1 \times M^2 \to M^3$. *Then the Jacobi identity for an intertwining operator is equivalent to the following two properties:*

(i) Commutativity:

$$(z_1 - z_2)^N Y_{M^3}(a, z_1) I(u, z_2) = (z_1 - z_2)^N I(u, z_2) Y_{M^2}(a, z_1) \quad \textit{for} \quad N \gg 0;$$

(ii) Associativity:

$$(z_0 + z_2)^N Y_{M^3}(a, z_0 + z_2) I(u, z_2) = (z_2 + z_0)^N I(Y_{M^1}(a, z_0)u, z_2) \quad \textit{for} \quad N \gg 0.$$

2.5 Automorphisms and twisted theory

In the theory of vertex operator algebras, automorphisms play an important role.

Definition 2.5.1. An automorphism σ of a vertex operator algebra V is a linear automorphism on V such that it fixes both the vacuum vector $\mathbb{1}$ and the Virasoro vector ω and preserves the vertex operator mapping $\sigma Y(a,z)b = Y(\sigma a, z)\sigma b$ for any $a, b \in V$. The group of automorphisms of V is denoted by $\mathrm{Aut}(V)$.

For a subgroup G of $\mathrm{Aut}(V)$, we can obtain the G-fixed point subalgebra $V^G := \{a \in V \mid \sigma a = a \text{ for any } \sigma \in G\}$, called the *$G$-orbifold* of V. It is an important problem to classify all irreducible V^G-modules in the orbifold conformal field theory, and this problem leads us to the notion of twisted modules. Let $\sigma \in \mathrm{Aut}(V)$ be of finite order $|\sigma|$. Then we obtain the eigenspace decomposition

$$V = V^0 \oplus V^1 \oplus \cdots \oplus V^{|\sigma|-1}, \quad \text{where } V^r := \{a \in V \mid \sigma a = e^{2\pi\sqrt{-1}/|\sigma|} \cdot a\}.$$

Then σ-twisted V-modules are defined as follows:

Definition 2.5.2. A σ-twisted V-module is a couple $(M, Y_M(\cdot, z))$ of a vector space M and a vertex operator mapping $Y_M(\cdot, z) : V \otimes M \to M((z^{1/|\sigma|}))$ such that

(i) $Y_M(\mathbb{1}, z) = \mathrm{id}_V$;

(ii) $Y_M(a,z)$ has a form $Y_M(a,z) = \sum_{n\in\mathbb{Z}} a_{(n+\frac{r}{|\sigma|})} z^{-n-1-r/|\sigma|}$ for $a \in V^r$, where $a_{(n+\frac{r}{|\sigma|})} \in \mathrm{End}(M)$;

(iii) The following twisted Jacobi identity holds for any $a \in V^r$ and $b \in V$:

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y_M(b, z_2)Y_M(a, z_1) \\ &\quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-r} Y_M(Y_V(a, z_0)b, z_2) \quad \text{on } M. \end{aligned}$$

A $\frac{1}{|\sigma|}\mathbb{Z}$*-graded* σ-twisted V-module is a σ-twisted V-module with a $\mathbb{Z}/|\sigma|\mathbb{Z}$-grading $M = M^0 \oplus M^1 \oplus \cdots M^{|\sigma|-1}$ such that $V^i \cdot M^j \subset M^{i+j}$ and each M^i is an $\mathbb{N}$-graded V^0-module. A *strong* σ-twisted V-module is a $\frac{1}{|\sigma|}\mathbb{Z}$-graded σ-twisted V-module which is also a strong module as a V^0-module. This completes the definition.

If we consider a trivial automorphism id_V, then we find that an id_V-twisted V-module is exactly a V-module. Therefore, sometimes a V-module is referred to as an untwisted module. As in the case of untwisted modules, we have the following (cf. see [Li2]).

Proposition 2.5.3. *Let M be a σ-twisted V-module and let $a \in V^r$, $b \in V$ and $v \in M$. Then the twisted Jacobi identity on M is equivalently replaced by the $L(-1)$-derivation property, the commutativity*

$$(z_1 - z_2)^N Y_M(a, z_1)Y_M(b, z_2) = (z_1 - z_2)^N Y_M(b, z_2)Y_M(a, z_1)$$

and the associativity

$$(z_0 + z_2)^{N+r/|\sigma|} Y_M(a, z_0 + z_2) Y_M(b, z_2) v = (z_2 + z_0)^{N+r/|\sigma|} Y_M(Y_V(a, z_0) b, z_2) v$$

with sufficiently large $N \in \mathbb{N}$.

2.6 Examples of VOAs

In this section we present some examples of vertex operator algebras and superalgebras. Here we only give constructions of underlying vector spaces (Fock spaces) and definitions of vertex operator maps on them, and the proofs of VOA structures are omitted; as the most of them will directly follow from a theory of local system given in Section 3.1.

2.6.1 Free bosonic VOA

Let $\mathfrak{h}$ be a finite dimensional vector space with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Viewing $\mathfrak{h}$ as an abelian Lie algebra with invariant bilinear form, we construct its affinization $\hat{\mathfrak{h}} := \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}c$ whose Lie bracket is defined as

$$[t^m \otimes a, t^n \otimes b] := \delta_{m+n,0} \langle a, b \rangle c \quad \text{for } a, b \in \mathfrak{h}, \quad [\hat{\mathfrak{h}}, c] = 0.$$

Set $\hat{\mathfrak{h}}^{\pm} := \oplus_{\pm m > 0} \mathbb{C} t^m \otimes \mathfrak{h}$ and $\hat{\mathfrak{h}}^0 := 1 \otimes \mathfrak{h} \oplus \mathbb{C}c$. Then we have a triangular decomposition $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^-$. We have a subalgebra $\mathfrak{h}^+ \oplus \mathbb{C}c \oplus \mathfrak{h}^-$ and this is often called the *Heisenberg algebra*. We often identify $\mathfrak{h}$ with $1 \otimes \mathfrak{h} \subset \hat{\mathfrak{h}}^0$. Let $k \in \mathbb{C}$ and $\alpha \in \mathfrak{h}^*$. Since the bilinear form on $\mathfrak{h}$ is non-degenerate, we can identify $\mathfrak{h}$ with its dual $\mathfrak{h}^*$. Let $\mathbb{C}e^{\alpha}$ be a one-dimensional representation of $\hat{\mathfrak{h}}^0$ defined as $h \cdot e^{\alpha} = \langle \alpha, h \rangle e^{\alpha}$ for $h \in \mathfrak{h}$ and $c \cdot e^{\alpha} = k e^{\alpha}$. With trivial action of $\mathfrak{h}^+$, we can extend $\mathbb{C}e^{\alpha}$ to be a $(\hat{\mathfrak{h}}^0 + \hat{\mathfrak{h}}^+)$-module. Then the induced module

$$M_{\mathfrak{h}}(k, \alpha) := \mathrm{Ind}_{\mathfrak{U}(\hat{\mathfrak{h}}^0 + \hat{\mathfrak{h}}^+)}^{\mathfrak{U}(\hat{\mathfrak{h}})} \mathbb{C}e^{\alpha} = \mathfrak{U}(\hat{\mathfrak{h}}) \underset{\mathfrak{U}(\hat{\mathfrak{h}}^0 + \hat{\mathfrak{h}}^+)}{\otimes} \mathbb{C}e^{\alpha}$$

is called the *Verma module* or the *highest weight module* with level k and highest weight α. Here and further $\mathfrak{U}(X)$ denotes the universal enveloping algebra for a Lie algebra X. We consider the case $k \neq 0$. In this case we see that $M_{\mathfrak{h}}(k, \alpha) \simeq M_{\mathfrak{h}}(1, \alpha)$ by changing $\langle \cdot, \cdot \rangle$ by $\langle \cdot, \cdot \rangle / k$. It is easy to prove that $M_{\mathfrak{h}}(1, \alpha)$ is an irreducible $\hat{\mathfrak{h}}$-module.

Let $\{h_1, \ldots, h_{\dim \mathfrak{h}}\}$ be a linear basis of $\mathfrak{h}$ and $\{h^1, \cdots, h^{\dim \mathfrak{h}}\}$ its dual basis. We show that $M_{\mathfrak{h}}(1, 0)$ has a structure of a simple VOA and $M_{\mathfrak{h}}(1, \alpha)$ are irreducible $M_{\mathfrak{h}}(1, 0)$-modules. For $a \in \mathfrak{h}$, let us write $a(m) = t^m \otimes a$ and set $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$. Note that we have a relation

$$(z_1 - z_2)^2 a(z_1) b(z_2) - (-z_2 + z_1)^2 b(z_2) a(z_1) = 0$$

for any $a, b \in \mathfrak{h}$, and $M_{\mathfrak{h}}(k, \alpha)$ has a linear basis

$$\{h_{i_1}(-n_1)\cdots h_{i_r}(-n_r)e_\alpha \mid n_1 \geq n_2 \geq \cdots \geq n_r \geq 0,\ r \in \mathbb{N}\}.$$

Using the above basis, we define a vertex operator on $M_{\mathfrak{h}}(1, \alpha)$ for each element in $M_{\mathfrak{h}}(1, 0)$. First, we set $Y_{M(1,\alpha)}(e^0, z) := \mathrm{id}_{M(1,\alpha)}$, and inductively we define

$$\begin{aligned}&Y_{M_{\mathfrak{h}}(1,\alpha)}(a(m)x, z)\\ &:= \mathrm{Res}_{z_0}\{(z_0 - z)^m a(z_0) Y_{M_{\mathfrak{h}}(1,\alpha)}(x, z) - (-z + z_0)^m Y_{M_{\mathfrak{h}}(1,\alpha)}(x, z) a(z_0)\}\end{aligned}$$

for $x \in M_{\mathfrak{h}}(1, 0)$. Put $\mathbb{1} = e^0$ and

$$\omega = \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} h^i(-1)h_i(-1)e^0.$$

Then $(M_{\mathfrak{h}}(1, 0), Y_{M_{\mathfrak{h}}(1,0)}(\cdot, z), \mathbb{1}, \omega)$ has a structure of a simple VOA with central charge $\dim \mathfrak{h}$, and $(M_{\mathfrak{h}}(1, \alpha), Y_{M_{\mathfrak{h}}(1,\alpha)}(\cdot, z))$ are inequivalent irreducible $M_{\mathfrak{h}}(1, 0)$-modules for all $\alpha \in \mathfrak{h}$. A VOA $M_{\mathfrak{h}}(1, 0)$ is called the *free bosonic VOA* and one of the simplest examples of VOAs. It is not difficult to check that $L(0)$ acts on $h_{i_1}(-n_1)\cdots h_{i_r}(-n_r)e^\alpha$ as a scalar $n_1 + \cdots + n_r + \langle\alpha, \alpha\rangle/2$. Thus we have a weight space decomposition $M_{\mathfrak{h}}(1, \alpha) = \oplus_{n\geq 0} M_{\mathfrak{h}}(1, \alpha)_{n+\langle\alpha,\alpha\rangle/2}$ with a q-character

$$q^{-\dim \mathfrak{h}/24}\mathrm{ch}_{M_{\mathfrak{h}}(1,\alpha)}(q) \quad = q^{\langle\alpha,\alpha\rangle/2} \cdot \eta(q)^{-\dim \mathfrak{h}},$$

where $\eta(q)$ denotes the Dedekind eta function $q^{1/24}\prod_{n\geq 1}(1 - q^n)$.

Since a free bosonic VOA contains a Heisenberg subalgebra, its representation theory is well-known:

Theorem 2.6.1. *([FLM, Theorem 1.7.2 and 1.7.3]) Every irreducible $\mathbb{N}$-graded $M_{\mathfrak{h}}(1, 0)$-module is isomorphic to $M_{\mathfrak{h}}(1, \alpha)$ for some $\alpha \in \mathfrak{h}$, and an $\mathbb{N}$-graded $M_{\mathfrak{h}}(1, 0)$-module W has a structure*

$$W \simeq M_{\mathfrak{h}}(1, 0) \otimes \Omega_W \quad \textit{with} \quad \Omega_W := \{v \in W \mid \hat{\mathfrak{h}}^+ \cdot v = 0\}.$$

A module W is completely reducible if and only if $\hat{\mathfrak{h}}^0$ acts on W semisimply.

Later, we will show that all modules $M_{\mathfrak{h}}(1, \alpha)$ are deformations of the 0-sector $M_{\mathfrak{h}}(1, 0)$ by using semisimple primary vectors, and based on this fact we can also prove the fusion rules $M_{\mathfrak{h}}(1, \alpha) \times M_{\mathfrak{h}}(1, \beta) = M_{\mathfrak{h}}(1, \alpha + \beta)$ for any $\alpha, \beta \in \mathfrak{h}$.

2.6.2 Lattice VOA

Let L be an integral lattice with a symmetric positive definite bilinear form $\langle \cdot, \cdot \rangle$. Then by using its complexification $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ we can construct a free bosonic VOA $M_{\mathfrak{h}}(1,0)$ and its irreducible modules $M_{\mathfrak{h}}(1,\alpha)$, $\alpha \in L$. We can define a VOA-structure on $V_L := \oplus_{\alpha \in L} M_{\mathfrak{h}}(1,\alpha)$. Before we define the vertex operator map on it, we have to construct a central extension of an abelian group L. It is shown in [FLM] that there is a unique central extension up to equivalence

$$1 \to \mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1 \rangle \to \hat{L} = \{\kappa^i e^\alpha \mid \alpha \in L,\ i = 0,1\} \to L \to 1$$

such that $e^\alpha \cdot e^\beta = \kappa^{\langle \alpha,\beta \rangle + \langle \alpha,\alpha \rangle \cdot \langle \beta,\beta \rangle} e^\beta \cdot e^\alpha$, where $e^\alpha \in \hat{L}$ is a section of $\alpha \in L$. It is convenient to introduce a notation $\mathbb{C}\{L\} := \mathbb{C}[\hat{L}]/(\kappa+1)\mathbb{C}[\hat{L}] \simeq \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in L\}$. Now we identify the highest weight vector $e^\alpha \in M_{\mathfrak{h}}(1,\alpha)$ with an element $e^\alpha \in \mathbb{C}\{L\}$ for each $\alpha \in L$. Let $\{\alpha_1, \dots, \alpha_{\mathrm{rank}(L)}\}$ be a $\mathbb{Z}$-basis of L and $\{\alpha^1, \dots, \alpha^{\mathrm{rank}(L)}\}$ its dual basis in $\mathfrak{h}$. Then V_L has the following linear basis

$$\{\alpha_{i_1}(-n_1)\alpha_{i_2}(-n_2) \cdots \alpha_{i_r}(-n_r) \otimes e^\beta \mid n_1 \geq n_2 \geq \cdots \geq n_r \geq 0,\ r \in \mathbb{N},\ \beta \in L\}$$

because V_L is isomorphic to $M_{\mathfrak{h}}(1,0) \otimes \mathbb{C}\{L\}$ as a vector space. For $\alpha \in \mathfrak{h}$, we define

$$E^\pm(\alpha, z) := \exp\left(\sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{\pm n} z^{\mp n} \right).$$

Then we define the vertex operator map $Y_{V_L}(\cdot, z)$ on $V_L \simeq M_{\mathfrak{h}}(1,0) \otimes \mathbb{C}\{L\}$ by

$$Y_{V_L}(e^\beta, z) := E^-(-\beta, z) E^+(-\beta, z)(1 \otimes \mathrm{ad}\, e^\beta) z^{\beta(0)},$$

for $\beta \in L$, where $\mathrm{ad}\, e^\beta$ is a left multiplication of e^β on $\mathbb{C}\{L\}$. Note that we have a relation

$$(z_1 - z_2)^{-\langle \alpha,\beta \rangle} Y_{V_L}(e^\alpha, z_1) Y_{V_L}(e^\beta, z_2) - (-1)^{\langle \alpha,\alpha \rangle \cdot \langle \beta,\beta \rangle} (-z_2 + z_1)^{-\langle \alpha,\beta \rangle} Y_{V_L}(e^\beta, z_2) Y_{V_L}(e^\alpha, z_1) = 0$$

for $\alpha, \beta \in L$. Then inductively we define

$$Y_{V_L}(a(m)x, z) := \mathrm{Res}_{z_0}\{(z_0 - z)^m a(z_0) Y_{V_L}(x, z) - (-z + z_0)^m Y_{V_L}(x, z) a(z_0)\}$$

for $\alpha \in \mathfrak{h}$ and $x \in V_L$, where $\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}$. Then it is shown in [FLM] that the quadruple $(V_L, Y_{V_L}(\cdot, z), \mathbb{1}, \omega)$, where $\mathbb{1}$ and ω are the vacuum vector and the Virasoro vector of $M_{\mathfrak{h}}(1,0) \subset V_L$, respectively, has a structure of a simple SVOA, and if $(L, \langle \cdot, \cdot \rangle)$ is an even lattice, then V_L is a simple VOA.

All irreducible V_L-modules are classified in [D1] and are in one-to-one correspondence with the quotient space L°/L, where $L^\circ := \{v \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid \langle v, L \rangle \in \mathbb{Z}\}$ is the dual lattice of L. More precisely, for a coset $\lambda + L \in L^\circ/L$, we can find a V_L-module structure on

$$V_{L+\lambda} = \oplus_{\alpha \in L+\lambda} M_{\mathfrak{h}}(1,\alpha).$$

Later we will prove this fact by using Li's method. The q-character of $V_{L+\lambda}$ is given by

$$q^{-\mathrm{rank}(L)/24}\mathrm{ch}_{V_{L+\lambda}}(q) = \sum_{\alpha\in L+\lambda} q^{\langle\alpha,\alpha\rangle/2}\cdot\eta(q)^{-\mathrm{rank}(L)} = \theta_{L+\lambda}(q)/\eta(q)^{\mathrm{rank}(L)},$$

where $\theta_{L+\lambda}(q) = \sum_{\alpha\in L+\lambda} q^{\langle\alpha,\alpha\rangle/2}$ is a theta series on $L+\lambda$.

2.6.3 Affine VOA

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ with an invariant bilinear form $\langle\cdot,\cdot\rangle$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We normalize $\langle\cdot,\cdot\rangle$ such that $\langle\theta,\theta\rangle = 2$ for the highest root $\theta^{\dagger}$. Let $\hat{\mathfrak{g}} = \mathbb{C}[t,t^{-1}]\otimes\mathfrak{g}\oplus\mathbb{C}c$ be its affinization, where its Lie bracket is given as follows:

$$[t^m\otimes a, t^n\otimes b] := t^{m+n}\otimes[a,b] + \delta_{m+n,0}\langle a,b\rangle c \quad \text{for } a,b\in\mathfrak{g}, \quad [\hat{\mathfrak{g}},c]=0.$$

Set $\hat{\mathfrak{g}}^{\pm} := \oplus_{n>0}\mathbb{C}t^{\pm n}\otimes\mathfrak{g}$ and $\hat{\mathfrak{g}}^0 := 1\otimes\mathfrak{g}\oplus\mathbb{C}c$. We identify $\mathfrak{g}$ with $1\otimes\mathfrak{g}\subset\hat{\mathfrak{g}}^0$. Let $\lambda\in\mathfrak{h}^*$ and $M(\lambda)$ a highest weight $\mathfrak{g}$-module with highest weight λ. By letting $\hat{\mathfrak{g}}^+$ act on $M(\lambda)$ trivially and c act on it as a scalar $\ell\in\mathbb{C}$, we may view $M(\lambda)$ as a $(\hat{\mathfrak{g}}^0+\hat{\mathfrak{g}}^+)$-module. Then we obtain a Verma module or a highest weight module for $\hat{\mathfrak{g}}$ of level ℓ as follows:

$$M_{\mathfrak{g}}(\ell,\lambda) := \mathrm{Ind}_{\mathfrak{U}(\hat{\mathfrak{g}}^0+\hat{\mathfrak{g}}^+)}^{\mathfrak{U}(\hat{\mathfrak{g}})} M(\lambda) = \mathfrak{U}(\hat{\mathfrak{g}}) \underset{\mathfrak{U}(\hat{\mathfrak{g}}^0+\hat{\mathfrak{g}}^+)}{\otimes} M(\lambda).$$

We show that there is a natural VOA structure on $M_{\mathfrak{g}}(\ell,0)$ if $\ell\neq h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Let $\{a_1,\dots,a_{\dim\mathfrak{g}}\}$ be a linear basis of $\mathfrak{g}$ and $\{a^1,\dots,a^{\dim\mathfrak{g}}\}$ its dual basis. Let $M(0)=\mathbb{C}v^0$. For $a\in\mathfrak{g}$ we denote $t^n\otimes a$ by $a(n)$, and define $a(z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$. Note that we have a relation

$$(z_1-z_2)^2 a(z_1)b(z_2) - (-z_2+z_1)^2 b(z_2)a(z_1) = 0$$

for $a,b\in\mathfrak{g}$, and $M_{\mathfrak{g}}(\ell,0)$ has a following linear basis:

$$\{a_{i_1}(-n_1)\cdots a_{i_r}(-n_r)v^0 \mid n_1\geq\cdots\geq n_r>0\}.$$

We define a vertex operator map $Y(\cdot,z)$ on $M_{\mathfrak{g}}(\ell,0)$ as follows. First, we set $Y(v^0,z) = \mathrm{id}_{M_{\mathfrak{g}}(\ell,0)}$ and inductively we define

$$Y(a(n)x,z) = \mathrm{Res}_{z_0}\{(z_0-z)^n a(z_0)Y(x,z) - (-z+z_0)^n Y(x,z)a(z_0)\}$$

for $a\in\mathfrak{g}$, $n\in\mathbb{Z}$ and $x\in M_{\mathfrak{g}}(\ell,0)$. Put $\mathbb{1}=v^0$ and

$$\omega = \frac{1}{2(\ell+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}} a^i(-1)a_i(-1)v^0 \in M_{\mathfrak{g}}(\ell,0).$$

†Then the Killing form on $\mathfrak{g}$ is given by $2h^\vee\langle\cdot,\cdot\rangle$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Then it is shown in [FZ] that the quadruple $(M_{\mathfrak{g}}(\ell,0), Y(\cdot,z), \mathbb{1}, \omega)$ satisfies all the axioms for a vertex operator algebra with central charge $\ell \dim \mathfrak{g}/(\ell + h^\vee)$. Moreover, if $\ell \neq 0$, then for any ideal I of $M_{\mathfrak{g}}(\ell,0)$ the quotient $\hat{\mathfrak{g}}$-module $M_{\mathfrak{g}}(\ell,0)/I$ also becomes a VOA. In particular, the unique irreducible quotient $L_{\mathfrak{g}}(\ell,0)$ has a structure of a simple VOA. This VOA is often called the *affine VOA* associated to a Lie algebra $\mathfrak{g}$.

By defining vertex operator map by a similar way, we can also verify that every highest weight $\hat{\mathfrak{g}}$-module $M_{\mathfrak{g}}(\ell,\lambda)$, $\lambda \in \mathfrak{h}^*$, becomes an $M_{\mathfrak{g}}(\ell,0)$-module. However, not all of them become modules for the affine VOA $L_{\mathfrak{g}}(\ell,0)$ in general. For example, if ℓ is a positive integer, then it is shown in [FZ] and [Li1] that only the integrable $\hat{\mathfrak{g}}$-modules afford $L_{\mathfrak{g}}(\ell,0)$-module structures.

2.6.4 Virasoro VOA

Let $\mathrm{Vir} = \oplus_{n\in\mathbb{Z}} \mathbb{C}L(n) \oplus \mathbb{C}\mathsf{c}$ be the Virasoro algebra with the following defining relations:

$$[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}\mathsf{c}, \quad [\mathrm{Vir}, \mathsf{c}] = 0.$$

Set $\mathrm{Vir}^\pm := \oplus_{n>0}\mathbb{C}L(\pm n)$ and $\mathrm{Vir}^0 := \mathbb{C}L(0) \oplus \mathbb{C}\mathsf{c}$. Then we obtain a triangular decomposition $\mathrm{Vir} = \mathrm{Vir}^+ \oplus \mathrm{Vir}^0 \oplus \mathrm{Vir}^-$ so that we can consider highest weight modules. Take any $c, h \in \mathbb{C}$, and define a semisimple Vir^0-module $\mathbb{C}v_{(c,h)}$ by $L(0)v_{(c,h)} = h \cdot v_{(c,h)}$ and $\mathsf{c}v_{(c,h)} = c \cdot v_{(c,h)}$. By letting Vir^+ act on $\mathbb{C}v_{(c,h)}$ trivially, we may consider $\mathbb{C}v_{(c,h)}$ as a $(\mathrm{Vir}^0 + \mathrm{Vir}^+)$-module. Then a *Verma module* or a *highest weight module* of the Virasoro algebra with central charge c and highest weight h is defined as

$$M_{\mathrm{Vir}}(c,h) := \mathrm{Ind}_{\mathfrak{U}(\mathrm{Vir}^0+\mathrm{Vir}^+)}^{\mathfrak{U}(\mathrm{Vir})}\mathbb{C}v_{(c,h)} = \mathfrak{U}(\mathrm{Vir}) \underset{\mathfrak{U}(\mathrm{Vir}^0+\mathrm{Vir}^+)}{\otimes} \mathbb{C}v_{(c,h)}.$$

By the famous PBW theorem, $M_{\mathrm{Vir}}(c,h)$ has a linear basis

$$\{L(-n_1)\cdots L(-n_k)v_{(c,h)} \mid n_1 \geq \cdots \geq n_k > 0,\ k \in \mathbb{Z}\}.$$

One can easily check that

$$L(0) \cdot L(-n_1)\cdots L(-n_k)v_{(c,h)} = (n_1 + \cdots + n_k + h) \cdot L(-n_1)\cdots L(-n_k)v_{(c,h)}$$

so that we have a q-character $\mathrm{ch}_{M_{\mathrm{Vir}}(c,h)} = q^h/\eta(q)$. One can also verify that $M_{\mathrm{Vir}}(c,h)$ affords a unique symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$ up to linearity, where the term "invariance" means

$$\langle L(n)a, b\rangle = \langle a, L(-n)b\rangle \quad \text{for any } a, b \in M_{\mathrm{Vir}}(c,h) \text{ and } n \in \mathbb{Z}.$$

The kernel of the above bilinear form is a unique maximal ideal of $M_{\mathrm{Vir}}(c,h)$, and is often denoted by $J(c,h)$. Then the quotient Vir-module $L_{\mathrm{Vir}}(c,h) := M_{\mathrm{Vir}}(c,h)/J(c,h)$ is a unique irreducible highest weight module with central charge c and highest weight h.

Structures of Verma modules $M_{\mathrm{Vir}}(c,h)$ have been studied so well and it is important to study singular vectors in $M_{\mathrm{Vir}}(c,h)$. A *singular vector* of $M_{\mathrm{Vir}}(c,h)$ is a vector w such that $L(n)w = 0$ for all $n \geq 0$. By the commutator formula, a vector $u \in M_{\mathrm{Vir}}(c,h)$ is singular if and only if $L(1)u = L(2)u = 0$. We may assume that every singular vector is homogeneous with respect to the action of $L(0)$. It is obvious that every singular vector u is contained in $J(c,h)$ if $u \notin \mathbb{C}v_{(c,h)}$. In particular, the irreducible quotient $L_{\mathrm{Vir}}(c,h)$ contains no singular vector but the highest weight vectors $\mathbb{C}v_{(c,h)}$.

Let us consider $M_{\mathrm{Vir}}(c,0)$, a Verma module with highest weight 0. In this case we always have a singular vector $L(-1)v_{(c,0)}$ of weight 1 for every $c \in \mathbb{C}$. By the PBW theorem, we know that the subalgebra of $M_{\mathrm{Vir}}(c,0)$ generated over $L(-1)v_{(c,0)}$ is isomorphic to a Verma module $M_{\mathrm{Vir}}(c,1)$. So we obtain a quotient module $M_{\mathrm{Vir}}(c,0)/M_{\mathrm{Vir}}(c,1)$. Denote by $\mathbb{1}$ the image of $v_{(c,0)}$ in this quotient. Then we have a relation $L(-1)\mathbb{1} = 0$. Now let us consider a generating series $\omega(z) := \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ of operators on $M_{\mathrm{Vir}}(c,0)/M_{\mathrm{Vir}}(c,1)$. By direct computation, we can show $(z_1-z_2)^4[\omega(z_1),\omega(z_2)] = 0$. Then by a theory of local systems, we can prove that $M_{\mathrm{Vir}}(c,0)/M_{\mathrm{Vir}}(c,1)$ has a unique vertex operator map $Y(\cdot,z)$ such that $\mathbb{1}$ is the vacuum vector and $\omega = L(-2)\mathbb{1}$ is the Virasoro vector such that $Y(L(-2)\mathbb{1},z) = \omega(z)$. The VOA $(M_{\mathrm{Vir}}(c,0)/M_{\mathrm{Vir}}(c,1), Y(\cdot,z), \mathbb{1}, \omega)$ is called the *Virasoro VOA*.

The structure of the Virasoro VOAs are deeply related to the structure of Verma modules, and the following fact is known (cf. [FF], see also [Ast]):

Proposition 2.6.2. *([FF]) For coprime integers $p,q \geq 2$, set $c_{p,q} = 1 - 6(p-q)^2/pq$. Then (i) $J(c,0)$ is generated by the singular vector $L(-1)v_{(c,0)}$ if $c \neq c_{p,q}$; (ii) $J(c_{p,q},0)$ is generated by two singular vector. One of them is $L(-1)v_{(c,0)}$. The other is denoted by $w_{p,q}$ whose weight is $(p-1)(q-1)$ and the monomial having maximal length in $w_{p,q}$ is $L(-2)^{(p-1)(q-1)/2}v_{(c,0)}$.*

By this fact, the Virasoro VOAs with central charge $c_{p,q}$ have a special importance. The irreducible quotient $L_{\mathrm{Vir}}(c_{p,q},0)$ are simple VOAs and called the *minimal series* or the *BPZ series* of the Virasoro VOAs. About this series, the following fact is known:

Theorem 2.6.3. *([Wan]) For coprime integers $p,q \geq 2$, set*

$$h_{r,s}^{p,q} := \frac{(sp-rq)^2-(p-q)^2}{4pq}. \tag{2.6.1}$$

Then every irreducible $L_{\mathrm{Vir}}(c_{p,q},0)$-module is isomorphic to $L_{\mathrm{Vir}}(c_{p,q},h_{r,s}^{p,q})$ for some $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$. Moreover, every $L_{\mathrm{Vir}}(c_{p,q},0)$-module is completely reducible.

In this article, the case $p = m+2$ and $q = m+3$, $m = 0, 1, 2, \ldots,$ are extremely significant. Set $c_m := c_{m+2,m+3} = 1 - 6/(m+2)(m+3)$ and $h_{r,s}^{(m)} := h_{r,s}^{m+2,m+3}$ for $m \in \mathbb{N}$, $1 \le r \le m+1$ and $1 \le s \le m+2$. Then it is shown in [FQS] [GKO] [KR] that the irreducible quotient $L(c_m, h_{r,s}^{(m)})$ affords a non-trivial unitary invariant bilinear form. So $L(c_m, 0)$, $m = 1, 2, \cdots$, are called the *unitary series* of the Virasoro VOA.

Chapter 3

Fundamental Topics

This chapter is devoted to present fundamental results on vertex operator algebras.

3.1 Simple VOAs

In this section we enjoy some properties of vertex operators. We will see that a simple vertex operator algebra has an aspect of a commutative associative algebra or sometimes a division ring.

Let $(V, Y(\cdot, z), \mathbb{1}, \omega)$ be a vertex operator algebra. A V-module is said to be irreducible if it has no non-trivial proper submodule. A VOA is said to be simple if it has no non-trivial proper ideal, or equivalently, if the adjoint module is irreducible. We should note that a left or a right ideal of a vertex operator algebra is also a two-sided ideal by the skew-symmetry.

For a subset A of V and that B of M, we define

$$A \cdot B := \mathrm{Span}_{\mathbb{C}}\{a_{(n)}b \mid a \in A,\, b \in B,\ n \in \mathbb{Z}\}.$$

The following simple lemma will be used frequently.

Lemma 3.1.1. *Let M be an $\mathbb{N}$-graded V-module. Let A^1, A^2 be subsets of V and B a subset of M. Then $A^1 \cdot (A^2 \cdot B) \subset (A^1 \cdot A^2) \cdot B$. In particular, $V \cdot v$ is a submodule of M for each $v \in M$.*

Proof: Use the associativity (2.3.2). ∎

For an $L(0)$-homogeneous $a \in V$, we define $o(a) := a_{(\mathrm{wt}(a)-1)}$ and extend linearly on V. The operator $o(a)$ is called a *zero-mode* of a and preserves every graded piece of an $\mathbb{N}$-graded module. As a corollary of Lemma 3.1.1, we have

Corollary 3.1.2. *Let $M = \oplus_{n\in\mathbb{N}} M(n)$ be an irreducible $\mathbb{N}$-graded V-module. Then $M(n) = \mathrm{Span}_{\mathbb{C}}\{o(a)v \mid a \in V\}$ for each $0 \neq v \in M(n)$. In particular, M has a countable dimension.*

The following lemma is a generalization of Schur's lemma.

Lemma 3.1.3. *Let A be an associative algebra over $\mathbb{C}$ and M an irreducible A-module with countable dimension. Then* $\mathrm{End}_A(M) = \mathbb{C}$.

Proof: Let $\varphi \in \mathrm{End}_A(M)$. We may assume $\varphi \neq 0$. First, we show that there is a scalar $\alpha \in \mathbb{C}$ such that $\varphi - \alpha$ is not invertible on M. Assume false. Then $\varphi - \alpha$ is invertible for all $\alpha \in \mathbb{C}$. Then $f(\varphi)$ is also invertible for any $f(x) \in \mathbb{C}[x]$ and so $f(\varphi)^{-1}$ is well-defined element in $\mathrm{End}_A(M)$. Take a non-zero element $v \in M$ and fix it. Then a mapping $\mathbb{C}(x) \ni f(x)/g(x) \mapsto f(\varphi) \cdot g(\varphi)^{-1} \cdot v \in M$ is well-defined and so we obtain a $\mathbb{C}$-linear homomorphism from $\mathbb{C}(x)$ to M. Since $\varphi - \alpha$ is invertible for all α, this homomorphism must be injective, which contradicts to the assumption that M is countable dimension. Thus we can take an $\alpha_0 \in \mathbb{C}$ such that $\varphi - \alpha_0$ is not invertible. Then at least one of the spaces $\mathrm{Ker}(\varphi - \alpha_0)$ or $\mathrm{Im}(\varphi - \alpha_0)$ is a proper subspace of M. If $\mathrm{Ker}(\varphi - \alpha_0) \neq 0$, then $\mathrm{Ker}(\varphi - \alpha_0) = M$ by the irreducibility and we are done. So we assume that $\mathrm{Ker}(\varphi - \alpha_0) = 0$ and $\mathrm{Im}(\varphi - \alpha_0) \neq 0$. Then again by the irreducibility we have $\mathrm{Im}(\varphi - \alpha_0) = M$ which means that $\varphi - \alpha_0$ is surjective. In this case we can also verify that $\varphi - \alpha_0$ is injective and hence invertible, which contradicts to the choice of α_0. Thus $\varphi = \alpha_0 \in \mathbb{C}$. ∎

Corollary 3.1.4. *Every irreducible $\mathbb{N}$-graded V-module has a $L(0)$-weight space decomposition.*

Proof: Let $M = \oplus_{n\in\mathbb{N}} M(n)$ be an irreducible $\mathbb{N}$-graded V-module with $M(0) \neq 0$. Then by Corollary 3.1.2 $M(0) = \mathrm{Span}_{\mathbb{C}}\{o(a)v \mid a \in V\}$ with a non-zero $v \in M(0)$. By the Jacobi identity, we have the following commutator formula:

$$[a_{(m)}, b_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}$$

for any $a, b \in V$. By substituting $L(0) = \omega_{(1)}$ into the above equality, we obtain

$$[L(0), a_{(m)}] = (\mathrm{wt}(a) - m - 1)a_{(m)}.$$

In particular, we have $[L(0), o(a)] = [\omega_{(1)}, a_{(\mathrm{wt}(a)-1)}] = 0$. Then there is an $h \in \mathbb{C}$ such that $L(0)$ acts on $M(0)$ as a scalar h by Lemma 3.1.3. Since $M = V \cdot v$, $M(n) = \mathrm{Span}_{\mathbb{C}}\{a_{(\mathrm{wt}(a)-1-n)}v \mid a \in V\}$. Then $L(0)$ acts on $M(n)$ as a scalar $n + h$. ∎

Proposition 3.1.5. *([DL]) Let M^i, $i = 1, 2, 3$, be V-modules and $I(\cdot, z)$ a V-intertwining operator of type $M^1 \times M^2 \to M^3$. Assume that there are subsets $S^i \subset M^i$ for $i = 1, 2$ such that $I(s^1, z)s^2 = 0$ for any $s^1 \in S^1$ and $s^2 \in S^2$. If M^i has no proper submodule containing S^i for $i = 1, 2$, then $I(\cdot, z) = 0$. In particular, if both M^1 and M^2 are irreducible, then $I(u, z)v = 0$ for some non-zero $u \in M^1$, $v \in M^2$ implies $I(\cdot, z) = 0$.*

Proof: Let $s^1 \in S^1$ and $s^2 \in S^2$. By assumption, we have $Y(a,z_1)I(s^1,z_2)s^2 = 0$ for any $a \in V$. Then by the commutativity there is an $N > 0$ such that $(z_1 - z_2)^N I(s^1,z_2)Y(a,z_1) = (z_1-z_2)^N Y(a,z_1)I(s^1,z_2)s^2 = 0$. Since there are finite negative powers of z_1 in $I(s^1,z_2)Y(a,z_1)s^2$, we have $I(s^1,z_2)Y(a,z_1)s^2 = 0$ for any $a \in V$. Then by the Jacobi identity we have

$$I(a_{(n)}s^1,z)s^2 = \mathrm{Res}_{z_0}\{(z_0-z)^n Y(a,z_0)I(s^1,z) - (-z+z_0)^n I(s^1,z)Y(a,z_0)\}s^2 = 0,$$

and hence $I^1(M^1,z)S^2 = 0$. Then by the commutator formula:

$$[a_{(n)}, I(u,z)] = \sum_{i=0}^{\infty} \binom{n}{i} z^{n-i} I(a_{(i)}u,z)$$

we have $I(u,z)a_{(n)}s^2 = a_{(n)}I(u,z)s^2 - [a_{(n)},I(u,z)]s^2 = 0$ for any $a \in V$ and $n \in \mathbb{Z}$. Therefore, $I(M^1,z)M^2 = 0$ and hence $I(\cdot,z) = 0$. ∎

Corollary 3.1.6. *Let V be a simple VOA.*
(1) For any $a,b \in V$, $Y(a,z)b = 0$ implies $a = 0$ or $b = 0$.
(2) Let W be an irreducible V-module. Then for non-zero $a^1,\ldots,a^k \in V$ and linearly independent $w^1,\ldots,w^k \in W$, $\sum_{i=1}^k Y(a^i,z)w^i \neq 0$.
(3) Let W be a V-module. Then $Y(a,z)w = 0$ implies $a = 0$ or $w = 0$.

Proof: (1) is clear from Proposition 3.1.5. Consider (2). If $\sum_i Y(a^i,z)w^i = 0$, then we can show that $\sum_i Y(a^i,z)Y(b^1,z_1)w^i = 0$ for any $b^1 \in V$ by a similar argument used in the proof of Proposition 3.1.5. Thus $\sum_i Y(a^i,z)Y(b^n,z_n)\cdots Y(b^1,z_1)w^i = 0$ for $b^1,\ldots,b^n \in V$. Since w^i are linearly independent in W and $\mathbb{C}$ is algebraically closed, we are reduced to the case $i = 1$ by the density theorem. This gives a contradiction and the assertion holds.

Now consider (3). Assume that $a \neq 0$ and $w \neq 0$. We may assume that $W = V \cdot w$. We can take a maximal submodule U of W which does not contain w by Zorn's lemma. Then the quotient W/U is an irreducible V-module and a relation $Y(a,z)w = 0$ in W yields a contradiction. ∎

3.2 A theory of local systems

In this section we review a theory of local systems in [Li1] [Li2] and their close relations to a theory of vertex operator algebras.

Let $M = M^0 \oplus M^1$ be a $\mathbb{Z}_2$-graded vector space. Then $\mathrm{End}(M) = (\mathrm{End}(M))^0 \oplus (\mathrm{End}(M))^1$ is also $\mathbb{Z}_2$-graded vector space where

$$(\mathrm{End}(M))^0 = \{a \in \mathrm{End}(M) \mid aM^i \subseteq M^i \quad \text{for } i = 0,1\},$$
$$(\mathrm{End}(M))^1 = \{a \in \mathrm{End}(M) \mid aM^0 \subseteq M^1,\ aM^1 \subseteq M^0\}.$$

Furthermore,

$$\mathrm{End}(M)[[z,z^{-1}]] = (\mathrm{End}(M))^0[[z,z^{-1}]] \oplus (\mathrm{End}(M))^1[[z,z^{-1}]]$$

is also a $\mathbb{Z}_2$-graded vector space. It is clear that the derivative operator $\partial_z := d/dz$ is an endomorphism of $(\mathrm{End}(M))[[z,z^{-1}]]$ preserving the $\mathbb{Z}_2$-grading.

Definition 3.2.1. Let M be a $\mathbb{Z}_2$-graded space and ∂_M a $\mathbb{Z}_2$-homogeneous endomorphism on M. A *field* on (M,∂_M) is a formal series $a(z)=\sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in \mathrm{End}(M)[[z,z^{-1}]]$ such that $a(z)v \in M((z))$ for any $u \in M$ and $[\partial_M, a(z)] = \partial_z a(z)$. We denote by $\mathcal{F}(M,\partial_M)$ the space of all fields on (M,∂_M).

It is obvious that $\mathcal{F}(M,\partial_M)$ has a $\mathbb{Z}_2$-grading $\mathcal{F}(M,\partial_M)^0 \oplus \mathcal{F}(M,\partial_M)^1$ and ∂_z is a $\mathbb{Z}_2$-homogeneous endomorphism of $\mathcal{F}(M,\partial_M)$. The space $\mathcal{F}(M,\partial_M)$ has the following algebraic operations:

Lemma 3.2.2. *Let $a(z), b(z)$ be $\mathbb{Z}_2$-homogeneous elements in $\mathcal{F}(M,\partial_M)$. Then*

$$a(z)\circ_n b(z) := \mathrm{Res}_{z_0}\{(z_0-z)^n a(z_0)b(z) - (-1)^{\varepsilon(a(z),b(z))}(-z+z_0)^n b(z)a(z_0)\} \qquad (3.2.1)$$

is also a $\mathbb{Z}_2$-homogeneous field on (M,∂_M) for all $n\in\mathbb{Z}$. If $a(z)\in\mathcal{F}(M,\partial_M)^i$ and $b(z)\in\mathcal{F}(M,\partial_M)^j$, where $i,j\in\mathbb{Z}_2$, then $a(z)\circ_n b(z)\in\mathcal{F}(M,\partial_M)^{i+j}$, that is, $\circ_n$ is a $\mathbb{Z}_2$-graded operation on $\mathcal{F}(M,\partial_M)$.

Proof: An easy exercise. ∎

The product $\circ_n$ in $\mathcal{F}(M,\partial_M)$ is called the *n-th normal ordered product*. For $\mathbb{Z}_2$-homogeneous $a(z), b(z)\in\mathcal{F}(M,\partial_M)$, we define

$$\begin{aligned}
Y(a(z),z_0)b(z) &:= \sum_{n\in\mathbb{Z}} a(z)\circ_n b(z) z_0^{-n-1}\\
&= \mathrm{Res}_{z_1}\left\{ z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right) a(z_1)b(z) - (-1)^{\varepsilon(a,b)} z_0^{-1}\delta\left(\frac{-z+z_1}{z_0}\right) b(z)a(z_1)\right\},
\end{aligned}$$

and extend linearly on $\mathcal{F}(M,\partial_M)$. Then we obtain a linear map

$$Y(\cdot,z) : \mathcal{F}(M,\partial_M)\otimes\mathcal{F}(M,\partial_M) \mapsto \mathcal{F}(M,\partial_M)((z)).$$

Lemma 3.2.3. *Let $a(z)\in\mathcal{F}(M,\partial_M)$. Then we have*
(1) $Y(I(z),z_0)a(z) = a(z)$ and $Y(a(z),z_0)I(z) = e^{z_0\partial_{z_0}}a(z) = a(z+z_0)$, where $I(z) :=$ $\mathrm{id}_M \in \mathcal{F}(M,\partial_M)^0$.
(2) $Y(\partial_z a(z),z_0) = [\partial_z, Y(a(z),z_0)] = \partial_{z_0} Y(a(z),z_0)$.

Proof: See the proof of Lemma 3.1.6 and Lemma 3.1.7 of [Li1]. ∎

By this lemma, the space $\mathcal{F}(M, \partial_M)$ together with $Y(\cdot, z)$ has a similar structure to a vertex superalgebra. However, the most important axiom, the commutativity, is lacked. So we have to consider suitable subspaces of $\mathcal{F}(M, \partial_M)$ to find vertex algebra structures.

Definition 3.2.4. Two $\mathbb{Z}_2$-homogeneous fields $a(z), b(z) \in \mathcal{F}(M, \partial_M)$ are said to be *mutually local* or simply *local* if there is an integer N such that

$$(z_1 - z_2)^n a(z_1) b(z_2) - (-1)^{\varepsilon(a,b)} (-z_2 + z_1)^n b(z_2) a(z_1) = 0$$

for all $n \geq N$. Among such N, the minimum one is called the *order of locality* of $a(z)$ and $b(z)$ and denoted by $N(a, b)$. A $\mathbb{Z}_2$-graded subspace of $\mathcal{F}(M, \partial_M)$ is called *local* if any two $\mathbb{Z}_2$-homogeneous fields in it are mutually local. A *local system* on (M, ∂_M) is a maximal local subspace of $\mathcal{F}(M, \partial_M)$.

Remark 3.2.5. By the maximality, every local system on (M, ∂_M) contains the identity field $I(z) = \mathrm{id}_M$.

Remark 3.2.6. If $(V, Y(\cdot, z), \mathbb{1}, \omega)$ is a vertex operator superalgebra and $(M, Y_M(\cdot, z))$ is a V-module, then the space $\{Y_M(a, z) \mid a \in V\}$ is a local subspace of $\mathcal{F}(M, L(-1))$.

Lemma 3.2.7. *If fields $a(z)$ and $b(z)$ are mutually local, then so are $a(z)$ and $\partial_z b(z)$.*

Proof: Let N be the order of locality between $a(z)$ and $b(z)$. Then

$$(z_1 - z_2)^{N+1} a(z_1) b(z_2) - (-1)^{\varepsilon(a,b)} (-z_2 + z_1)^{N+1} b(z_2) a(z_1) = 0.$$

Differentiating the above with respect to z_2, we obtain

$$(z_1 - z_2)^{N+1} a(z_1) \partial_{z_2} b(z_2) - (-1)^{\varepsilon(a,b)} (-z_2 + z_1)^{N+1} \partial_{z_2} b(z_2) a(z_1) = 0.$$

∎

The following famous lemma is known as Dong's lemma.

Lemma 3.2.8. *Let $a(z), b(z), c(z)$ be $\mathbb{Z}_2$-homogeneous mutually local fields. Then $a(z) \circ_n b(z)$ and $c(z)$ are also mutually local for all $n \in \mathbb{Z}$. The order of locality is bounded as follows:*

$$N(a \circ_n b, c) \leq \begin{cases} |N(a,b)| + |N(b,c)| + |N(c,a)| & \text{if } \ n \geq 0, \\ |N(a,b)| + |N(b,c)| + |N(c,a)| - n & \text{if } \ n < 0. \end{cases}$$

Proof: See, for example, the proof of Proposition 3.2.7 of [Li1]. ∎

Proposition 3.2.9. *Let V be a local system of fields on (M, ∂_M). Then for any $a(z)$ and $b(z)$ in V, $Y(a(z), z_0)$ and $Y(b(z), z_0)$ are mutually local fields on (V, ∂_{z_0}).*

Proof: See the proof of Proposition 3.2.9 of [Li1]. ∎

By this proposition, the vertex operator map $Y(\cdot, z)$ defines a vertex superalgebra structure on every local system.

Theorem 3.2.10. *Let (M, ∂_M) be a $\mathbb{Z}_2$-graded space with a $\mathbb{Z}_2$-homogeneous endomorphism ∂_M, and let V be a local system of fields on (M, ∂_M). Then $(V, Y(\cdot, z), I(z), \partial_z)$ is a vertex superalgebra and M is a V-module.*

Corollary 3.2.11. *Let (M, ∂_M) be a $\mathbb{Z}_2$-graded space with a $\mathbb{Z}_2$-homogeneous endomorphism ∂_M, and let S be a set of mutually local fields on (M, ∂_M). Let A be a local system on (M, ∂_M) containing S. Then the subspace generated by S and $I(\cdot, z)$ under the normal ordered products forms a vertex superalgebra in A with M as a module.*

The following theorem enable us to adopt another definition of a vertex algebra.

Theorem 3.2.12. *Let V be a vertex superalgebra. Then to give a V-module (M, ∂_M) is equivalent to give a vertex superalgebra homomorphism from V to some local system of fields on (M, ∂_M).*

Proof: See the proof of Proposition 3.2.13 of [Li1]. ∎

By this theorem, we may adopt the following definition of a vertex superalgebra:

> "A vertex superalgebra V is a subspace of fields $\mathcal{F}(M, \partial_M)$ on a $\mathbb{Z}_2$-graded vector space (M, ∂_M) such that (i) any two fields in V are local; (ii) $V \circ_n V \subset V$ for all $n \in \mathbb{Z}$; (iii) $\mathbb{1} \in V$."

(We do not have to assume that fields in V are closed under ∂_z as $a(z) \circ_{-2} \mathbb{1} = \partial_z a(z)$.) Actually, if V is a vertex superalgebra, then the subspace $\{Y(a, z) \mid a \in V\}$ of fields on (V, ∂_V) satisfies all the conditions above. On the other hand, if V is a vertex superalgebra in the above sense, then by the 1:1 state-field correspondence $a(z) \in V \longleftrightarrow Y(a(z), z_0) \in \mathcal{F}(V, \partial_z)$, V carries a structure of an axiomatic vertex superalgebra.

Uniqueness Theorem. The following theorem is extremely useful in identifying a field with one of the fields of a vertex operator algebra.

Theorem 3.2.13. *([G][K]) Let V be a vertex superalgebra and let $t(z)$ be a field on V which is mutually local with all the fields $Y(a, z)$, $a \in V$. Suppose that $t(z)\mathbb{1} = e^{z\partial_V} b$ for some $b \in V$. Then $t(z) = Y(b, z)$.*

Proof: Since the derivation ∂_V does not change the $\mathbb{Z}_2$-parity, we note that $\varepsilon(t, a) = \varepsilon(b, a)$ for any $a \in V$. By the assumption of locality we have:

$$(z_1 - z_2)^N t(z_1) Y(a, z_2)\mathbb{1} = (-1)^{\varepsilon(t,a)} (-z_2 + z_1)^N Y(a, z_2) t(z_1)\mathbb{1}$$

for sufficient large $N > 0$. Since $Y(a,z)\mathbb{1} = e^{z\partial_V}a$, we obtain:

$$(z_1 - z_2)^N t(z_1)e^{z_2\partial_V}a = (-1)^{\varepsilon(t,a)}Y(a,z_2)e^{z_1\partial_V}b = (-1)^{\varepsilon(t,a)}Y(a,z_2)Y(b,z_1)\mathbb{1}.$$

By taking sufficiently large N, we get

$$(z_1 - z_2)^N t(z_1)Y(a,z_2)\mathbb{1} = (-1)^{\varepsilon(t,a)}Y(a,z_2)Y(b,z_1)\mathbb{1} = (-1)^{\varepsilon(t,a)}Y(b,z_1)Y(a,z_2)\mathbb{1}.$$

Letting $z_2 = 0$ and dividing by z_1^N, we get $t(z)a = Y(b,z)a$ for any $a \in V$. ∎

Existence Theorem. The following theorem allows one to construct vertex superalgebras.

Theorem 3.2.14. *(Theorem 4.5 of [K]) Let V be a vector superspace, let $\mathbb{1}$ be an even vector of V and ∂_V an even endomorphism of V. Let $\{a^\alpha(z) = \sum_{n\in\mathbb{Z}} a^\alpha_{(n)}z^{-n-1}\}_{\alpha\in I}$ be a collection of $\mathbb{Z}_2$-homogeneous fields on V such that*
(i) $[\partial_V, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all $\alpha \in I$,
(ii) $\partial_V\mathbb{1} = 0$, $a^\alpha(z)\mathbb{1} = a^\alpha_{(-1)}\mathbb{1}$, $\alpha \in I$, where the $a^\alpha_{(-1)}\mathbb{1}$ are linearly independent in V,
(iii) $a^\alpha(z)$ and $a^\beta(z)$, $\alpha, \beta \in I$, are mutually local,
(iv) the vectors $a^{\alpha_1}_{(j_1)}\cdots a^{\alpha_n}_{(j_n)}\mathbb{1}$, $j_k \in \mathbb{Z}$, $n \geq 0$, span V.
Then the formula

$$Y_V(a^{\alpha_1}_{(j_1)}\cdots a^{\alpha_n}_{(j_n)}\mathbb{1}, z) := a^{\alpha_1}(z) \circ_{j_1} Y_V(a^{\alpha_2}_{(j_2)}\cdots a^{\alpha_n}_{(j_n)}\mathbb{1}, z) \tag{3.2.2}$$

defines a unique structure of a vertex superalgebra on V such that $\mathbb{1}$ is the vacuum vector, ∂_V is the derivation and $Y(a^\alpha_{(-1)}\mathbb{1}, z) = a^\alpha(z)$ for all $\alpha \in I$.

Proof: Choose a basis among the vectors of the form (iv) and define the vertex operator $Y(a,z)$ by formula (3.2.2). By (iii) and Dong's lemma, the locality axioms hold. Therefore, V has a structure of a vertex superalgebra which depends on a choice of linear basis of V at now. If we choose another basis among the monomials (iv) we get possibly different structure of a vertex superalgebra on V, which we denote by $Y'(\cdot, z)$. But all the fields of this new structure are mutually local with those of the old structure and satisfy $Y'(a,z)\mathbb{1} = e^{z\partial_V}a$. Then by Theorem 3.2.13 it follows that these vertex superalgebra structure coincide. Thus (3.2.2) is well-defined and we obtain the uniqueness of the structure. ∎

3.3 Free vertex algebras

In the previous section, we have seen that a vertex algebra is essentially a space of mutually local fields on a vector space. We also know that the Jacobi identity, the main axiom of a vertex algebra, is equivalent to the commutativity of vertex operators or in other words the locality of fields. By these facts, it is a quiet natural problem to construct a vertex algebra from a set of generators and a locality function among them. More precisely, we would like to construct a universal vertex algebra which is determined only by generators and their locality, that is, it has only the minimum relation to be a vertex algebra and thus shall have a universal property. Such a vertex algebra is called a *free vertex algebra* and studied by Roitman [R1] [R2]. In this section we give a construction of free vertex algebras as an application of a theory of local systems.

First, let us consider locality functions. Let $(V = V^0 \oplus V^1, Y(\cdot, z), \mathbb{1}, \partial)$ be a vertex operator superalgebra. For any $\mathbb{Z}_2$-homogeneous $a, b \in V$, denote by $N(a, b)$ the order of locality between $Y(a, z)$ and $Y(b, z)$. Then we have $(z_1 - z_2)^n Y(a, z_1)Y(b, z_2) - (-1)^{\varepsilon(a,b)}(-z_2 + z_1)^n Y(b, z_2)Y(a, z_1) = 0$ for $n \geq N(a, b)$. Then by the Jacobi identity we have

$$
\begin{aligned}
Y(a_{(n)}b, z_2) &= \mathrm{Res}_{z_0} z_0^n Y(Y(a, z_0)b, z_2) \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^n z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y(Y(a, z_0)b, z_2) \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^n \left\{ z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)Y(b, z_2) \right. \\
&\qquad \left. - (-1)^{\varepsilon(a,b)} z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y(b, z_2)Y(a, z_1) \right\} \\
&= \mathrm{Res}_{z_1} \left\{ (z_1 - z_2)^n Y(a, z_1)Y(b, z_2) - (-1)^{\varepsilon(a,b)}(-z_2 + z_1)^n Y(b, z_2)Y(a, z_1) \right\} \\
&= 0
\end{aligned}
$$

for $n \geq N(a, b)$. Applying the above to the vacuum, we obtain $a_{(n)}b = 0$ for $n \geq N(a, b)$. Conversely, if $a_{(n)}b = 0$ for all $n \geq N$, then one can similarly show that $(z_1 - z_2)^N Y(a, z_1)Y(b, z_2) - (-1)^{\varepsilon(a,b)}(-z_2 + z_1)^N Y(b, z_2)Y(a, z_1) = 0$. Thus the order of locality is the minimum N such that $a_{(n)}b = 0$ for all $n \geq N$. By the skew-symmetry $Y(a, z)b = (-1)^{\varepsilon(a,b)} e^{z\partial} Y(b, -z)a$, one can easily see that $N(a, b) = N(b, a)$. This observation lead us to the following fact.

Lemma 3.3.1. *Let $a \in V$ be $\mathbb{Z}_2$-homogeneous. Then $a \in V^0$ if and only if $N(a, a)$ is even.*

Proof: By definition of the locality, we have $a_{(N(a,a)-1)}a \neq 0$. On the other hand, the skew-symmetry implies

$$
a_{(N(a,a)-1)}a = (-1)^{\varepsilon(a,a)} \sum_{j \geq 0} \frac{(-1)^{N(a,a)+j}}{j!} \partial^j a_{(N(a,a)-1+j)}a = (-1)^{N(a,a)+\varepsilon(a,a)} a_{(N(a,a)-1)}a.
$$

Thus $N(a, a) \equiv \varepsilon(a, a) \mod 2$. ∎

Now we start to give a construction of a free vertex algebra. Let $\mathcal{B} = \mathcal{B}^0 \sqcup B^1$ be a set of symbols, $N : \mathcal{B} \times \mathcal{B} \to \mathbb{Z}$ be a function such that $N(a,a) \in 2\mathbb{Z}$ for all $a \in \mathcal{B}^0$ and $N(a,a) \in 2\mathbb{Z}+1$ for all $a \in B^1$. Let $\mathcal{A}$ be an associative algebra generated by symbols $\{\partial, a(n) \mid a \in \mathcal{B}, n \in \mathbb{Z}\}$ with the relation $[\partial, a(n)] = -na(n-1)$. Let $\mathbb{1}$ be a symbol and $\mathcal{F}$ a quotient $\mathcal{A}$-module of a free $\mathcal{A}$-module $\mathcal{A} \cdot \mathbb{1}$ factored by a relation $\partial\mathbb{1} = 0$. Then $\mathcal{F}$ has a linear basis $X = \{a_1(n_1)\cdots a_k(n_k)\mathbb{1} \mid a_i \in \mathcal{B}, n_i \in \mathbb{Z}, k \geq 0\}$. We define a function $N' : \mathcal{B} \times X \to \mathbb{Z}$ as follows. First, we set $N'(a, \mathbb{1}) = 0$ for $a \in \mathcal{B}$. Then for $k \geq 1$ we inductively define $N'(a, a_1(n_1)\cdots a_k(n_k)\mathbb{1})$ by $|N(a,a_1)| + N'(a, a_2(n_2)\cdots a_k(n_k)\mathbb{1}) + N'(a_1, a_2(n_2)\cdots a_k(n_k)\mathbb{1})$ if $n_1 \geq 0$, and by $|N(a,a_1)| + N'(a, a_2(n_2)\cdots a_k(n_k)\mathbb{1}) + N'(a_1, a_2(a_2)\cdots a_k(n_k)\mathbb{1}) - n_1$ if $n_1 < 0$. Let I be a left $\mathcal{A}$-ideal of $\mathcal{F}$ generated by $a(m)x$, $a \in \mathcal{B}$, $x \in X$ and $m \geq N'(a,x)$. For $a \in \mathcal{B}$, set $a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$. Then $a(z)$, $a \in \mathcal{B}$, are fields on $(\mathcal{F}/I, \partial)$. Let J be a left $\mathcal{A}$-ideal of $\mathcal{F}/I$ generated by coefficients of $(z_1-z_2)^{N(a,b)}a(z_1)b(z_2)v - (-1)^{\varepsilon(a,b)}(-z_2+z_1)^{N(a,b)}b(z_2)a(z_1)v$, $a, b \in \mathcal{B}$ and $v \in \mathcal{F}/I$, and set $\bar{\mathcal{F}} := (\mathcal{F}/I)/J$. Then $a(z)$, $a \in \mathcal{B}$, are mutually local fields on $(\bar{\mathcal{F}}, \partial)$ and hence they generate a vertex superalgebra. Then by Theorem 3.2.14, we have a unique vertex superalgebra structure $(\bar{\mathcal{F}}, Y(\cdot,z), \bar{\mathbb{1}}, \partial)$ such that $Y(\overline{a(-1)\mathbb{1}}, z) = a(z)$ for $a \in \mathcal{B}$, where $\bar{x}$ denotes the image of an element $x \in X$ in $\bar{\mathcal{F}}$.

To emphasize the generators and the locality function, we denote $\bar{\mathcal{F}}$ by $\mathcal{F}_N(\mathcal{B})$ as in [R1] [R2]. By our canonical construction, $\mathcal{F}_N(\mathcal{B})$ has the following universal property: Let $V = V^0 \oplus V^1$ be a vertex superalgebra and S a subset of $V^0 \sqcup V^1$. Let N_S be the locality function on S. Then there exists a unique vertex superalgebra epimorphism from $\mathcal{F}_{N_S}(S)$ to a subalgebra of V generated by S. By this fact, $\mathcal{F}_N(\mathcal{B})$ is called the *free vertex superalgebra defined by the set of generators $\mathcal{B}$ and the locality bound N*. Note that the order of locality between $Y_{\mathcal{F}_N(\mathcal{B})}(a(-1)\mathbb{1}, z)$ and $Y_{\mathcal{F}_N(\mathcal{B})}(b(-1)\mathbb{1}, z)$ for $a, b \in \mathcal{B}$ are less than or equal to $N(a,b)$, and we do not know whether it is exactly equal to $N(a,b)$ at now. So our construction determines only bounds for orders of locality. A precise description of the order of locality is deeply studied by Roitman in [R1] and [R2].

3.4 Invariant bilinear form

Let $(V, Y(\cdot,z), \mathbb{1}, \omega)$ be a VOA and $W = \oplus_{n\in\mathbb{N}} W(n)$ an $\mathbb{N}$-graded V-module such that each homogeneous component $W(n)$ is of finite dimension. We set the *restricted dual* of W by means of a direct sum $W^* := \oplus_{n\in\mathbb{N}} W(n)^*$. Since $W(n)$ are finite dimensional, we see that $(W^*)^*$ is naturally isomorphic to W. In this section we define a V-module structure on the dual space W^*. Let $\langle \cdot \mid \cdot \rangle : W^* \times W \to \mathbb{C}$ be a canonical pairing. We define the *adjoint vertex operator* $Y_M^*(\cdot,z)$ by

$$\langle\, Y_M^*(a,z)\nu \mid w \,\rangle = \langle\, \nu \mid Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})w \,\rangle \tag{3.4.1}$$

for $a \in V$, $w \in W$ and $\nu \in W^*$. If we write $Y_M^*(a,z) = \sum_{n\in\mathbb{Z}} a^*_{(n)} z^{-n-1}$, then

$$\langle\, a^*_{(n)}\nu \mid w \,\rangle = \sum_{i\geq 0} \frac{(-1)^{\mathrm{wt}(a)}}{i!} \left\langle\, \nu \mid (L(1)^i a)_{(2\mathrm{wt}(a)-n-i-2)} w \,\right\rangle . \tag{3.4.2}$$

The following is proved in Theorem 5.2.1 of [FHL]:

Theorem 3.4.1. *Let $(W, Y_M(\cdot,z))$ be an $\mathbb{N}$-graded V-module with finite dimensional homogeneous subspaces. Then $(W^*, Y_M^*(\cdot,z))$ is also an $\mathbb{N}$-graded V-module. Moreover, $(W^*)^* \simeq W$ as V-modules.*

We call $(W^*, Y_M^*(\cdot,z))$ the *dual module* of $(M, Y_M(\cdot,z))$. The pairing has an invariant property for the Virasoro algebra:

$$\langle\, L(n)\nu \mid w \,\rangle = \langle\, \nu \mid L(-n)w \,\rangle.$$

If W has an $L(0)$-weight space decomposition $W = \oplus_{n\geq 0} W_{n+h}$, then so does $W^* = \oplus_{n\geq 0} W^*_{n+h}$ and by the above invariance we have $\langle\, W^*_{m+h} \mid W_{n+h} \,\rangle = 0$ unless $m = n$.

If $W^* \simeq W$ as a V-module, then such a module is called *self dual*. Clearly, a self dual module has a V-invariant bilinear form on it. Now let us consider the case $W = V$. By definition, an invariant bilinear form $\langle \cdot \mid \cdot \rangle$ on V is a bilinear form satisfying the following property:

$$\langle\, Y_V(a,z)b^1 \mid b^2 \,\rangle = \langle\, b^1 \mid Y_V(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})b^2 \,\rangle.$$

The existence of an invariant bilinear form is equivalent to the existence of a V-module isomorphism from V^* to V. About this isomorphism, the following theorem has been established in Theorem 3.1 of [Li3].

Theorem 3.4.2. *The space of invariant bilinear forms on V is isomorphic to the space $(V_0/L(1)V_1)^* = \mathrm{Hom}_{\mathbb{C}}(V_0/L(1)V_1, \mathbb{C})$.*

Proof: Let $\mathcal{F}$ be the space of invariant bilinear forms on V and $\langle \cdot \mid \cdot \rangle$ an element in $\mathcal{F}$. Take $a, b \in V$ arbitrary. Then

$$\begin{aligned} \langle\, a \mid b \,\rangle &= \langle\, a_{(-1)}\mathbb{1} \mid b \,\rangle = \mathrm{Res}_z z^{-1} \langle\, Y_V(a,z)\mathbb{1} \mid b \,\rangle \\ &= \mathrm{Res}_z z^{-1} \langle\, \mathbb{1} \mid Y_V(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})b \,\rangle. \end{aligned}$$

Let $\Phi_{\langle\cdot|\cdot\rangle}$ be a linear functional on V_0 defined by $\Phi_{\langle\cdot|\cdot\rangle}(x) = \langle\, \mathbb{1} \mid x \,\rangle$ for $x \in V_0$. Then by the above equality $\langle \cdot \mid \cdot \rangle$ is completely determined by $\Phi_{\langle\cdot|\cdot\rangle}$. Since $L(-1)\mathbb{1} = \omega_{(0)}\mathbb{1} = 0$, we have $\Phi_{\langle\cdot|\cdot\rangle}(L(1)a) = \langle\, \mathbb{1} \mid L(1)a\,\rangle = \langle\, L(-1)\mathbb{1} \mid a\,\rangle = 0$ for any $a \in V$. Therefore, $L(1)V_1 \subset \mathrm{Ker}_{V_0}\Phi_{\langle\cdot|\cdot\rangle}$ and so we may view $\Phi_{\langle\cdot|\cdot\rangle}$ as a linear functional on $V_0/L(1)V_1$. Thus we obtain a linear map $\Phi : \mathcal{F} \ni \langle\cdot|\cdot\rangle \mapsto \Phi_{\langle\cdot|\cdot\rangle} \in \mathrm{Hom}_{\mathbb{C}}(V_0/L(1)V_1, \mathbb{C})$.

On the other hand, if $f \in \mathrm{Hom}_{\mathbb{C}}(V_0/L(1)V_1, \mathbb{C})$, then we can regard f as an element of V^* which vanishes on $L(1)V_1$ and on V_n for all $n \neq 0$. Let $\langle\,\cdot \mid \cdot\,\rangle$ be a natural pairing between V^* and V. For any $a \in V$, we have $\langle\, L(-1)f \mid a\,\rangle = \langle\, f \mid L(1)a\,\rangle = 0$ and hence $L(-1)f = 0$ in V^*. Then the next lemma tells us that a linear map $\psi : V \ni a \mapsto a_{(-1)}f \in V^*$ is a V-isomorphism. Namely, V as a V-module is isomorphic to V^*. Thus, we obtain a linear map $\Psi : \mathrm{Hom}_{\mathbb{C}}(V_0/L(1)V_1, \mathbb{C}) \to \mathcal{F}$. One can easily verify that $\Psi \circ \Phi = \mathrm{id}_{\mathcal{F}}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathrm{Hom}_{\mathbb{C}}(V_0/L(1)V_1,\mathbb{C})}$. Thus $\mathcal{F} \simeq \mathrm{Hom}_{\mathbb{C}}(V_0/L(1)V_1, \mathbb{C})$. ∎

In the proof above, we find a vector v in a module such that $L(-1)v = 0$. Such a vector is often called a *vacuum-like vector* and has an important property.

Lemma 3.4.3. *([Li3, Proposition 3.3]) For a V-module M, set $M^{\mathrm{vac}} := \{v \in M \mid L(-1)v = 0\}$. Then*
(i) For any $v \in M^{\mathrm{vac}}$, $Y_M(a, z)v = e^{zL(-1)}a_{(-1)}v$ for all $a \in V$. In particular, $a_{(n)}v = 0$ for all $n \geq 0$.
(ii) A linear map $\pi : \mathrm{Hom}_V(V, M) \ni \phi \mapsto \phi(\mathbb{1}) \in M^{\mathrm{vac}}$ is isomorphism. In particular, $V \ni a \mapsto a_{(-1)}v \in M$ defines an embedding of adjoint module V into M for each $v \in M^{\mathrm{vac}}$.

Proof: (i): Let $a \in V$ and $v \in M^{\mathrm{vac}}$. Then there exists an N such that $a_{(n)}v = 0$ for all $n \geq N$ and $a_{(N-1)}v \neq 0$. If $N > 0$, then by $[L(-1), a_{(n)}] = -na_{(n-1)}$, we have $0 = L(-1)a_{(N)}v = [L(-1), a_{(N)}]v + a_{(N)}L(-1)v = -Na_{(N-1)}v$, and hence $a_{(N-1)}v = 0$, which is a contradiction. So $a_{(n)}v = 0$ for $n \geq 0$. The equality $Y_M(a, z)v = e^{zL(-1)}a_{(-1)}v$ will follow from (ii).

(ii): If $\phi \in \mathrm{Hom}_V(V, M)$, then clearly $\phi(\mathbb{1}) \in M^{\mathrm{vac}}$. Let $a, b \in V$ and $u \in M^{\mathrm{vac}}$. Then by (i)

$$(a_{(n)}b)_{(-1)}u = \sum_{i\geq 0}(-1)^i\binom{n}{i}\{a_{(n-i)}b_{(-1+i)} - (-1)^n b_{(n-1-i)}a_{(i)}\}u = a_{(n)}b_{(-1)}u$$

and hence $\phi_u : V \ni a \mapsto a_{(-1)}u \in M$ is a V-homomorphism and $\pi(\phi_u) = \phi_u(\mathbb{1}) = u$. So π is epimorphism. On the other hand, if $\pi(\phi) = 0$, then $\phi(a) = \phi(a_{(-1)}\mathbb{1}) = a_{(-1)}\phi(\mathbb{1}) = 0$ for all $a \in V$ so that π is also injective. Thus π is an isomorphism. ∎

Corollary 3.4.4. *If V is a VOA of CFT-type and $L(1)V_1 = 0$, then V has a unique invariant bilinear form up to linearity.*

Remark 3.4.5. By the corollary above, we can show that all of the examples, that is, the free bosonic VOA $M_{\mathfrak{h}}(1, 0)$, the lattice VOA V_L, the affine VOA $L_{\mathfrak{g}}(\ell, 0)$ and the Virasoro VOA $L_{\mathrm{Vir}}(c, 0)$ have a unique invariant bilinear form up to linearity.

The following are proved in [FHL] and [Li3].

Proposition 3.4.6. *Let V be a VOA and M an $\mathbb{N}$-graded V-module.*
(1) ([FHL]) If V has an invariant bilinear form, then it is automatically symmetric.
(2) ([Li3]) If M is irreducible and has an invariant bilinear form, then it is either symmetric or skew-symmetric.

3.5 Zhu algebra

In this section we present a tool to study representations of a VOA, called the Zhu algebra. There are many variants of Zhu algebras; Zhu algebra for twisted representation, Zhu algebras for higher degrees, etc. Here we treat the most fundamental case. For references, see [Z] [Wan] [DLM1] [DLM6] [DLM7] [MT] [Y1]. Let us consider a Zhu algebra for a vertex operator superalgebra. Let V be an SVOA. For simplicity, we assume that V is of CFT-type throughout this section. The following definition is due to Zhu [Z] and Kac-Wang [KW].

Definition 3.5.1. We define the bilinear maps $* : V \otimes V \to V$ and $\circ : V \otimes V \to V$ as follows:

$$a * b \;:=\; \begin{cases} \mathrm{Res}_z Y(a,z)\dfrac{(1+z)^{\mathrm{wt}(a)}}{z} b & \text{if } a \in V^0, \\ 0 & \text{if } a \in V^1, \end{cases}$$

$$a \circ b \;:=\; \begin{cases} \mathrm{Res}_z Y(a,z)\dfrac{(1+z)^{\mathrm{wt}(a)}}{z^2} b & \text{if } a \in V^0, \\ \mathrm{Res}_z Y(a,z)\dfrac{(1+z)^{\mathrm{wt}(a)-\frac{1}{2}}}{z} b & \text{if } a \in V^1. \end{cases}$$

Extend to $V \otimes V$ linearly, denote by $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$.

Remark 3.5.2. It follows from the definition that $a \circ \mathbb{1} = a$ for $a \in V^1$. So V^1 is contained in $O(V)$. We also note that $O(V^0) \subset O(V)$, where $O(V^0)$ is the kernel of the Zhu algebra $A(V^0)$ for a VOA V^0. Therefore, $A(V)$ is a quotient algebra of $A(V^0)$.

The algebra $A(V)$ is called the *Zhu algebra* attached to V. Zhu algebras have a powerful role in the representation theory of SVOAs. In [Z] and [KW] one can find the following 1:1 correspondence theorem.

Theorem 3.5.3. *([Z] [KW])*
(1) $O(V)$ is a two-sided ideal of V under the multiplication $*$. Moreover, the quotient algebra $(A(V), *)$ is associative.

(2) $\mathbb{1} + O(V)$ is the unit element of $A(V)$ and $\omega + O(V)$ is in the center of $A(V)$.
(3) Let $M = \oplus_{n\in\frac{1}{2}\mathbb{N}} M(n)$ be an $\frac{1}{2}\mathbb{N}$-graded V-module. Then the top level $M(0)$ is an $A(V)$-module via $a + O(V) \mapsto o(a) = a_{\mathrm{wt}(a)-1}$.
(4) Given an $A(V)$-module (W, π), there exists an $\frac{1}{2}\mathbb{N}$-graded V-module $M = \oplus_{n\in\frac{1}{2}\mathbb{N}} M(n)$ such that the $A(V)$-module $M(0)$ and W are isomorphic. Moreover, this gives a bijective correspondence between the set of irreducible $A(V)$-modules and the set of irreducible $\frac{1}{2}\mathbb{N}$-graded V-modules.

Example. Let $\bar{M}_{\mathrm{Vir}}(c, 0) := M_{\mathrm{Vir}}(c, 0)/M_{\mathrm{Vir}}(c, 1)$ be the Virasoro VOA with central charge c. In [Wan], it was proved that $A(\bar{M}_{\mathrm{Vir}}(c, 0)) \simeq \mathbb{C}[x]$. For BPZ-series $c_{p,q} = 1 - 6(p-q)^2/pq$ with coprime integers $p, q \geq 2$, it is also proved in [Wan] that $A(L_{\mathrm{Vir}}(c_{p,q}, 0)) \simeq \mathbb{C}[x]/\langle \prod_{r=1}^{p-1} \prod_{s=1}^{r} (x - h_{r,s}^{p,q})\rangle$ where $h_{r,s}^{p,q}$ are defined as in (2.6.1).

For a V-module U, we can construct an $A(V)$-bimodule $A(U)$ by a similar way (cf. [FZ] [Li6] [KW]).

Definition 3.5.4. For an $\frac{1}{2}\mathbb{N}$-graded V-module U, we define bilinear operations $a \circ u$, $a * u$ and $u * a$ for homogeneous $a \in V$ and $u \in U$ as follows:

$$\begin{aligned}
&a \circ u := \mathrm{Res}_z Y(a, z)\frac{(1+z)^{\mathrm{wt}(a)-\frac{r}{2}}}{z^{2-r}}u, \quad \text{for } a \in V^r,\ r = 0, 1,\\
&a * u := \mathrm{Res}_z Y(a, z)\frac{(1+z)^{\mathrm{wt}(a)}}{z}u, \quad \text{for } a \in V^0,\\
&u * a := \mathrm{Res}_z Y(a, z)\frac{(1+z)^{\mathrm{wt}(a)-1}}{z}u, \quad \text{for } a \in V^0,\\
&a * u = u * a = 0, \quad \text{for } a \in V^1,
\end{aligned}$$

and extend linearly. We also define $O(U) \subset U$ to be the linear span of elements of the form $a \circ u$ and $A(U)$ to be the quotient space $U^0/\left(O(U) \cap U^0\right)$.

Remark 3.5.5. If V is an SVOA, then our definition of $A(U)$ differs from the Kac-Wang's original one. Namely, we define $A(U)$ to be a quotient space of U^0. See [Y1] for the validity of this change.

Let $I(\cdot, z)$ be a V-intertwining operator of type $M^1 \times M^2 \to M^3$. By definition, we know that $z^{h_1+h_2-h_3}I(\cdot, z) \in \mathrm{Hom}_{\mathbb{C}}(M^2, M^3)[[z, z^{-1}]]$. It is convenient to set $I(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1-h_1-h_2+h_3}$ and $\deg(u) := \mathrm{wt}(u) - h_1$ for $u \in M^1$.

Theorem 3.5.6. *([FZ] [KW]) $A(U)$ is an $A(V)$-bimodule under the action $*$.*

Theorem 3.5.7. *(1) ([FZ] [KW]) For a V-intertwining operator $I(\cdot, z)$ of type $U \times M^1 \to M^2$, define the zero-mode operator by $o^I(u) := u_{\deg(u)-1}$. Then we have a linear injection from $\binom{M^2}{U\ M^1}_V$ to $\mathrm{Hom}_{A(V)}\big(A(U) \otimes_{A(V)} M^1(0), M^2(0)\big)$ by a mapping: $I(\cdot, z) \mapsto o^I$.*

(2) (Theorem 2.11 of [Li6]) Suppose that every $\frac{1}{2}\mathbb{N}$-graded V-module is completely reducible. Then the linear map $I(\cdot, z) \mapsto o^I$ given in (1) defines a linear isomorphism between $\binom{M^2}{U\ M^1}_V$ and $\mathrm{Hom}_{A(V)}\big(A(U) \otimes_{A(V)} M^1(0), M^2(0)\big)$.

Remark 3.5.8. As pointed out in [Li6], the assumption on completely reducibility in (2) of the theorem above is necessary. This property is known as the *rationality*, and will be discussed in the next section.

We give an application of Zhu algebras. Let $(V^i, Y^i(\cdot, z), \mathbb{1}^i, \omega^i)$, $i = 1, 2$, be VOAs. Then the tensor product $V^1 \otimes_{\mathbb{C}} V^2$ naturally carries a structure of a VOA with vertex operator map

$$(Y^1 \otimes Y^2)(a \otimes b, z) := Y^1(a, z) \otimes Y^2(b, z)$$

and the vacuum vector is $\mathbb{1}^1 \otimes \mathbb{1}^2$ and the Virasoro vector is $\omega^1 \otimes \mathbb{1}^2 + \mathbb{1}^1 \otimes \omega^2$. Similarly, let $(M^i, Y_{M^i}(\cdot, z))$ be a V^i-module for $i = 1, 2$. Then $M^1 \otimes M^2$ is a $V^1 \otimes V^2$-module with vertex operator map

$$Y_{M^1 \otimes M^2}(a \otimes b, z) := Y_{M^1}(a, z) \otimes Y_{M^2}(b, z).$$

Since both $V^1 \otimes \mathbb{1}^2$ and $\mathbb{1}^1 \otimes V^2$ are mutually commutative subalgebras of $V^1 \otimes V^2$, one can easily verify the following.

Lemma 3.5.9. *Let V^i be VOA and M^i a V^i-module for $i = 1, 2$.*
(1) $M^1 \otimes_{\mathbb{C}} M^2$ is an irreducible $V^1 \otimes V^2$-module if and only if each M^i is an irreducible V^i-module for $i = 1, 2$.
(2) $A(V^1 \otimes_{\mathbb{C}} V^2) \simeq A(V^1) \otimes_{\mathbb{C}} A(V^2)$ as an associative algebra.
(3) $A(M^1 \otimes_{\mathbb{C}} M^2) \simeq A(M^1) \otimes_{\mathbb{C}} A(M^2)$ as an $A(V^1 \otimes_{\mathbb{C}} V^2)$-module.

Corollary 3.5.10. *Let V^i, $i = 1, 2$, be VOAs and let $M^{i,j}$, $j = 1, 2, 3$, be strong V^i-modules. If all $\mathbb{N}$-graded V^i-modules are completely reducible, then we have the following isomorphism:*

$$\binom{M^{1,3} \otimes_{\mathbb{C}} M^{2,3}}{M^{1,1} \otimes_{\mathbb{C}} M^{1,2} \quad M^{2,1} \otimes_{\mathbb{C}} M^{2,2}}_{V^1 \otimes_{\mathbb{C}} V^2} \simeq \binom{M^{1,3}}{M^{1,1}\ M^{1,2}}_{V^1} \bigotimes_{\mathbb{C}} \binom{M^{2,3}}{M^{2,1}\ M^{2,2}}_{V^2}.$$

The following assertion was first established in [FHL].

Proposition 3.5.11. *([FHL]) Let V^i, $i = 1, 2$, be VOAs. If all $\mathbb{N}$-graded V^1-modules are completely reducible, then any irreducible $V^1 \otimes_{\mathbb{C}} V^2$-module is isomorphic to a tensor product of modules, necessarily irreducible, for the V^i.*

Proof: Let M be an irreducible $\mathbb{N}$-graded $V^1 \otimes V^2$-module. Since M as a V^1-module is completely reducible, M is a direct sum of copies of an irreducible V^1-module, say W^1, since M is an irreducible $V^1 \otimes V^2$-module. So as a linear space, we have a decomposition $M = W^1 \otimes \mathrm{Hom}_{V^1}(W^1, M)$. Since actions of V^1 and those of V^2 on M is mutually commutative, V^2 naturally acts on the space of multiplicity $\mathrm{Hom}_{V^1}(W^1, M)$, which must be irreducible. Thus M is a tensor product of irreducible modules. ∎

3.6 Rationality and C_2-cofiniteness

A Zhu algebra works so powerful when the associated VOA is rational.

Definition 3.6.1. A VOA V is called *rational* if every $\mathbb{N}$-graded V-module is completely reducible. V is called *regular* if every V-module is completely reducible and every irreducible V-module is a strong module.

Definition 3.6.2. A rational VOA V is said to be *holomorphic* if the adjoint module V is a unique irreducible V-module.

Clearly, a regular VOA is rational by definition. Rational VOAs enjoy many "nice" properties.

Theorem 3.6.3. *([Z] [DLM1]) Let V be a rational VOA.*
(1) Zhu Algebra $A(V)$ is a finite dimensional semisimple associative algebra.
(2) Every irreducible $\mathbb{N}$-graded V-module is a strong module.
(3) There are finitely many inequivalent irreducible strong V-modules.

Zhu algebras are introduced in [Z] to prove the modular invariance property of the space of q-characters. In [Z], another condition was also introduced to prove it, which is now called the C_2-cofiniteness condition.

Definition 3.6.4. Let V be a VOA. A V-module M is said to be C_2-*cofinite* if the space $C_2(M) = \mathrm{Span}_{\mathbb{C}}\{a_{(-2)}v \mid a \in V,\ v \in M\}$ has a finite co-dimension in M. If V as a V-module is C_2-cofinite, then we say V satisfies C_2-cofinite condition.

Remark 3.6.5. Almost all examples of rational VOAs satisfy the C_2-cofinite condition. A good reference is [DLM2].

A C_2-cofinite VOA also enjoys many "nice" properties.

Theorem 3.6.6. *([AM] [DLM2] [M9]) If V is C_2-cofinite, then the top weight of any irreducible V-module is a rational number.*

Theorem 3.6.7. *([ABD]) Let V be a VOA of CFT-type. Then V is regular if and only if V is rational and satisfies C_2-cofinite condition.*

In the proof of the above statement, we use certain spanning set for a module.

Theorem 3.6.8. *Let V be a C_2-cofinite VOA of CFT-type. Let A be a finite dimensional subspace of V such that $L(0)$ acts on A and $V = A + C_2(V)$.*
(i) ([GN]) $V = \mathrm{Span}_{\mathbb{C}}\{a^1_{(-n_1)}a^2_{(-n_2)}\cdots a^r_{(-n_r)}\mathbb{1} \mid a^i \in A,\ n_1 > n_2 > \cdots > n_r > 0,\ r \geq 0\}$.
(ii) ([Bu] [M9]) Let W be a V-module and $w \in W$. Then there exists an integer T such that $V \cdot w = \mathrm{Span}_{\mathbb{C}}\{a^1_{(-n_1)}\cdots a^r_{(-n_r)}w \mid a^i \in A,\ n_1 > \cdots > n_r > T,\ r \geq 0\}$.

As a corollary of the theorem above, we have:

Corollary 3.6.9. *Assume that V is C_2-cofinite and of CFT-type. Then every strong V-module is C_n-cofinite for all $n \geq 2$.*

There is a representation theoretic characterization of the C_2-cofinite condition.

Theorem 3.6.10. *([M9]) Let V be a VOA of CFT-type. Then the following are equivalent:*
(1) V is C_2-cofinite.
(2) Every V-module is a direct sum of generalized eigenspaces of $L(0)$.
(3) Every V-module is an $\mathbb{N}$-graded module such that each homogeneous component is a direct sum of generalized eigenspaces of $L(0)$.
(4) V is finitely generated and every V-module is an $\mathbb{N}$-graded module.

Let $\sigma \in \mathrm{Aut}(V)$ be of finite order. We have similar results in σ-twisted theory.

Definition 3.6.11. A VOA V is called *σ-rational* if every $\frac{1}{|\sigma|}\mathbb{N}$-graded V-module is completely reducible. V is called *σ-regular* if every V-module is a direct sum of irreducible strong V-modules.

As in the untwisted case, it is shown in [DLM1] that every irreducible $\frac{1}{|\sigma|}\mathbb{N}$-graded V-module is a strong module. The following slight generalizations have been established in [Y2].

Theorem 3.6.12. *([Y2]) Let V be a C_2-cofinite VOA of CFT-type.*
(1) Let A be a finite dimensional subspace of V such that A is closed under the actions of σ and $L(0)$, and $V = A + C_2(V)$. Let W be a σ-twisted V-module. Then for each $w \in W$ there exists an $T \in \frac{1}{|\sigma|}\mathbb{Z}$ such that

$$V \cdot w = \mathrm{Span}_{\mathbb{C}}\{a^1_{(-n_1)}\cdots a^r_{(-n_r)}w \mid a^i \in A,\ n_1 > \cdots > n_r > T,\ n_i \in \tfrac{1}{|\sigma|}\mathbb{Z}\}.$$

(2) V is σ-regular if and only if V is σ-rational.

Here we give a proof of (1), which is a good exercise of the σ-twisted Jacobi identity. Let $V = V^0 \oplus \cdots \oplus V^{|\sigma|-1}$ be eigenspace decomposition of σ with $V^r = \{a \in V \mid \sigma a = e^{2\pi\sqrt{-1}r/|\sigma|}a\}$. It is know that The σ-twisted Jacobi identity is equivalent to the following two identities:
(1) The iterate formula:

$$\begin{aligned}(a_{(p)}b)_{(q+\frac{r}{|\sigma|}+\frac{s}{|\sigma|})} &= \sum_{i=0}^{N}\sum_{j=0}^{\infty}(-1)^j\binom{-r/|\sigma|}{i}\binom{p+i}{j} \\ &\times\left\{a_{(p+i-j+\frac{r}{|\sigma|})}b_{(q-i+j+\frac{s}{|\sigma|})}-(-1)^{p+i}b_{(p+q-j+\frac{s}{|\sigma|})}a_{(j+\frac{r}{|\sigma|})}\right\},\end{aligned} \tag{3.6.1}$$

where $a \in V^r$ and $b \in V^s$, and N is a positive integer such that $a_{(n)}b = 0$ for all $n \geq N$.
(2) The commutator formula:

$$[a(m), b(n)] = \sum_{i=0}^{\infty}\binom{m}{i}(a_{(i)}b)_{(m+n-i)}. \tag{3.6.2}$$

We recall the σ-twisted universal enveloping algebra $\mathfrak{U}^\sigma(V)$ of V in [DLM1]. As a tensor product of two vertex algebras $\mathbb{C}[t^{\pm\frac{1}{|\sigma|}}]$ and V, $\hat{V} := \mathbb{C}[t^{\pm\frac{1}{|\sigma|}}]\otimes_{\mathbb{C}} V$ carries a structure of a vertex algebra and $\mathfrak{g}_V := \hat{V}/(\frac{d}{dt}\otimes 1 + 1\otimes L(-1))\hat{V}$ forms a Lie algebra under the 0-th product induced from $\hat{V}$. Define a linear isomorphism $\hat{\sigma}$ on $\hat{V}$ by $\hat{\sigma}(t^n\otimes a) := e^{-2\pi\sqrt{-1}n}t^n\otimes\sigma a$. Then $\hat{\sigma}$ defines an automorphism of a vertex algebra $\hat{V}$ and hence it gives rise to an automorphism of a Lie algebra $\mathfrak{g}_V$. Denote by $\mathfrak{g}_V^\sigma$ the $\hat{\sigma}$-invariants of $\mathfrak{g}_V$, which is a Lie subalgebra of $\mathfrak{g}_V$. Then the σ-twisted universal enveloping algebra $\mathfrak{U}^\sigma(V)$ is defined to be the universal enveloping algebra for $\mathfrak{g}_V^\sigma$. The algebra $\mathfrak{U}^\sigma(V)$ has a universal property such that for any σ-twisted V-module M, the mapping $a(n)\in\mathfrak{U}^\sigma(V)\mapsto a_{(n)} = \mathrm{Res}_z Y_M(a,z)z^n \in \mathrm{End}(M)$ gives a representation of $\mathfrak{U}^\sigma(V)$ on M. It is clear that $\mathfrak{g}_V^g$ is spanned by images of elements $t^{n+\frac{r}{|\sigma|}}\otimes a$ with $a\in V^r$, $0\leq r\leq|\sigma|-1$. We denote the image of $t^{n+\frac{r}{|\sigma|}}\otimes a$ in $\mathfrak{g}_V^\sigma$ by $a(n+\frac{r}{|\sigma|})$. By definition, we have the following commutator relation:

$$[a(m), b(n)] = \sum_{i=0}^{\infty}\binom{m}{i}(a_{(i)}b)(m+n-i). \tag{3.6.3}$$

Definition 3.6.13. For a monomial $x^1(n_1)\cdots x^k(n_k)$ in $\mathfrak{U}^\sigma(V)$, we define its *length* by k, *degree* by $\mathrm{wt}(x^1)+\cdots+\mathrm{wt}(x^k)$ and *weight* by $(\mathrm{wt}(x^1)-n_1-1)+\cdots+(\mathrm{wt}(x^k)-n_k-1)$.

Let W be a g-twisted V-module generated by one element $w\in W$. In this case, a linear map $\phi_w : x^1(m_1)\cdots x^k(m_k)\in\mathfrak{U}^g(V)\mapsto x^1_{(m_1)}\cdots x^k_{(m_k)}w\in W = V\cdot w$ gives a surjection.

The proof of the following assertion comes from Lemma 2.4 of [M9].

Lemma 3.6.14. *Let V be a C_2-cofinite VOA of CFT type and W a σ-twisted V-module generated by a non-zero element w, i.e., $W = V \cdot w$. Let U be a finite dimensional subspace of V such that both $L(0)$ and g act on U and $V = U + C_2(V)$. Then the image $\phi_w(X) \in W$ of any monomial $X = x^1(m_1) \cdots x^k(m_k)$ in $\mathfrak{U}^\sigma(V)$ can be expressed as a linear combination of images of monomials $\alpha^1(n_1) \cdots \alpha^s(n_s)$ in $\mathfrak{U}^\sigma(V)$ such that $\deg \alpha^1(n_1) \cdots \alpha^s(n_s)$ is less or equal to $\deg X$, $\mathrm{wt}\, \alpha^1(n_1) \cdots \alpha^s(n_s) = \mathrm{wt}\, X$ and $n_1 < \cdots < n_s < T$, where T is a fixed element in $\frac{1}{|\sigma|}\mathbb{Z}$ such that $\phi_w(\beta(m)) = 0$ for all $\beta \in U$ and $m \geq T$.*

Proof: We divide the proof into several steps.

Claim 1. *We can express the image $\phi_w(X)$ of any monomial $X = x^1(m_1) \cdots x^k(m_k) \in \mathfrak{U}^g(V)$ in the following form:*

$$\phi_w(X) = \phi_w(A) + \phi_w(B),$$

where A is a linear combination of monomials $\alpha^1(n_1) \cdots \alpha^k(n_k) \in \mathfrak{U}^g(V)$ with $\alpha^i \in U$ such that $\deg \alpha^1(n_1) \cdots \alpha^k(n_k) = \deg X$ and $\mathrm{wt}\alpha^1(n_1) \cdots \alpha^k(n_k) = \mathrm{wt}X$, and B is a sum of monomials whose degrees are less than $\deg X$ and weights are equal to $\mathrm{wt}X$.

We prove the claim above by induction on $r = \deg X$. The case $r = 0$ is clear. Assume that the claim is true for $r-1$. Without loss, we may assume that both $L(0)$ and g act on x^i, $1 \leq i \leq k$ semisimply and none of them is the vacuum. Then, by inductive assumption, $\phi_w(x^2(m_2) \cdots x^k(m_k))$ can be expressed a linear combination of images of monomials as stated. Therefore, we may assume that $x^2, \cdots, x^k$ are contained in U. Since $V = U + C_2(V)$, we can write $x^1 = \alpha^1 + \sum_i a^i_{(-2)} b^i$ with $L(0)$-homogeneous $\alpha^1 \in U$ and $a^i, b^i \in V$ such that $\mathrm{wt}(\alpha^1) = \mathrm{wt}(a^i_{(-2)} b^i) = \mathrm{wt}(x^1)$. Then $X = \alpha^1(m_1) x^2(m_2) \cdots x^k(m_k) + \sum_i (a^i_{(-2)} b^i)(m_1) x^2(m_2) \cdots x^k(m_k)$. Then using (3.6.1) we can rewrite the image of second term in the desired form because $\mathrm{wt}(a^i) + \mathrm{wt}(b^i) < \mathrm{wt}(a^i_{(-2)} b^i)$. This completes the proof of Claim 1.

Claim 2. *Let $A = \alpha^1(m_1) \cdots \alpha^k(m_k) \in \mathfrak{U}^\sigma(V)$ be a monomial with $\alpha^i \in U$ and σ a permutation on the set $\{1, 2, \ldots, k\}$. Then we have the following equality in W:*

$$\phi_w \left(\alpha^{\sigma(1)}(n_{\sigma(1)}) \cdots \alpha^{\sigma(k)}(n_{\sigma(k)}) \right) = \phi_w(A) + \phi_w(B),$$

where B is a sum of monomials whose degrees are less than $\deg A$ and weights are equal to $\mathrm{wt}A$.

Again we proceed by induction on $r = \deg A$. The case $r = 0$ is obvious. Assume that the assertion is correct for $\deg A = r - 1$. If $m_i > m_j$ for some $i < j$, then using the commutator formula (3.6.3) we can rearrange A to be as asserted since $\mathrm{wt}(\alpha^i_{(p)} \alpha^j) < \mathrm{wt}(\alpha^i) + \mathrm{wt}(\alpha^j)$ for $p \geq 0$. Thus, Claim 2 holds.

Claim 3. *Let $A = \alpha^1(m_1)\cdots\alpha^k(m_k) \in \mathfrak{U}^g(V)$ be a monomial with $\alpha^i \in U$ and $m_1 \le \cdots \le m_k < T$. Then the image $\phi_w(A)$ of A can be expressed in the following form:*

$$\phi_w(A) = \phi_w(B) + \phi_w(C),$$

where B is a sum of monomials $\beta^1(n_1)\cdots\beta^s(n_s)$ with $\beta^j \in U$ such that $n_1 < \cdots < n_s$, $s \le k$, $\deg\beta^1(n_1)\cdots\beta^s(n_s) = \deg A$ and $\mathrm{wt}\beta^1(n_1)\cdots\beta^s(n_s) = \mathrm{wt}A$, and C is a sum of monomials whose degrees are less than $\deg A$ and weights are equal to $\mathrm{wt}A$.

We show that if the assertion is not correct then keeping both degree and weight of A we can make m_1 in a monomial A infinitely larger. We define an ordering on $\mathbb{N}\times\mathbb{N}$. For $(r_1, s_1), (r_2, s_2) \in \mathbb{N}\times\mathbb{N}$, we define $(r_1, s_1) > (r_2, s_2)$ if $r_1 > r_2$, or $r_1 = r_2$ and $s_1 > s_2$. By this ordering, $\mathbb{N}\times\mathbb{N}$ becomes a well-ordered set and hence we can perform an induction on $(\deg A, \mathrm{length}A) \in \mathbb{N}\times\mathbb{N}$, Clearly, the assertion is clear for $(\mathbb{N}, 0)$, $(\mathbb{N}, 1)$ and $(0, \mathbb{N})$. So we assume that the assertion is true for all elements in $\mathbb{N}\times\mathbb{N}$ smaller than (r, s) with $r > 0$, $s > 0$. Then, by inductive assumption, we may assume that $m_2 < \cdots < m_k < T$. If $m_1 < m_2$, then we are done. So we have to consider the case $m_1 = m_2$ and the case $m_1 > m_2$. But, the following argument shows that the latter case can be reduced to the former case. Assume that $m_1 > m_2$. Then A can be replaced by a linear combination of $A' = \alpha^2(m_2)\alpha^1(m_1)\alpha^3(m_3)\cdots\alpha^k(m_k)$ and monomials whose degrees are smaller than $\deg A$ and weights are the same as $\mathrm{wt}A$. Then applying Claim 1 and Claim 2 together with inductive assumption to A', we can replace A by a monomial $A'' = (\alpha^1)'(m_1')\cdots(\alpha^k)'(m_k')$ such that $(\alpha^i)' \in U$, $m_1' > m_1$, $m_1' \le \cdots \le m_k' < T$, $\deg A'' = \deg A$ and $\mathrm{wt}A'' = \mathrm{wt}A$. Then, repeating this procedure, we will reach the case $m_1 = m_2 < m_3 < \cdots < m_k$.

Now let us consider the case $m_1 = m_2 < m_3 < \cdots < m_k$. In this case, both α^1 and α^2 are contained in the same eigenspace, say V^r. Write $m_1 = n + \frac{r}{|\sigma|}$. Using the iterate formula (3.6.1) on $(\alpha^1_{(-1)}\alpha^2)_{(2n+1+\frac{2r}{|\sigma|})}$, we get

$$\begin{aligned}
&\phi_w(\alpha^1(m_1)\alpha^2(m_1)\alpha^3(m_3)\cdots\alpha^k(m_k)) \\
&\quad= \lambda\phi_w((\alpha^1_{(-1)}\alpha^2)(2m_1+1)\alpha^3(m_3)\cdots\alpha^k(m_k)) \\
&\quad+ \textstyle\sum_{i>0}\mu_i\phi_w(\alpha^1(m_1+i)\alpha^2(m_1-i)\alpha^3(m_3)\cdots\alpha^k(m_k)) \\
&\quad+ \displaystyle\sum_{i>0}\mu_i'\phi_w(\alpha^2(m_1+i)\alpha^2(m_1-i)\alpha^3(m_3)\cdots\alpha^k(m_k)) + \phi_w(X),
\end{aligned} \tag{3.6.4}$$

where X is a sum of monomials whose degrees are less than $\deg A$ and weights are equal to $\mathrm{wt}A$. (Note that in the expansion of $(\alpha^1_{(-1)}\alpha^2)_{(2m_1+1)}$, we can make the coefficient of $\alpha^1(m_1)\alpha^2(m_2)$ non-zero by choosing suitable N in (3.6.1).) The first term in the right-hand side of (3.6.4) has smaller length than that of A so that by induction together with Claim 1 and Claim 2 we may omit this term. The second and third terms in the right-hand side of (3.6.4) shall be reduced to the case $m_1 > m_2$. Therefore, we obtain a procedure which makes m_1 infinitely larger with keeping $\deg A$ and $\mathrm{wt}A$, which must stop in finite steps. Thus, we get Claim 3 and hence we complete the proof of the lemma. ∎

3.7 Associativity of fusion products

In this section we discuss elementary results on intertwining operators. Let V be a VOA and M^i strong V-modules for $i = 1, 2, 3$. Let $I(\cdot, z)$ a V-intertwining operator of type $M^1 \times M^2 \to M^3$. We define a *transpose* intertwining operator ${}^tI(\cdot, z)$ of $I(\cdot, z)$ of type $M^2 \times M^1 \to M^3$ by means of

$$ {}^tI_\pm(v^2, z)v^1 := e^{zL(-1)} I(v^1, \zeta_\pm z)v^2, \quad \text{where } \zeta_\pm = e^{\pm\pi\sqrt{-1}}, \tag{3.7.1} $$

for $v^1 \in M^1$ and $v^2 \in M^2$.

Proposition 3.7.1. *([FHL]) For $I(\cdot, z) \in \binom{M^3}{M^1\ M^2}_V$, its transpose ${}^tI_\pm(\cdot, z) \in \binom{M^3}{M^2\ M^1}_V$. Moreover, ${}^t({}^tI_\pm)_\mp(\cdot, z) = I(\cdot, z)$. Thus, as a linear space, $\binom{M^3}{M^2\ M^1}_V$ is naturally isomorphic to $\binom{M^3}{M^1\ M^2}_V$.*

On the other hand, define a *contragredient* intertwining operator $I^*_\pm(\cdot, z)$ of $I(\cdot, z)$ of type $M^1 \times (M^3)^* \to (M^2)^*$ by means of

$$ \langle I^*_\pm(v^1, z)\nu^3, v^2\rangle = \langle \nu^3, I(e^{zL(1)}(\zeta_\pm z^{-2})^{L(0)}v^1, z^{-1})v^2\rangle, \quad \text{where } \zeta_\pm = e^{\pi\sqrt{-1}}, \tag{3.7.2} $$

for $v^1 \in M^1$, $v^2 \in M^2$ and $\nu \in M^3$.

Proposition 3.7.2. *([FHL]) For $I(\cdot, z) \in \binom{M^3}{M^1\ M^2}_V$, its contragredient operator $I^*_\pm(\cdot, z) \in \binom{(M^2)^*}{M^1\ (M^3)^*}_V$. Moreover, $(I^*_\pm)^*_\mp(\cdot, z) = I(\cdot, z)$. Thus, as a linear space, $\binom{(M^2)^*}{M^1\ (M^3)^*}_V$ is naturally isomorphic to $\binom{M^3}{M^1\ M^2}_V$.*

Now we introduce a concept of the fusion product. Here we adopt a definition due to Li [Li7].

Definition 3.7.3. Let M^1, M^2 be V-modules. A *tensor product* or a *fusion product* for the ordered pair (M^1, M^2) is a pair $(M^1 \boxtimes_V M^2, F(\cdot, z))$ consisting of a V-module $M^1 \boxtimes_V M^2$ and a V-intertwining operator $F(\cdot, z)$ of type $M^1 \times M^2 \to M^1 \boxtimes_V M^2$ satisfying the following universal property: For any V-module U and any intertwining operator $I(\cdot, z)$ of type $M^1 \times M^2 \to U$, there exists a unique V-homomorphism ψ from $M^1 \boxtimes_V M^2$ to U such that $I(\cdot, z) = \psi F(\cdot, z)$.

Remark 3.7.4. It follows from the universal property that if a tensor product exists, then it is unique up to isomorphism.

It is shown in [HL1]–[HL4] and [Li7] that if a VOA V is rational, then a tensor product for any two V-modules always exists. So in the rest of this section we assume that V is rational. Then there are finitely many irreducible V-modules by the previous section. Let $C = \{M^i \mid i = 1, 2, \cdots n\}$ be the set of all inequivalent V-modules. Note that all M^i are strong V-modules. Consider an $\mathbb{N}$-algebra $\mathcal{A} = \oplus_{i=1}^n \mathbb{N}\, M^i$ formally spanned by C equipped with a product

$$M^i \times M^j = \sum_{k=1}^n N_{ij}^k \cdot M^k, \quad \text{where } N_{ij}^k := \dim \binom{M^k}{M^i\ M^j}_V. \tag{3.7.3}$$

Then $\mathcal{A}$ is called the *fusion algebra* or *Verlinde algebra* of V. The coefficient N_{ij}^k is often referred to as the *fusion rule* (of type $M^i \times M^j \to M^k$). It follows from definition that the fusion rule (3.7.3) coincides with the irreducible decomposition of a fusion product:

$$M^i \boxtimes_V M^j \simeq \bigoplus_{k=1}^n (M^k)^{\oplus N_{ij}^k} \simeq \bigoplus_{k=1}^n M^k \underset{\mathbb{C}}{\otimes} \binom{M^k}{M^i\ M^j}_V \text{ (linearly).}$$

By Proposition 3.7.1, the fusion algebra is always commutative. By Lemma 2.4.3, there is a canonical isomorphism $\binom{M^j}{V\ M^j}_V \simeq \mathrm{Hom}_V(M^i, M^j)$ so that $V \times M^i = M^i$. Namely, V is a unit element in the fusion algebra.

Recently, the associativity of fusion products was established in [H1]–[H4] (see also [DLM4]), if V is rational, C_2-cofinite and of CFT-type:

Theorem 3.7.5. *Assume that V is rational C_2-cofinite VOA of CFT-type.*
(1) (Associativity) Let $I^1(\cdot, z)$ and $I^2(\cdot, z)$ be V-intertwining operators of types $M^1 \times M^4 \to M^5$ and $M^2 \times M^3 \to M^4$, respectively, where M^i, $i = 1, \dots, 5$, are arbitrary strong V-modules. Then there exists a strong V-module M^6 and V-intertwining operators $J^1(\cdot, z)$ and $J^2(\cdot, z)$ of types $M^6 \times M^3 \to M^5$ and $M^1 \times M^2 \to M^6$, respectively, such that the following equality holds for any $v^i \in M^i$ for $i = 1, 2, 3$ and $\nu \in (M^5)^*$:

$$\langle \nu, I^1(v^1, z_1) I^2(v^2, z_2) v^3 \rangle = \langle \nu, J^1(J^2(v^1, z_0) v^2, z_2) v^3 \rangle|_{z_0 = z_1 - z_2},$$

where the left hand side and the right hand side of the above equality converge in the domain $|z_1| > |z_2| > 0$ and $|z_2| > |z_0| > 0$, respectively, for any choices of $\log z_1$, $\log z_2$ and $\log(z_1 - z_2)$ in the definition of $z^r = e^{\log z}$ for $r \in \mathbb{R}$, and the equality above means that the left hand side and the right hand side are analytic extensions of each other.
(2) The assertion (1) still holds if we exchange the role of $I^1(\cdot, z), I^2(\cdot, z)$ and that of $J^1(\cdot, z), J^2(\cdot, z)$.
(3) (Commutativity) Let $I^1(\cdot, z)$ and $I^2(\cdot, z)$ be V-intertwining operators of type $W^1 \times W^4 \to W^5$ and $W^2 \times W^3 \to W^4$, respectively, where W^i, $i = 1, \dots, 5$ are arbitrary strong

V-modules. Then there exist a strong V-module W^6 and V-intertwining operators $J^1(\cdot, z)$ and $J^2(\cdot, z)$ of types $W^2 \times W^6 \to W^5$ and $W^1 \times W^3 \to W^6$, respectively, such that the multivalued analytic function

$$\langle \nu, I^1(v^1, z_1) I^2(v^2, z_2) v^3 \rangle$$

of z_1 and z_2 in the region $|z_1| > |z_2| > 0$ and the multivalued analytic function

$$\langle \nu, J^1(v^2, z_2) J^2(v^1, z_1) v^3 \rangle$$

of z_1 and z_2 in the region $|z_2| > |z_1| > 0$ are analytic extensions of each other.
(4) For any strong V-modules N^i, $i = 1, 2, 3$, the space of V-intertwining operators of type $N^1 \times N^2 \to N^3$ is finite dimensional.

Corollary 3.7.6. *If V is rational C_2-cofinite VOA of CFT-type, then the associated fusion algebra is a finite dimensional commutative associative $\mathbb{N}$-algebra with a unit V.*

3.8 Coset construction

In this section we consider the coset construction of vertex operator algebras.

Let $(V, Y_V(\cdot, z), \mathbb{1}_V, \omega_V)$ be a VOA with central charge c_V. A vector $e \in V$ is called a *conformal vector* with central charge c_e if its component operators $L^e(n)$, $n \in \mathbb{Z}$, of $Y_V(e, z) = \sum_{n\in\mathbb{Z}} L^e(n) z^{-n-2}$ defines a representation of the Virasoro algebra on V with central charge c_e:

$$[L^e(m), L^e(n)] = (m-n) L^e(m+n) + \delta_{m+n,0} \frac{m^3 - m}{12} c_e.$$

A graded subspace $U = \oplus_{n\geq 0} U_n$ of V, where $U_n = U \cap V_n$, is said to be a sub VOA of V if $\mathbb{1}_V \in U$ and there is a conformal vector $\omega_U \in U_2$ such that $(U, Y_V(\cdot, z)|_U, \mathbb{1}_V, \omega_U)$ satisfies all the axioms of a VOA. We simply write (U, e) for $(U, Y_V(\cdot, z)|_U, \mathbb{1}_V, e)$. For a sub VOA (U, e), we want to its commutant subalgebra. Before we give the definition of the commutant subalgebra, we prove the following lemmas.

Lemma 3.8.1. *For $a, b \in V$, $[Y_V(a, z), Y_V(b, z)] = 0$ if and only if $a_{(i)} b = 0$ for all $i \geq 0$.*

Proof: The equation $[Y_V(a, z), Y_V(b, z)] = 0$ asserts that the order of locality of a and b is less than or equal to 0. So the assertion is obvious from Section 3.3. ∎

Lemma 3.8.2. *For a subset S of V, the subspace $S^c = \{a \in V \mid a_{(i)} v = 0$ for any $v \in S,\ i \geq 0\}$ forms a subalgebra in V.*

Proof: The assertion immediately follows from the iterate formula

$$(a_{(n)}b)_{(i)} = \sum_{j=0}^{\infty}(-1)^j \binom{n}{j} \{a_{(n-j)}b_{(i+j)} - (-1)^n b_{(n+i-j)}a_{(j)}\}.$$

∎

By the lemmas above, we define the *commutant subalgebra* of a sub VOA (U, e) by

$$\mathrm{Com}_V(U) := \{a \in V \mid a_{(i)}v = 0 \text{ for any } v \in U,\ i \geq 0\}. \tag{3.8.1}$$

However, the definition of the commutant subalgebra is completely determined only by the Virasoro vector e of U as we will see below.

Proposition 3.8.3. *Let (U, e) be a sub VOA of V. Then* $\mathrm{Com}_V(U) = \mathrm{Ker}_V e_{(0)}$.

Proof: Let $a \in \mathrm{Com}_V(U)$. Then $e_{(0)}a = e_{(0)}a_{(-1)}\mathbb{1} = a_{(-1)}e_{(0)}\mathbb{1} = 0$. Thus $a \in \mathrm{Ker}_V e_{(0)}$. Conversely, let $a \in \mathrm{Ker}_V e_{(0)}$ and $v \in U$. Then we can find an integer N such that $v_{(n)}a = 0$ for all $n \geq N$ and $v_{(N-1)}a \neq 0$. If $N > 0$, then $e_{(0)}v_{(N)}a = v_{(N)}e_{(0)}a + [e_{(0)}, v_{(N)}]a = -Nv_{(N-1)}a = 0$, which is a contradiction. So $Y_V(v, z)a \in V[[z]]$. Then by the skew-symmetry we have $Y_V(b, z)v = e^{zL(-1)}Y_V(v, -z)b \in V[[z]]$. Thus $b_{(i)}v = 0$ for all $i \geq 0$ and hence $b \in \mathrm{Com}_V(U)$. ∎

Next, we consider the condition that $\omega_V - e$ defines a Virasoro vector of $\mathrm{Com}_V(U)$. A decomposition $\omega_V = \omega^1 + \omega^2$ of the Virasoro element of V is called *orthogonal* if both ω^1 and ω^2 are conformal vectors and their component operators are mutually commutative, i.e., $[Y(\omega^1, z_1), Y(\omega^2, z_2)] = 0$. Given a conformal vector e, we can verify whether $\omega_V = e + (\omega_V - e)$ is orthogonal or not by the following lemma:

Lemma 3.8.4. *([FZ, Theorem 5.1]) Assume that V is of CFT-type. Then for a conformal vector $e \in V$, $\omega_V = e + (\omega_V - e)$ is orthogonal if and only if $(\omega_V)_{(2)}e = 0$.*

Proof: By the $L(-1)$-derivation property we have $\omega_{(0)}e = e_{(-2)}\mathbb{1}$. On the other hand, $e_{(0)}e = e_{(0)}e_{(-1)}\mathbb{1} = L^e(-1)L^e(-2)\mathbb{1} = L^e(-2)L^e(-1)\mathbb{1} + L^e(-3)\mathbb{1} = e_{(-1)}e_{(0)}\mathbb{1} + e_{(-2)}\mathbb{1} = e_{(-2)}\mathbb{1}$. Note that $e_{(n)} = L^e(n-1)$ and $a_{(n)}\mathbb{1} = 0$ for any $a \in V$ and $n \geq 0$. Thus $(\omega_V - e)_{(0)}e = 0$. Since $(\omega_V)_{(1)}e = e_{(1)}e = 2e$, $(\omega_V - e)_{(1)}e = 0$. By assumption, $(\omega_V - e)_{(2)}e = 0$. As V is of CFT-type, we have $(\omega_V - e)_{(3)}e \in \mathbb{C}\mathbb{1}$ and $(\omega_V - e)_{(n)}e = 0$ for $n \geq 4$. Then by the skew-symmetry formula

$$a_{(n)}b = \sum_{i=0}^{\infty} \frac{(-1)^{n+i+1}}{i!} L(-1)^i b_{(n+i)}a,$$

we have $e_{(0)}(\omega_V - e) = \sum_{0 \leq i \leq 3}(-1)^{i+1}L(-1)^i(\omega_V - e)_{(i)}e/i! = 0$. Thus $\omega_V - e \in \mathrm{Com}_V(e)$. ∎

Proposition 3.8.5. *Let $\omega_V = e + (\omega_V - e)$ is an orthogonal decomposition. Then*
(i) $(\mathrm{Ker}_V e_{(0)}, \omega_V - e)$ *is a sub VOA of* V.
(ii) $\mathrm{Ker}_V e_{(0)}$ *is a unique maximal sub VOA of* V *whose Virasoro vector is* $\omega_V - e$.

Proof: (i): We have shown that $\mathrm{Ker}_V e_{(0)}$ is a subalgebra of V. It is clear that $\mathbb{1}_V \in \mathrm{Ker}_V e_{(0)}$ and $\omega_V - e \in \mathrm{Ker}_V e_{(0)}$. So we have to prove that $\omega_V - e$ is a Virasoro vector of $\mathrm{Ker}_V e_{(0)}$. Let $a \in \mathrm{Ker}_V e_{(0)}$. Then $e_{(0)}a = 0$ so that we have the $L(-1)$-derivation: $Y_V((\omega_V - e)_{(0)}a, z) = Y_V((\omega_V)_{(0)}a, z) = \partial_z Y_V(a, z)$. Since $e_{(1)}a = 0$, we have $(\omega_V - e)_{(1)}a = (\omega_V)_{(1)}a$ so that $(\omega_V - e)_{(1)}$ acts on $\mathrm{Ker}_V e_{(0)}$ semisimply with a graded decomposition $\mathrm{Ker}_V e_{(0)} = \oplus_{n\in\mathbb{Z}}(\mathrm{Ker}_V e_{(0)} \cap V_n)$. Thus, $\omega_V - e$ is the Virasoro vector of $\mathrm{Ker}_V e_{(0)}$.

(ii): Let $(W, \omega_V - e)$ be a sub VOA of V. For each $x \in W$, we have $Y_V(e_{(0)}x, z) = Y_V((\omega_V)_{(0)}x, z) + Y_V((\omega_V - e)_{(0)}x, z) = \partial_z Y_V(x, z) - \partial_z Y_V(x, z) = 0$. In particular, $e_{(0)}x = \mathrm{Res}_z Y_V(e_{(0)}x, z)\mathbb{1}_V = 0$. Thus $x \in \mathrm{Ker}_V e_{(0)}$ and M is a subalgebra of $\mathrm{Ker}_V e_{(0)}$. ∎

By this proposition, if we have an orthogonal decomposition $\omega_V = \omega^1 + \omega^2$, then we have a two mutually commuting sub VOAs $U^1 = (\mathrm{Ker}_V \omega^2_{(0)}, \omega^1)$ and $U^2 = (\mathrm{Ker}_V \omega^1_{(0)}, \omega^2)$. Denote by T the subalgebra generated by U^1 and U^2. If V is of CFT-type, then all of U^1, U^2, $U^1 \otimes_{\mathbb{C}} U^2$ and T are of CFT-type. It is easy to see that a linear map $U^1 \otimes_{\mathbb{C}} U^2 \ni a \otimes b \mapsto a_{(-1)}b \in T$ is a VOA-epimorphism as $a_{(-1)}b = b_{(-1)}a$. However, this epimorphism is not isomorphism in general. Here we give an example. Let V^1 and V^2 be VOAs such that both of then are not simple. Then we can take proper ideals I^i of V^i for $i = 1, 2$. Then $I^1 \otimes I^2$ is a proper ideal of $V^1 \otimes_{\mathbb{C}} V^2$ which does not contain neither V^1 nor V^2. Then by setting $V = V^1 \otimes_{\mathbb{C}} V^2 / I^1 \otimes_{\mathbb{C}} I^2$, $U^1 = V^1$ and $U^2 = V^2$, we have an exact sequence:

$$0 \to I^1 \underset{\mathbb{C}}{\otimes} I^2 \to U^1 \underset{\mathbb{C}}{\otimes} U^2 \to V = T \to 0.$$

However, if one of U^i, $i = 1, 2$, is simple, then the epimorphism is actually isomorphism:

Lemma 3.8.6. *Let V be a VOA and let V^1, V^2 be two sub VOAs of V such that V^1 and V^2 generate V and $[Y_V(a^1, z_1), Y_V(a^2, z_2)] = 0$ for any $a^i \in V^i$, $i = 1, 2$. Suppose that V^1 is simple. Then V is isomorphic to $V^1 \otimes_{\mathbb{C}} V^2$.*

Proof: As we have seen, a linear map $V^1 \otimes V^2 \ni a \otimes b \mapsto a_{(-1)}b \in V$ is an epimorphism. Assume that this is not injective. In this case, since $V^1 \otimes V^2$ as a V^1-module is isomorphic to a direct sum of copies of V^1 and V^1 is an irreducible V^1-module, the kernel of the above epimorphism contains a non-zero element of the form $\mathbb{1} \otimes a \in V^1 \otimes V^2$. However, we have $\mathbb{1} \otimes a \mapsto \mathbb{1}_{(-1)}a = a \neq 0$, a contradiction. Thus V is isomorphic to $V^1 \otimes V^2$. ∎

Remark 3.8.7. If both V and V^1 are simple, then it is very likely to happen that V^2 is also simple. In fact, this is true in many (almost all known) examples. However, there is no systematic theory on this point.

By this lemma, if one of U^i, $i = 1, 2$, is simple, then V has a sub VOA isomorphic to $U^1 \otimes U^2$ whose Virasoro vector is the same as that of V. In this case, the pair (U^1, U^2) is often called a *commutant pair*. For a sub VOA V^1 of V, the association $V^1 \rightsquigarrow \mathrm{Com}_V(V^1)$ is called the *commutant construction* or *coset construction*. Many important VOAs are constructed by the coset construction. One of the most famous examples is the GKO-construction [GKO] of the unitary Virasoro VOAs. We will consider this topic later.

Intertwining operators for sub VOAs. We consider a relation with vertex operator map and intertwining operators for sub VOAs. Let (U, e) be a sub VOA of (V, ω).

Lemma 3.8.8. *We have $(\omega - e)_{(0)}u = 0$ and $[u_{(m)}, (\omega - e)_{(1)}] = 0$ for all $u \in U$ and $m \in \mathbb{Z}$.*

Proof: Let $u \in U$. Since e is the Virasoro vector of U, we have $e_{(0)}u = u_{(-2)}\mathbb{1}$ and thus $(\omega - e)_{(0)}u = \omega_{(0)}u - e_{(0}u = u_{(-2)}\mathbb{1} - u_{(-2)}\mathbb{1} = 0$. By the definition of a sub VOA, we have $\omega_{(1)}u = e_{(1)}u$. Then by the commutator formula we have $[(\omega - e)_{(1)}, u_{(m)}] = ((\omega - e)_{(0)}u)_{(m+1)} + ((\omega - e)_{(1)}u)_{(m)} = 0$. ∎

Let $(M, Y_M(\cdot, z))$ be a strong V-module. Assume that M as a U-module is completely reducible. Then we have a decomposition

$$M = \bigoplus_{\lambda \in \Lambda} W^\lambda \otimes \mathrm{Hom}_U(W^\lambda, M)$$

where $\{W^\lambda \mid \lambda \in \Lambda\}$ is the set of inequivalent irreducible U-submodules of M. Write $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ and $Y_M(e, z) = \sum_{n\in\mathbb{Z}} L^e(n)z^{-n-2}$. Since both $L(0)$ and $L^e(0)$ acts on M diagonally and $L(0) - L^e(0)$ acts on each $\mathrm{Hom}_U(W^\lambda, M)$ by Lemma 3.8.8, the space $\mathrm{Hom}_U(W^\lambda, M)$ splits into a direct sum of eigenspaces for $L(0) - L^e(0)$. Therefore, $z^{L^e(0)}$ and $z^{L(0)-L^e(0)}$ are well-defined operators on M. Let $A_\lambda = \{v^{\lambda,\alpha} \mid \alpha \in P_\lambda\}$ be a linear basis of $\mathrm{Hom}_U(W^\lambda, M)$ consisting of eigenvectors for $L(0) - L^e(0)$ for $\lambda \in \Lambda$. Let $\pi_{\lambda,\alpha} : M \to W^\lambda \otimes v^{\lambda,\alpha}$ be a projection map. Clearly, $\pi_{\lambda,\alpha}$ is a U-homomorphism. Define a linear map $I^\gamma_{\alpha\beta}(\cdot, z)$ of type $W^{\lambda_1} \times W^{\lambda_2} \to W^{\lambda_3} \otimes v^{\lambda_3,\gamma}$ by

$$I^\gamma_{\alpha\beta}(a, z)b := z^{-L(0)+L^e(0)}\pi_{\lambda_3,\gamma}Y_M(z^{L(0)-L^e(0)}a \otimes v^{\lambda_1,\alpha}, z)z^{L(0)-L^e(0)}b \otimes v^{\lambda_2,\beta} \tag{3.8.2}$$

for $a \in W^{\lambda_1}$ and $b \in W^{\lambda_2}$, $v^{\lambda_1,\alpha} \in A_{\lambda_1}$, $v^{\lambda_2,\beta} \in A_{\lambda_2}$ and $v^{\lambda_3,\gamma} \in A_{\lambda_3}$.

Proposition 3.8.9. *The map $I^\gamma_{\alpha\beta}(\cdot, z)$ is a U-intertwining operator of type $W^{\lambda_1} \times W^{\lambda_2} \to W^{\lambda_3}$.*

Proof: Clearly, $I^{\gamma}_{\alpha\beta}(\cdot,z)$ satisfies the lower truncated condition. Since $Y_M(\cdot,z)$ satisfies the Jacobi identity on M and $L(0)-L^e(0)\in \mathrm{Hom}_U(W^\lambda,M)$ for any $\lambda\in\Lambda$, $I^{\gamma}_{\alpha\beta}(\cdot,z)$ also satisfies the Jacobi identity. By the term $z^{\pm(L(0)-L^e(0))}$ inserted as above, we can also verify that $I^{\gamma}_{\alpha\beta}(\cdot,z)$ satisfies the $L^e(-1)$-derivation property. Therefore, $I^{\gamma}_{\alpha\beta}(\cdot,z)$ satisfies all the axioms for a U-intertwining operator. ∎

Now assume that $\omega=e+(\omega-e)$ is an orthogonal decomposition. Then $U^c:=\mathrm{Ker}_V e_{(0)}$ is a sub VOA with the Virasoro vector $\omega-e_{(0)}$ by Proposition 3.8.5. Therefore, V contains a sub VOA $U\otimes U^c$. Since the action of U^c on M commutes with that of U on M, U^c naturally acts on each space $\mathrm{Hom}_U(W^\lambda,M)$. Namely, $\mathrm{Hom}_U(W^\lambda,M)$ are U^c-modules for all $\lambda\in\Lambda$. For simplicity, we set $(W^\lambda)^c:=\mathrm{Hom}_U(W^\lambda,M)$ and $M^\lambda:=W^\lambda\otimes(W^\lambda)^c$. Then M^λ is a $U\otimes U^c$-submodule of M and M as a $U\otimes U^c$-module has a decomposition $M=\oplus_{\lambda\in\Lambda}M^\lambda$. Set $\pi_\lambda=\prod_{\alpha\in A_\lambda}\pi_{\lambda,\alpha}$. Then π_λ is a projection map from M to M^λ and so is a $U^1\otimes U^2$-homomorphism. Therefore, $\pi_{\lambda_3}Y_M(\cdot,z)$ restricted on $M^{\lambda_1}\otimes M^{\lambda_2}$ defines a $U\otimes U^c$-intertwining operator of type

$$W^{\lambda_1}\otimes(W^{\lambda_1})^c\times W^{\lambda_2}\otimes(W^{\lambda_2})^c\to W^{\lambda_3}\otimes(W^{\lambda_3})^c.$$

Therefore, the fusion rules for U-modules partially determine the V-module structure on M. If both U and U^c are rational, then by Corollary 3.5.10 we have the following isomorphism:

$$\begin{pmatrix} W^{\lambda_3} \\ W^{\lambda_1}\ W^{\lambda_2}\end{pmatrix}_U \otimes \begin{pmatrix} (W^{\lambda_3})^c \\ (W^{\lambda_1})^c\ (W^{\lambda_2})^c\end{pmatrix}_{U^c} \simeq \begin{pmatrix} W^{\lambda_3}\otimes(W^{\lambda_3})^c \\ W^{\lambda_1}\otimes(W^{\lambda_1})\ W^{\lambda_2}\otimes(W^{\lambda_2})^c\end{pmatrix}_{U\otimes U^c}.$$

However, it is usually a difficult problem to show that the commutant U^c of U is rational even if both V and U are rational. But the following theorem says that if U is rational then we always have the isomorphism as follows.

Theorem 3.8.10. *([ADL, Theorem 2.10]) Let U^i, $i=1,2$, be VOAs. Let W^i, $i=1,2,3$, be U^1-modules on which $L^1(0)$ acts semisimply and let X^i, $i=1,2,3$, be U^2-modules on which $L^2(0)$ acts semisimply, where $L^1(0)$ and $L^2(0)$ are the grade-keeping operators of the Virasoro vectors of U^1 and U^2, respectively. If the fusion rules* $\dim\binom{W^3}{W^1\ W^2}_{U^1}$ *is finite, then the following linear isomorphism holds:*

$$\begin{pmatrix} W^3 \\ W^1\ W^2\end{pmatrix}_{U^1}\otimes\begin{pmatrix} X^3 \\ X^1\ X^2\end{pmatrix}_{U^2}\simeq\begin{pmatrix} W^3\otimes X^3 \\ W^1\otimes X^1\ W^2\otimes X^2\end{pmatrix}_{U^1\otimes U^2}.$$

3.9 Quantum Galois theory

In this section we review the theory of the quantum Galois theory, one of the most beautiful results on vertex operator algebras.

Let V be a simple VOA and G a finite subgroup of $\mathrm{Aut}(V)$. Then the fixed point subalgebra $V^G := \{a \in V \mid ga = a \text{ for any } g \in G\}$ is called the *G-orbifold subalgebra* or simply *orbifold* of V. The study of V^G was initiated in [DVVV] in physical point of view and was begun by [DM1] in mathematical point of view. Let $\mathrm{Irr}(G)$ be the set of all inequivalent characters of G. For each $\chi \in \mathrm{Irr}(G)$, we fix an irreducible $\mathbb{C}[G]$-module M_χ affording the character χ. Then we have a decomposition

$$V = \bigoplus_{\chi \in \mathrm{Irr}(G)} M_\chi \otimes \mathrm{Hom}_G(M_\chi, V).$$

Set $V_\chi := \mathrm{Hom}_G(M_\chi, V)$. We note $V^G = V_{1_G}$ for the principal character $1_G \in \mathrm{Irr}(G)$. Each V_χ is a V^G-module since $\mathbb{C}[G]$ commutes with V^G. Thus V is a module of $\mathbb{C}[G] \otimes_{\mathbb{C}} V^G$. The following Schur-Weyl type duality theorem is shown in [DM1] and [DLM3].

Theorem 3.9.1. *With reference to the above setting, we have:*
(1) For each $\chi \in \mathrm{Irr}(G)$, $V_\chi \neq 0$.
(2) All V_χ, $\chi \in \mathrm{Irr}(G)$, are irreducible V^G-modules. In particular, V^G is a simple VOA.
(3) $V_\lambda \simeq V_\mu$ as V^G-modules if and only if $M_\lambda \simeq M_\mu$ as $\mathbb{C}[G]$-modules.

By this theorem, we see that $\mathbb{C}[G]$ and V^G forms a dual pair on V. Moreover, by using the theorem above, we can derive the following Galois correspondence established in [HMT]:

Theorem 3.9.2. *(Quantum Galois theorem) Assume that V is a simple VOA. Let $\Phi : H \mapsto V^H$ be the map which associates to a subgroup H of G the sub VOA V^H of V. Then Φ induces a bijection between the subgroups of G and the sub VOAs of V which contains V^G.*

By theorem 3.9.1 and 3.9.2, the representation of the group G plays an important role to study V as a V^G-module.

There is a generalization of Theorem 3.9.1 for modules. Let $g, h \in G$. For a g-twisted V-module $(M, Y_M(\cdot, z))$, we can define another module structure. Define the *h-conjugate vertex operator* $Y_M^h(\cdot, z)$ by

$$Y_M^h(a, z) := Y_M(ha, z).$$

Then one can easily check that $(M, Y_M^h(\cdot, z))$ is an $h^{-1}gh$-twisted V-module. We usually denote $(M, Y_M^h(\cdot, z))$ by $M \circ h$ for short. It is obvious that if M is irreducible then so is $M \circ h$, and if g and h commutes, then $M \circ h$ is again a g-twisted V-module. If $M \circ h \simeq M$, then M is refereed to as *h-stable*. This is equivalent to the condition that there is a V-isomorphism $\phi(h) : M \to M$ such that $\phi(h) Y_M(a, z) v = Y_M(ha, z) \phi(h) v$ for all $a \in V$ and $v \in M$. The linear isomorphism $\phi(h)$ is called *h-stabilizing automorphism.* If M is irreducible, then the stabilizing automorphism is unique up to linearity.

Now consider the case $g = h = 1$. As we explained, there is an action of G on the set of equivalent classes of irreducible V-modules. Take an orbit $\mathcal{M}$. Then for $M^1, M^2 \in \mathcal{M}$, there is $g \in G$ such that $M^1 \circ g \simeq M^2$. Consider a direct sum $\oplus_{M\in\mathcal{M}} M$. Then as a generalization of Theorem 3.9.1, the following theorem has been established in [DY]:

Theorem 3.9.3. *For each G-orbit $\mathcal{M}$, there is a central extension $1 \to \mathbb{C} \to \tilde{G} \to G \to 1$ and an action of $\tilde{G}$ on $\oplus_{M\in\mathcal{M}} M$ which extends certain linear representation of the central component $\mathbb{C}$ of $\tilde{G}$. Moreover, under the action above, $V^G \otimes \mathbb{C}[\tilde{G}]$ forms a dual pair on $\oplus_{M\in\mathcal{M}} M$.*

To describe the central extension $\tilde{G}$ and the action of $\mathbb{C}[\tilde{G}]$ on $\oplus_{M\in\mathcal{M}} M$ precisely, we have to introduce the twisted associative algebra $\mathcal{A}_\lambda(G, \mathcal{M})$ as in [DY], where λ is a suitable 2-cocycle on G. We leave the accurate definition of them to [DY]. The twisted algebra $\mathcal{A}_\lambda(G, \mathcal{M})$ is refereed to as *generalized twisted double* in [DY], and we will find some representations of the twisted algebras in the theory of simple current extensions of vertex operator algebras.

Remark 3.9.4. Theorem 3.9.3 has been generalized to include twisted modules in [MT].

Chapter 4

Simple Current Extensions of Vertex Operator Algebras

In this chapter we study a theory of simple current extensions of vertex operator algebras. First, we briefly consider how a vertex operator algebra is extended to a larger algebra with suitable grading. Let $(V, Y_V(\cdot,z), \mathbb{1}_V, \omega_V)$ be a vertex operator algebra. An *extension* of V is a vertex operator (super)algebra $(W, Y_W(\cdot,z), \mathbb{1}_W, \omega_W)$ such that (V, ω_V) is a sub VOA of W with $\mathbb{1}_W = \mathbb{1}_V$ and $\omega_W = \omega_V$. Assume that there is an extension $W = \oplus_{\alpha\in A} W^\alpha$ of V graded by a monoid A such that $W^{1_A} \simeq V$, where 1_A is the unit element of A, and W has an A-graded structure $W^\alpha \cdot W^\beta \subset W^{\alpha\beta}$ for $\alpha, \beta \in A$. Then by the skew-symmetry we have $\alpha\beta = \beta\alpha$ for any $\alpha, \beta \in A$ and so A is a commutative monoid. Moreover, if W is simple, then for each $\alpha \in A$ there is a $\beta \in A$ such that $W^\alpha \cdot W^\beta = W^{1_A}$. Namely, A is an abelian group. Thus, in the study of extensions of vertex operator algebras, extensions graded by abelian groups come to play a central role. In the case where A is abelian and W is a simple VOA, the duality in Theorem 3.9.1 implies all W^α, $\alpha \in A$, are inequivalent irreducible $W^{1_A} = V$-modules as the fixed point subalgebra W^A is exactly equal to W^{1_A}. And if we further assume that V is rational, then there are finitely many inequivalent irreducible V-modules so that A is a finite abelian group. By the argument above, we have come to the following situation: V^0 is a simple and rational VOA and $\{V^\alpha \mid \alpha \in D\}$ is a set of irreducible V^0-modules indexed by a finite abelian group D such that the direct sum $V_D = \oplus_{\alpha\in D} V^\alpha$ forms a VOA with D-grading $V^\alpha \cdot V^\beta \subset V^{\alpha+\beta}$. In the following, we study such an extension V_D.

4.1 Simple currents

Let V be a rational VOA. Then the fusion product $M^1 \boxtimes_V M^2$ exists and determined uniquely up to isomorphism for any two (strong) V-modules M^1 and M^2. It is shown in Theorem 3.6.3 that there are finitely many inequivalent irreducible V-modules. Let

$\{W^i \mid i = 1, \dots, N\}$ be the set of all inequivalent irreducible V-module. Denote by N_{ij}^k the fusion rule of type $W^i \times W^j \to W^k$, that is, the dimension of V-intertwining operators of type $W^i \times W^j \to W^k$. Then by definition we have

$$W^i \boxtimes_V W^j \simeq \bigoplus_{k=1}^N W^k \underset{\mathbb{C}}{\otimes} \binom{W^k}{W^i \ W^j}_V \simeq \bigoplus_{k=1}^N (W^k)^{\oplus N_{ij}^k}.$$

Definition 4.1.1. A strong V-module U is called *simple current* if $U \boxtimes_V W$ is not zero and irreducible for every irreducible V-module W.

By definition, we can easily verify that every simple current module is irreducible. The following simple lemma is useful to determine whether a module is a simple current or not:

Lemma 4.1.2. *Assume that the fusion algebra for V is associative. If two V-modules U and X satisfy the equality $U \times X = V$ in the fusion algebra for V, the both U and X are irreducible and simple currents.*

Proof: Clearly, both U and X are not zero. Decompose U into a direct sum $U = \oplus_i U^{(i)}$ of irreducible V-modules. Then $U \times X = V$ implies that there is an irreducible component $U^{(i_0)}$ such that $U^{(i)} \times X = \delta_{i,i_0} V$. Then $U = V \times U = (U^{(i_0)} \times X) \times U = U^{(i_0)} \times (U \times X) = U^{(i_0)} \times V = U^{(i_0)}$. Thus $U = U^{(i_0)} \neq 0$ and U is irreducible. Symmetrically, X is also irreducible. Now let W be an irreducible V-module. Then $U \times W \neq 0$ because $X \times (U \times W) = (X \times U) \times W = V \times W = W$. If $U \times W$ is not irreducible, then so is $X \times (U \times W)$. However, $X \times (U \times W) = (X \times U) \times W = V \times W = W$ is irreducible, $U \times W$ is not zero and must be irreducible. Thus U is a simple current. Similarly, X is also a simple current. ∎

Remark 4.1.3. If V is C_2-cofinite and of CFT-type, then the fusion algebra for V is associative so that we can apply the lemma above.

4.2 Simple current extensions

Let V^0 be a simple rational VOA and D a finite abelian group. Let $\{V^\alpha \mid \alpha \in D\}$ be a set of irreducible V^0-modules indexed by D.

Lemma 4.2.1. *Assume that a direct sum $\oplus_{\alpha\in D} V^\alpha$ carries a structure of a VOA such that $0 \neq V^\alpha \cdot V^\beta \subset V^{\alpha+\beta}$. It is simple if and only if all V^α, $\alpha \in D$, are inequivalent irreducible V^0-modules.*

Proof: Assume that V_D is simple. Then the automorphism group of V_D contains a group isomorphic to the dual group D^* of an abelian group D because V_D is D-graded. It is clear that the D^*-invariants of V_D is exactly V^0. Therefore, by Theorem 3.9.1, each V^α is an irreducible V^0-modules.

Conversely, if $\{V^\alpha \mid \alpha \in D\}$ is a set of inequivalent irreducible V^0-modules such that $V_D = \oplus_{\alpha\in D} V^\alpha$ forms a D-graded vertex operator algebra, then V_D must be simple because of the density theorem. ∎

By the lemma above, we introduce the following definition.

Definition 4.2.2. A *D-graded extension* V_D of V^0 is a simple VOA with the shape $V_D = \oplus_{\alpha\in D} V^\alpha$ whose vacuum element and Virasoro element are given by those of V^0 and vertex operations in V_D satisfies $Y(u^\alpha, z)v^\beta \in V^{\alpha+\beta}((z))$ for any $u^\alpha \in V^\alpha$ and $v^\beta \in V^\beta$.

By the following lemma we can obtain a uniqueness of VOA-structure of a D-graded extension.

Proposition 4.2.3. *([DM3, Proposition 5.3]) Suppose that the space of V^0-intertwining operators of type $V^\alpha \times V^\beta \to V^{\alpha+\beta}$ is one dimensional. Then the VOA structure of a D-graded extension V_D of V^0 over $\mathbb{C}$ is unique.*

Proof: Assume that we have two vertex operator maps $Y^1(\cdot, z)$ and $Y^2(\cdot, z)$ on $\oplus_{\alpha\in D} V^\alpha$ such that both of them provide D-graded extensions of V^0. Then by assumption there are non-zero scalars $c(\alpha, \beta) \in \mathbb{C}$ such that $Y^2(x^\alpha, z)x^\beta = c(\alpha, \beta)Y^1(x^\alpha, z)x^\beta$ for any $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$. Since $Y^i(\mathbb{1}, z) = \mathrm{id}_{V_D}$ for $i = 1, 2$, $c(0, \alpha) = 1$ for all $\alpha \in D$. By the skew-symmetry, we have $c(\alpha, \beta) = c(\beta, \alpha)$. Moreover, by the commutativity

$$(z_1 - z_2)^{N_1} Y^i(x^\alpha, z_1)Y^i(x^\beta, z_2)x^\gamma = (z_1 - z_2)^{N_1} Y^i(x^\beta, z_2)Y^i(x^\alpha, z_1)x^\gamma$$

and the associativity

$$(z_0 + z_2)^{N_2} Y^i(x^\alpha, z_0 + z_2)Y^i(x^\beta, z_2)x^\gamma = (z_2 + z_0)^{N_2} Y^i(Y^i(x^\alpha, z_0)x^\beta, z_2)x^\gamma,$$

we have the following relations:

$$c(\alpha, \beta + \gamma)c(\beta, \gamma) = c(\beta, \alpha + \gamma)c(\alpha, \gamma) = c(\alpha, \beta)c(\alpha + \beta, \gamma).$$

Namely, $c(\cdot, \cdot) \in H^2(D, \mathbb{C}^*)$ and hence defines an abelian central extension of D by $\mathbb{C}$. Since D is a direct sum of finite cyclic groups and $\mathbb{C}$ is algebraically closed, such an extension splits by Theorem 3.2 of [Kar]. This means that there exists a coboundary $t : D \to \mathbb{C}^*$ such that $c(\alpha, \beta) = t(\alpha)t(\beta)/t(\alpha + \beta)$ for all $\alpha, \beta \in D$. Now define a linear map ψ from $(V_D, Y^1(\cdot, z))$ to $(V_D, Y^2(\cdot, z))$ by $\psi(x^\alpha) = t(\alpha)x^\alpha$ for $x^\alpha \in V^\alpha$. Then $\psi Y^1(x^\alpha, z)x^\beta = t(\alpha+$

$\beta)Y^1(x^\alpha,z)x^\beta = c(\alpha,\beta)t(\alpha+\beta)Y^2(x^\alpha,z)x^\beta = t(\alpha)t(\beta)Y^2(x^\alpha,z)x^\beta = Y^2(\psi x^\alpha,z)\psi x^\beta$. Therefore, ψ defines an isomorphism between two vertex operator algebra structures $(V_D, Y^1(\cdot,z))$ and $(V_D, Y^2(\cdot,z))$. ∎

Now we present a definition of simple current extensions.

Definition 4.2.4. A D-graded extension $V_D = \oplus_{\alpha\in D} V^\alpha$ is said to be a *D-graded simple current extension* if all V^α, $\alpha \in D$, are simple current V^0-modules.

The VOA structure of a simple current extension is unique over $\mathbb{C}$ by Proposition 4.2.3. But it is usually a difficult problem to determine whether a module is a simple current or not. On this problem, We can use Lemma 4.1.2.

Proposition 4.2.5. *Let V^0 be a simple rational C_2-cofinite VOA of CFT-type, and let $V_D = \oplus_\alpha V^\alpha$ be a D-graded extension of V^0. If we have the fusion rules $V^\alpha \times V^\beta = V^{\alpha+\beta}$ for all $\alpha,\beta \in D$, then V_D is a D-graded simple current extension of V^0.*

Proof: By Corollary 3.7.6, the fusion algebra for V^0 is associative. Then the assertion directly follows from Lemma 4.1.2. ∎

Next we introduce a notion of equivalent extensions. Let σ be an automorphism of V^0 and denote by $(V^\alpha)^\sigma$ the σ-conjugate V^0-module of V^α for $\alpha \in D$. If we have a D-graded extension $V_D = \oplus_{\alpha\in D} V^\alpha$ of V^0, then we can construct another D-graded extension $(V_D)^\sigma = \oplus_{\alpha\in D}(V^\alpha)^\sigma$ in the following way. By definition, there exist linear isomorphisms $\varphi_\alpha : V^\alpha \to (V^\alpha)^\sigma$ such that $Y_{(V^\alpha)^\sigma}(a,z)\varphi_\alpha = \varphi_\alpha Y_{V^\alpha}(\sigma a, z)$ for all $a \in V^0$. For $a \in V^\alpha$ and $b \in V^\beta$, define the vertex operation in $(V_D)^\sigma = \oplus_{\alpha\in D}(V^\alpha)^\sigma$ by

$$Y_{(V_D)^\sigma}(\varphi_\alpha a, z)\varphi_\beta b := \varphi_{\alpha+\beta} Y_{V_D}(a,z)b.$$

Since $Y_{(V_D)^\sigma}(\cdot,z)|_{(V^\alpha)^\sigma\times(V^\beta)^\sigma}$ is a V^0-intertwining operator of type $(V^\alpha)^\sigma \times (V^\beta)^\sigma \to (V^{\alpha+\beta})^\sigma$, $((V_D)^\sigma, Y_{(V_D)^\sigma}(\cdot,z))$ also forms a D-graded extension of V^0. Moreover, if V_D is a D-graded simple current extension of V^0, then so is $(V_D)^\sigma$. We call $(V_D)^\sigma$ the *σ-conjugate* of V_D. It is clear from its construction that V_D and $(V_D)^\sigma$ are isomorphic as VOAs even if $\{V^\alpha \mid \alpha \in D\}$ and $\{(V^\alpha)^\sigma \mid \alpha \in D\}$ are distinct sets of inequivalent V^0-modules. Therefore, we introduce the following definition.

Definition 4.2.6. Two D-graded simple current extensions $V_D = \oplus_{\alpha\in D} V^\alpha$ and $\tilde{V}_D = \oplus_{\alpha\in D}\tilde{V}^\alpha$ are said to be *equivalent* if there exists a VOA-isomorphism $\Phi : V_D \to \tilde{V}_D$ such that $\Phi(V^\alpha) = \tilde{V}^\alpha$ for all $\alpha \in D$.

The following are clear from its definition.

Lemma 4.2.7. *Let σ be an automorphism of V^0. Let V_D be a D-graded extension of V^0 and $(V_D)^\sigma$ the σ-conjugate of V_D. Then the V_D and $(V_D)^\sigma$ form equivalent D-graded extensions of V^0.*

Lemma 4.2.8. *Suppose that V_D is a D-graded extension of V^0. For an automorphism $\sigma \in \mathrm{Aut}(V^0)$, assume that there is an automorphism Ψ on V_D such that $\Psi(V^0) = V^0$ and $\Psi|_{V^0} = \sigma$. Then as sets of inequivalent irreducible V^0-modules, $\{\Psi^{-1}V^\alpha \mid \alpha \in D\}$ and $\{(V^\alpha)^\sigma \mid \alpha \in D\}$ are the same.*

Proof: Denote $Y_{V_D}(\cdot, z)|_{V^0 \otimes V^\alpha}$ by $Y_\alpha(\cdot, z)$. By definition, we can take linear isomorphisms $\varphi_\alpha : V^\alpha \to (V^\alpha)^\sigma$ such that $Y_{(V^\alpha)^\sigma}(a, z)\varphi_\alpha = \varphi_\alpha Y_\alpha(\sigma a, z)$ for all $a \in V^0$. Define $\Psi_\alpha : \Psi^{-1}V^\alpha \to (V^\alpha)^\sigma$ by $\Psi_\alpha = \varphi_\alpha \circ \Psi|_{\Psi^{-1}V^\alpha}$. Then for $a \in V^0$ we have

$$\begin{aligned} &Y_{(V^\alpha)^\sigma}(a, z)\Psi_\alpha = Y_{(V^\alpha)^\sigma}(a, z)\varphi_\alpha\Psi = \varphi_\alpha Y_\alpha(\sigma a, z)\Psi \\ &= \varphi_\alpha Y_\alpha(\Psi a, z)\Psi = \varphi_\alpha \Psi Y_{V_D}(a, z)|_{\Psi^{-1}V^\alpha} = \Psi_\alpha Y_{V_D}(a, z)|_{\Psi^{-1}V^\alpha}. \end{aligned}$$

Therefore, Ψ_α is a V^0-isomorphisms. Hence, we get the assertion. ∎

Conversely, we have the following lifting property which is due to Shimakura [Sh]:

Proposition 4.2.9. *([Sh]) Let $V_D = \oplus_{\alpha \in D} V^\alpha$ be a D-graded simple current extension of V^0. If $\sigma \in \mathrm{Aut}(V^0)$ satisfies $\{(V^\alpha)^\sigma \mid \alpha \in D\} = \{V^\alpha \mid \alpha \in D\}$, then there is a lifting $\tilde{\sigma} \in \mathrm{Aut}(V_D)$ of σ such that $\tilde{\sigma}V^0 = V^0$ and $\tilde{\sigma}|_{V^0} = \sigma$. This lifting is unique up to multiples of D^*.*

Proof: Let $\varphi_\alpha : V^\alpha \to (V^\alpha)^\sigma$ be a canonical linear isomorphism such that

$$Y_{(V^\alpha)^\sigma}(x^0, z)\varphi_\alpha = \varphi_\alpha Y_{V^\alpha}(\sigma x^0, z)$$

for all $x^0 \in V^0$. By assumption, there is a permutation $\rho : D \to D$ such that $(V^\alpha)^\sigma \simeq V^{\rho(\alpha)}$ as V^0-modules. Since all V^α, $\alpha \in D$, are simple current V^0-modules, the permutation satisfies the group homomorphism condition: $\rho(\alpha + \beta) = \rho(\alpha) + \rho(\beta)$ for any $\alpha, \beta \in D$. Take a V^0-isomorphism $\psi_\alpha : (V^\alpha)^\sigma \xrightarrow{\sim} V^{\rho(\alpha)}$ for each $\alpha \in D$ and define a linear isomorphism Φ on V_D by $\Phi|_{V^\alpha} = \psi_\alpha \circ \varphi_\alpha$. Let $Y_{V_D}(\cdot, z)$ be the vertex operator map on V_D. We define a new vertex operator map on V_D by

$$\tilde{Y}(\cdot, z) := \Phi^{-1} Y_{V_D}(\Phi\, \cdot, z)\Phi.$$

Then $(V_D, \tilde{Y}(\cdot, z))$ also forms a D-graded simple current extension of V^0. Then by the proof of Proposition 4.2.3, there is a 2-coboundary $t : D \to \mathbb{C}^*$ such that

$$\tilde{Y}(\cdot, z) = \frac{t(\alpha + \beta)}{t(\alpha)t(\beta)} Y_{V_D}(\cdot, z). \tag{4.2.1}$$

We may choose the coboundary t such that $t(0) = 1$. Now define a linear isomorphism $\tilde{\Phi}$ on V_D by $\tilde{\Phi}|_{V^\alpha} = t(\alpha)\Phi|_{V^\alpha}$ for $\alpha \in D$. Then by (4.2.1) we have

$$\tilde{\Phi} Y_{V_D}(x^\alpha, z)x^\beta = Y_{V_D}(\tilde{\Phi}x^\alpha, z)\tilde{\Phi}x^\beta$$

for any $x^\alpha \in V^\alpha$ and $x^\beta \in V^\beta$. Namely, $\tilde{\Phi} \in \mathrm{Aut}(V_D)$. Since $\tilde{\Phi}|_{V^0} = \sigma^{-1} \cdot \mathrm{id}_{V^0}$, the desired automorphism is given by $\tilde{\Phi}^{-1}$. The rest of the assertion is clear by Schur's lemma ∎

4.3 Twisted algebra $A_\lambda(D, \mathcal{S}_W)$

We keep the setup of Section 4.2. Let W be an irreducible V^0-module. In this subsection we describe a construction of the twisted algebra from W and V^α, $\alpha \in D$. Since all V^α, $\alpha \in D$, are simple current V^0-modules, all $V^\alpha \boxtimes_{V^0} W$, $\alpha \in D$, are also irreducible V^0-modules. By the results of Huang [H1] [H4], fusion products among V^0-modules satisfy the associativity. Therefore $V^\alpha \boxtimes_{V^0} W \neq 0$ for all $\alpha \in D$ and $D_W := \{\alpha \in D \mid V^\alpha \boxtimes_{V^0} W \simeq W\}$ forms a subgroup of D. Set $\mathcal{S}_W := D/D_W$. Then $\mathcal{S}_W$ naturally admits an action of D. By definition, D_W acts on $\mathcal{S}_W$ trivially. Let $s \in \mathcal{S}_W$ and take a representative $\alpha \in D$ such that $s = \alpha + D_W$. We should note that irreducible V^0-modules $V^\alpha \boxtimes_{V^0} W$ and $V^\beta \boxtimes_{V^0} W$ are isomorphic if and only if $\alpha - \beta \in D_W$. Thus, the equivalent class of $V^\alpha \boxtimes_{V^0} W$ is independent of choice of a representative α in $\alpha + D_W$ and hence determined uniquely. So for each $s \in \mathcal{S}_W$, we define $W^s := V^\alpha \boxtimes_{V^0} W$ after fixing a representative $\alpha \in D$ such that $s = \alpha + D_W$.

Let $\alpha, \beta, \gamma \in D$ and $s \in \mathcal{S}_W$. It follows from the associativity of fusion products that $V^\alpha \boxtimes_{V^0} W^s = W^{s+\alpha}$, where $s+\alpha$ denotes the action of $\alpha \in D$ to $s \in \mathcal{S}_W$. Take basis $I_s^\alpha(\cdot, z)$ of the 1-dimensional spaces of V^0-intertwining operators of type $V^\alpha \times W^s \to W^{s+\alpha}$. By an associative property of V^0-intertwining operators (cf. [H1] [H4]), there are (non-zero) scalars $\lambda_s(\alpha, \beta) \in \mathbb{C}$ such that the following equality holds:

$$\langle \nu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w^s \rangle = \lambda_s(\alpha, \beta) \langle \nu, I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha, z_0) x^\beta, z_2) w^s \rangle |_{z_0 = z_1 - z_2},$$

where $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$, $w^s \in W^s$ and $\nu \in (W^{s+\alpha+\beta})^*$. We normalize intertwining operators $I_s^0(\cdot, z)$ to satisfy $I_s^0(\mathbb{1}, z) = \mathrm{id}_{W^s}$. In other words, $I_s^0(\cdot, z)$ are vertex operators on V^0-modules W^s. By considering

$$\langle \nu', I_{s+\beta+\gamma}^\alpha(x^\alpha, z_1) I_{s+\gamma}^\beta(x^\beta, z_2) I_s^\gamma(x^\gamma, z_3) w^s \rangle,$$

we can deduce a relation

$$\lambda_{s+\gamma}(\alpha, \beta) \lambda_s(\alpha + \beta, \gamma) = \lambda_s(\alpha, \beta + \gamma) \lambda_s(\beta, \gamma).$$

By the normalization $I_s^0(\mathbb{1}, z) = \mathrm{id}_{W^s}$, the $\lambda_s(\cdot, \cdot)$'s above also satisfy a condition $\lambda_s(0, \alpha) = \lambda_s(\alpha, 0) = 1$ for all $\alpha \in D$ and $s \in \mathcal{S}_W$. Using $\lambda_s(\alpha, \beta)$, we introduce the twisted algebra. Let $q(s)$, $s \in \mathcal{S}_W$, be formal symbols and $\mathbb{C}\mathcal{S}_W := \oplus_{s \in \mathcal{S}_W} \mathbb{C} q(s)$ a linear space spanned by them. We define a multiplication on $\mathbb{C}\mathcal{S}_W$ by $q(s) \cdot q(t) := \delta_{s,t} q(s)$. Then $\mathbb{C}\mathcal{S}_W$ becomes a semisimple commutative associative algebra isomorphic to $\mathbb{C}^{\oplus |\mathcal{S}_W|}$. Let $U(\mathbb{C}\mathcal{S}_W) := \{\sum_{s \in \mathcal{S}_W} \mu_s q(s) \mid \mu_s \in \mathbb{C}^*\}$ be the set of units in $\mathbb{C}\mathcal{S}_W$. Then $U(\mathbb{C}\mathcal{S}_W)$ forms a multiplicative group in $\mathbb{C}\mathcal{S}_W$. Define an action of $\alpha \in D$ on $\mathbb{C}\mathcal{S}_W$ by $q(s)^\alpha := q(s - \alpha)$. Then $U(\mathbb{C}\mathcal{S}_W)$ is a multiplicative right D-module. Set $\bar{\lambda}(\alpha, \beta) = \sum_{s \in \mathcal{S}_W} \lambda_s(\alpha, \beta)^{-1} q(s) \in$

$U(\mathbb{C}\mathcal{S}_W)$. Then $\bar{\lambda}(\cdot,\cdot)$ defines a 2-cocycle $D \times D \to U(\mathbb{C}\mathcal{S}_W)$ because it satisfies a 2-cocycle condition

$$\bar{\lambda}(\alpha,\beta)^\gamma \cdot \bar{\lambda}(\alpha+\beta,\gamma) = \bar{\lambda}(\alpha,\beta+\gamma) \cdot \bar{\lambda}(\beta,\gamma).$$

Since the space of V^0-intertwining operators of type $V^\alpha \times W^s \to W^{s+\alpha}$ is 1-dimensional, $\bar{\lambda}$ is unique up to 2-coboundaries. Namely, $\bar{\lambda}$ defines an element of the second cohomology group $H^2(D, U(\mathbb{C}\mathcal{S}_W))$. Let $\mathbb{C}[D] = \oplus_{\alpha \in D} \mathbb{C}e^\alpha$ be the group ring of D and set

$$A_\lambda(D,\mathcal{S}_W) := \mathbb{C}[D] \underset{\mathbb{C}}{\otimes} \mathbb{C}\mathcal{S}_W = \left\{ \sum \mu_{\alpha,s} e^\alpha \otimes q(s) \mid \alpha \in D, s \in \mathcal{S}_W, \mu_{\alpha,s} \in \mathbb{C} \right\}$$

and define a multiplication $*$ on $A_\lambda(D,\mathcal{S}_W)$ by

$$e^\alpha \otimes q(s) * e^\beta \otimes q(t) := \lambda_t(\alpha,\beta)^{-1} e^{\alpha+\beta} \otimes q(s)^\beta \cdot q(t).$$

Then $A_\lambda(D,\mathcal{S}_W)$ equipped with the product $*$ forms an associative algebra with the unit element $\sum_{s \in \mathcal{S}_W} e^0 \otimes q(s)$. We call $A_\lambda(D,\mathcal{S}_W)$ the *twisted algebra* associated to a pair $(D,\mathcal{S}_W)$.

Remark 4.3.1. The algebra $A_\lambda(D,\mathcal{S}_W)$ is called the *generalized twisted double* in [DY]. It naturally appears in the orbifold theory, and has been studied in many papers. For reference, see [DVVV] [DM3] [DY] [Mas].

Take an $s \in \mathcal{S}_W$ and set $\mathbb{C}[D] \otimes q(s) := \oplus_{\alpha \in D} \mathbb{C}e^\alpha \otimes q(s)$. Then $\mathbb{C}[D] \otimes q(s)$ is a subalgebra of $A_\lambda(D,\mathcal{S}_W)$. It has a subalgebra $\mathbb{C}[D_W] \otimes q(s) := \oplus_{\alpha \in D_W} \mathbb{C}e^\alpha \otimes q(s)$ which is isomorphic to the twisted group algebra $\mathbb{C}^{\lambda_s}[D_W]$ of D_W associated to a 2-cocycle $\lambda_s(\cdot,\cdot)^{-1} \in Z^2(D_W,\mathbb{C})$. There is a one-to-one correspondence between the category of $\mathbb{C}^{\lambda_s}[D_W]$-modules and the category of $A_\lambda(D,\mathcal{S}_W)$-modules given as below:

Theorem 4.3.2. *([Mas] [DY, Theorem 3.5]) The functors*

$$\begin{array}{llll} \mathrm{Ind}_{\mathbb{C}^{\lambda_s}[D_W]}^{A_\lambda(D,\mathcal{S}_W)}: & M \in \mathbb{C}^{\lambda_s}[D_W]\text{-Mod} & \mapsto & \mathbb{C}[D] \otimes q(s) \bigotimes_{\mathbb{C}[D_W] \otimes q(s)} M \in A_\lambda(D,\mathcal{S}_W)\text{-Mod}, \\ \mathrm{Red}_{\mathbb{C}^{\lambda_s}[D_W]}^{A_\lambda(D,\mathcal{S}_W)}: & N \in A_\lambda(D,\mathcal{S}_W)\text{-Mod} & \mapsto & e^0 \otimes q(s) N \in \mathbb{C}^{\lambda_s}[D_W]\text{-Mod} \end{array}$$

define equivalences between the module categories $\mathbb{C}^{\lambda_s}[D_W]$-Mod *and* $A_\lambda(D,\mathcal{S}_W)$-Mod. *In particular, $A_\lambda(D,\mathcal{S}_W)$ is a semisimple algebra.*

4.4 Representation theory

We keep the setup of the previous section. In this section we study V_D-modules.

4.4.1 Untwisted modules

Let M be an indecomposable V_D-module. Since V^0 is regular, we can find an irreducible V^0-submodule W of M. We use the same notation for D_W, $\mathcal{S}_W$, $A_\lambda(D,\mathcal{S}_W)$ and $\mathbb{C}^{\lambda_s}[D_W]$ as previously. We should note that the definition of D_W is independent of the choice of an irreducible component W. One can show the following.

Lemma 4.4.1. *All $V^\alpha \cdot W = \{\sum a_n w \mid a \in V^\alpha, w \in W, n \in \mathbb{Z}\}$, $\alpha \in D$, are non-trivial irreducible V^0-submodules of M.*

Proof: Recall the associativity and the commutativity of vertex operators. Let x, y be any elements in a VOA and v be any element in a module. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$(z_1 - z_2)^{N_1} Y(x, z_1)Y(y, z_2)v = (z_1 - z_2)^{N_1} Y(y, z_2)Y(x, z_1)v,$$
$$(z_0 + z_2)^{N_2} Y(x, z_0 + z_2)Y(y, z_2)v = (z_2 + z_0)^{N_2} Y(Y(x, z_0)y, z_2)v.$$

The first equality is called the commutativity and the second is called the associativity of vertex operators. An integer N_1 depends on x and y, whereas N_2 does not only on x and y but also on v. Using the associativity, we can show that $V^\alpha \cdot (V^\beta \cdot W) \subset (V^\alpha \cdot V^\beta) \cdot W = V^{\alpha+\beta} \cdot W$. In particular, all $V^\alpha \cdot W$, $\alpha \in D$, are V^0-submodules. We show that $V^\alpha \cdot W$ is not zero and then we prove that it is irreducible. If $V^\alpha \cdot W = 0$, then by the iterate formula

$$(a_{(m)}b)_{(n)} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} \{a_{(m-i)}b_{(n+i)} - (-1)^m b_{(m+n-i)}a_{(i)}\},$$

we obtain $V^{n\alpha} \cdot W = 0$ for $n = 1, 2, \ldots$. But D is a finite abelian, we arrive at $V^0 \cdot W = 0$, a contradiction. Therefore, $V^\alpha \cdot W \neq 0$ for all $\alpha \in D$. Next, assume that there exists a proper non-trivial V^0-submodule X in $V^\alpha \cdot W$. Then we have $V^{-\alpha} \cdot X \subset V^{-\alpha} \cdot (V^\alpha \cdot W) \subset (V^{-\alpha} \cdot V^\alpha) \cdot W = V^0 \cdot W = W$ and hence we get $V^{-\alpha} \cdot X = W$ because W is irreducible. Then we obtain $V^\alpha \cdot W = V^\alpha \cdot (V^{-\alpha} \cdot X) \subset (V^\alpha \cdot V^{-\alpha}) \cdot X = V^0 \cdot X = X$, a contradiction. Therefore, $V^\alpha \cdot W$ is a non-trivial and irreducible V^0-submodule of M. ∎

Remark 4.4.2. We note that in the proof above we do not use the condition that V^α are simple currents. Thus Lemma 4.4.1 is true even if V_D is just a D-graded extension.

Let $D = \bigsqcup_{i=1}^{n} (t^i + D_W)$ be a coset decomposition of D with respect to D_W. Set $V_{D_W + t^i} := \oplus_{\alpha \in D_W} V^{\alpha + t^i}$. Then V_{D_W} forms a sub VOA of V_D, which is a D_W-graded simple current extension of V^0, and $V_D = \oplus_{i=1}^{n} V_{D_W + t^i}$ forms a D/D_W-graded simple current extension of V_{D_W}. As we have assumed that M is an indecomposable V_D-module, every irreducible V^0-submodule is isomorphic to one of $V^{t^i} \boxtimes_{V^0} W$, $i = 1, \ldots, n$.

Remark 4.4.3. Let $M_{D_W+t^i}$ be the sum of all irreducible V^0-submodules of M isomorphic to $V^{t^i} \boxtimes_{V^0} W$. Then we have the following decomposition of M into a direct sum of isotypical V^0-components with a D/D_W-grading:

$$M = \bigoplus_{t=1}^{n} M_{D_W+t^i}, \quad V_{D_W+t^i} \cdot M_{D_W+t^j} = M_{D_W+t^i+t^j}.$$

In particular, if M is irreducible under V_D, then each $M_{D_W+t^i}$ is an irreducible V_{D_W}-module. Thus, viewing V_D as a D/D_W-graded simple current extension of V_{D_W}, we can regard M as a D/D_W-stable V_D-module (for the D/D_W-stability, see Definition 4.2.8 below).

For each $s \in \mathcal{S}_W$, we set $W^s = V^s \boxtimes_{V^0} W$ by abuse of notation (because it is well-defined). Since all V^α, $\alpha \in D$, are simple current V^0-modules, there are unique V^0-intertwining operators $I_s^\alpha(\cdot, z)$ of type $V^\alpha \times W^s \to W^{s+\alpha}$ up to scalar multiples. We choose $I_s^0(\cdot, z)$ to satisfy the condition $I_s^0(\mathbb{1}, z) = \mathrm{id}_{W^s}$, i.e., $I_s^0(\cdot, z)$ defines the vertex operator on a V^0-module W^s for each $s \in \mathcal{S}_W$. Then, by Huang [H1] [H4], there exist scalars $\lambda_s(\alpha, \beta) \in \mathbb{C}$ such that

$$\langle \nu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w\rangle = \lambda_s(\alpha, \beta)\langle \nu, I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha, z_0)x^\beta, z_2)w\rangle|_{z_0=z_1-z_2}, \qquad (4.4.1)$$

where $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$, $w \in W^s$ and $\nu \in (W^{s+\alpha+\beta})^*$. Then by the same procedure as in the previous section, we can find the 2-cocycle $\bar{\lambda}(\cdot, \cdot) \in H^2(D, U(\mathbb{C}\mathcal{S}_W))$ and construct the twisted algebra $A_\lambda(D, \mathcal{S}_W)$. By assumption, M is a direct sum of some copies of W^s, $s \in \mathcal{S}_W$, as a V^0-modules so that we have $M \simeq \oplus_{s\in\mathcal{S}_W} W^s \otimes \mathrm{Hom}_{V^0}(W^s, M)$. Set $U^s := \mathrm{Hom}_{V^0}(W^s, M)$. Clearly, all of U^s, $s \in \mathcal{S}_W$, are not zero because of Lemma 4.4.1. On $W^s \otimes U^s$, the vertex operator of $x^\alpha \in V^\alpha$ can be written as

$$Y_M(x^\alpha, z)|_{W^s\otimes U^s} = I_s^\alpha(x^\alpha, z) \otimes \phi_s(\alpha) \qquad (4.4.2)$$

with some $\phi_s(\alpha) \in \mathrm{Hom}_{\mathbb{C}}(U^s, U^{s+\alpha})$. Using (4.4.1) we can show that

$$\phi_{s+\beta}(\alpha)\phi_s(\beta) = \lambda_s(\alpha, \beta)^{-1}\phi_s(\alpha + \beta)$$

and hence we can define an action of $A_\lambda(D, \mathcal{S}_W)$ on $\oplus_{s\in\mathcal{S}_W} U^s$ by $e^\alpha \otimes q(s) \cdot \mu := \delta_{s,t}\phi_s(\alpha)\mu$ for $\mu \in U^t$.

Lemma 4.4.4. *Under the action above, $\oplus_{s\in\mathcal{S}_W} U^s$ becomes an $A_\lambda(D, \mathcal{S}_W)$-module.*

Then we have

Proposition 4.4.5. *Suppose that M is irreducible under V_D. Then U^s is an irreducible $\mathbb{C}^{\lambda_s}[D_W]$-module for every $s \in \mathcal{S}_W$. Moreover, $\oplus_{s\in\mathcal{S}_W} U^s$ is an irreducible $A_\lambda(D, \mathcal{S}_W)$-module.*

Proof: Since $W^s \otimes U^s$ is an irreducible V_{D_W}-submodule of M, U^s is an irreducible $\mathbb{C}^{\lambda_s}[D_W]$-module. Moreover, there is a canonical $A_\lambda(D, \mathcal{S}_W)$-homomorphism from $P = A_\lambda(D, \mathcal{S}_W) \otimes_{\mathbb{C}[D_W]\otimes q(s)} U^s = \{\mathbb{C}[D] \otimes q(s)\} \otimes_{\mathbb{C}[D_W]\otimes q(s)} U^s$ to $\oplus_{s\in\mathcal{S}_W} U^s$. As

$$P = \bigoplus_{t\in\mathcal{S}_W} e^t \otimes q(s) \underset{\mathbb{C}[D_W]\otimes q(s)}{\otimes} U^s$$

and is irreducible under $A_\lambda(D, \mathcal{S}_W)$ by Theorem 4.3.2, we see that $U^t = e^t \otimes q(s) \otimes U^s$ and hence $P = \oplus_{s\in\mathcal{S}_W} U^s$. Consequently, $\oplus_{s\in\mathcal{S}_W} U^s$ is also irreducible under $A_\lambda(D, \mathcal{S}_W)$. ∎

Corollary 4.4.6. *If M is an irreducible V_D-module, then irreducible V^0-components W^s and W^t have the same multiplicity in M for all $s, t \in \mathcal{S}_W$.*

Proof: Because U^s and U^t have the same dimension. ∎

Theorem 4.4.7. *Let V^0 be a rational C_2-cofinite vertex operator algebra of CFT-type, and let $V_D = \oplus_{\alpha\in D} V^\alpha$ be a D-graded simple current extension of V^0. Then an indecomposable V_D-module M is completely reducible under V_D. Consequently, V_D is regular. As a V^0-module, an irreducible V_D-submodule of M has the shape $\oplus_{s\in\mathcal{S}_W} W^s \otimes U^s$ with each U^s an irreducible $\mathbb{C}[D_W] \otimes q(s) \simeq \mathbb{C}^{\lambda_s}[D_W]$-module for all $s \in \mathcal{S}_W$. Moreover, all U^t, $t \in \mathcal{S}_W$, are determined by one of them, say U^s, by the following rule:*

$$U^t \simeq \mathrm{Red}^{A_\lambda(D,\mathcal{S}_W)}_{\mathbb{C}^{\lambda_t}[D_W]} \mathrm{Ind}^{A_\lambda(D,\mathcal{S}_W)}_{\mathbb{C}^{\lambda_s}[D_W]} U^s.$$

Proof: Since $M = \oplus_{s\in\mathcal{S}_W} W^s \otimes \mathrm{Hom}_{V^0}(W^s, M)$ and the space $\oplus_{s\in\mathcal{S}_W} \mathrm{Hom}_{V^0}(W^s, M)$ carries a structure of a module for a semisimple algebra $A_\lambda(D, \mathcal{S}_W)$ by Proposition 4.4.5, M is also a completely reducible V_D-module because of (4.4.2). Since V^0 is regular, all V_D-modules are completely reducible. So V_D is also regular. Now assume that M is an irreducible V_D-module. The decomposition is already shown. It remains to show that U^t is determined by U^s by the rule as stated. It is shown in the proof of Proposition 4.4.5 that $U^t = e^t \otimes q(s) \otimes U^s$. It is easy to see that $e^t \otimes q(s) \otimes U^s = \mathrm{Red}^{A_\lambda(D,\mathcal{S}_W)}_{\mathbb{C}^{\lambda_t}[D_W]} \mathrm{Ind}^{A_\lambda(D,\mathcal{S}_W)}_{\mathbb{C}^{\lambda_s}[D_W]} U^s$. The proof is completed. ∎

By the theorem above and Theorem 4.3.2, the number of inequivalent irreducible V_D-modules containing W as a V^0-submodule is equal to $\dim_{\mathbb{C}} Z(\mathbb{C}^{\lambda_s}[D_W])$. In particular, if $D_W = 0$, then the structure of a V_D-module containing W is uniquely determined by its V^0-module structure. For convenience, we introduce the following notion.

Definition 4.4.8. An irreducible V_D-module N is said to be *D-stable* if $D_W = 0$ for some irreducible V^0-submodule W.

It is obvious that the definition of the D-stability is independent of the choice of an irreducible V^0-submodule W. Let N^i, $i = 1, 2, 3$, be irreducible D-stable V_D-modules, and let W^i be irreducible V^0-submodules of N^i for $i = 1, 2, 3$. Set $W^{i,\alpha} := V^\alpha \boxtimes_{V^0} W^i$. Then $W^{i,\alpha} \simeq W^{i,\beta}$ as V^0-modules if and only if $\alpha = \beta$, and N^i as a V^0-module is isomorphic to $\oplus_{\alpha \in D} W^{i,\alpha}$. We expect that the following lemma would be used in our future study.

Theorem 4.4.9. *(The lifting property) Let $N^i = \oplus_{\alpha\in D} W^{i,\alpha}$ be as above. Then for a V^0-intertwining operator $I(\cdot, z)$ of type $W^{1,0} \times W^{2,0} \to W^{3,0}$, there is a V_D-intertwining operator $\tilde{I}(\cdot, z)$ of type $N^1 \times N^2 \to N^3$ such that the restriction of $\tilde{I}(\cdot, z)$ on $W^{1,0} \otimes W^{2,0}$ is equal to $I(\cdot, z)$. In particular, we have the following isomorphism:*

$$\binom{N^3}{N^1\ N^2}_{V_D} \simeq \bigoplus_{\alpha \in D} \binom{W^{3,\alpha}}{W^{1,0}\ W^{2,0}}_{V^0}.$$

In the above, $\binom{N^3}{N^1\ N^2}_{V_D}$ denotes the space of V_D-intertwining operators of type $N^1 \times N^2 \to N^3$.

Proof: It is obvious from Proposition 3.1.5 that the above linear map is injective. So we only have to prove the first assertion. By assumption, we have D-graded decompositions $X = \oplus_{\alpha\in D} X^\alpha$, $W = \oplus_{\alpha \in D} W^\alpha$ and $T = \oplus_{\alpha\in D} T^\alpha$ such that all X^α, W^α and T^α, $\alpha \in D$, are irreducible V^0-submodules. By Theorem 3.7.5, there exist V^0-intertwining operators $I^{\alpha,0}(\cdot, z)$ and $I^{0,\alpha}(\cdot, z)$ of type $X^\alpha \times W^0 \to T^\alpha$ and $X^0 \times W^\alpha \to T^\alpha$, respectively such that

$$\iota_{20}^{-1}\langle t^*, I^{\alpha,0}(Y(u^\alpha, z_0)x^0, z_2)w^0\rangle|_{z_0=z_1-z_2} = \iota_{12}^{-1}\langle t^*, Y(u^\alpha, z_1)I^{0,0}(x^0, z_2)w^0\rangle \tag{4.4.3}$$

and

$$\iota_{12}^{-1}\langle t^*, Y(u^\alpha, z_1)I^{0,0}(x^0, z_2)w^0\rangle = \iota_{21}^{-1}\langle t^*, I^{0,\alpha}(x^0, z_2)Y(u^\alpha, z_1)w^0\rangle \tag{4.4.4}$$

because all V^α are simple current V^0-modules, where $u^\alpha \in V^\alpha$, $x^0 \in X^0$, $w^0 \in W^0$, $t^* \in T^*$, and $\iota_{12}^{-1} f(z_1, z_2)$ denotes the formal power expansion of an analytic function $f(z_1, z_2)$ in the domain $|z_1| > |z_2| > 0$ (cf. [FHL]). Then, again by Theorem 3.7.5, we can find V^0-intertwining operators $I^{\alpha,\beta}(\cdot, z)$ of type $X^\alpha \times W^\beta \to T^{\alpha+\beta}$ such that

$$\iota_{12}^{-1}\langle t^*, Y(u^\alpha, z_1)I^{0,\beta}(x^0, z_2)w^\beta\rangle = \iota_{20}^{-1}\langle t^*, I^{\alpha,\beta}(Y(u^\alpha, z_0)x^0, z_2)w^\beta\rangle|_{z_0=z_1-z_2}. \tag{4.4.5}$$

We claim that $\tilde{I}(x^\alpha, z)w^\beta := I^{\alpha,\beta}(x^\alpha, z)w^\beta$ defines a V_D-intertwining operator of type $X \times W \to T$. We only need to show the associativity and the commutativity of $\tilde{I}(\cdot, z)$. Let $v^\beta \in V^\beta$ and $w^\gamma \in W^\gamma$. Then we have

$$\begin{aligned}
&\iota_{120}^{-1}\langle t^*, Y(u^\alpha, z_1)I^{\beta,\gamma}(Y(v^\beta, z_0)x^0, z_2)w^\gamma\rangle|_{z_0=z_3-z_2} \\
&= \iota_{132}^{-1}\langle t^*, Y(u^\alpha, z_1)Y(v^\beta, z_3)I^{0,\gamma}(x^0, z_2)w^\gamma\rangle \\
&= \iota_{342}^{-1}\langle t^*, Y(Y(u^\alpha, z_4)v^\beta, z_3)I^{0,\gamma}(x^0, z_2)w^\gamma\rangle|_{z_4=z_1-z_3} \\
&= \iota_{240}^{-1}\langle t^*, I^{\alpha+\beta,\gamma}(Y(Y(u^\alpha, z_4)v^\beta, z_0)x^0, z_2)w^\gamma\rangle|_{z_4=z_1-z_3, z_0=z_3-z_2} \\
&= \iota_{260}^{-1}\langle t^*, I^{\alpha+\beta,\gamma}(Y(u^\alpha, z_6)Y(v^\beta, z_0)x^0, z_2)w^\gamma\rangle|_{z_6=z_1-z_2, z_0=z_3-z_2}
\end{aligned}$$

and hence we obtain the following associativity:

$$\langle t^*, Y(u^\alpha, z_1)I^{\beta,\gamma}(x^\beta, z_2)w^\gamma\rangle = \langle t^*, I^{\alpha+\beta,\gamma}(Y(u^\alpha, z_0)x^\beta, z_2)w^\gamma\rangle|_{z_0=z_1-z_2}. \tag{4.4.6}$$

Next we prove the commutativity of $I^{\alpha,\beta}(\cdot, z)$. We have

$$\begin{aligned}
&\iota_{201}^{-1}\langle t^*, I^{\beta,\alpha}(Y(v^\beta, z_0)x^0, z_2)Y(u^\alpha, z_1)w^0\rangle|_{z_0=z_3-z_2}\\
&= \iota_{321}^{-1}\langle t^*, Y(v^\beta, z_3)I^{0,\alpha}(x^0, z_2)Y(u^\alpha, z_1)w^0\rangle\\
&= \iota_{312}^{-1}\langle t^*, Y(v^\beta, z_3)Y(u^\alpha, z_1)I^{0,0}(x^0, z_2)w^0\rangle\\
&= \iota_{132}^{-1}\langle t^*, Y(u^\alpha, z_1)Y(v^\beta, z_3)I^{0,0}(x^0, z_2)w^0\rangle\\
&= \iota_{342}^{-1}\langle t^*, Y(Y(u^\alpha, z_4)v^\beta, z_3)I^{0,0}(x^0, z_2)w^0\rangle|_{z_4=z_1-z_3}\\
&= \iota_{204}^{-1}\langle t^*, I^{\alpha+\beta,0}(Y(Y(u^\alpha, z_4)v^\beta, z_0)x^0, z_2)w^0\rangle|_{z_0=z_3-z_2,z_4=z_1-z_3}\\
&= \iota_{250}^{-1}\langle t^*, I^{\alpha+\beta,0}(Y(u^\alpha, z_5)Y(v^\beta, z_0)x^0, z_2)w^0\rangle|_{z_0=z_3-z_2,z_5=z_1-z_2}\\
&= \iota_{120}^{-1}\langle t^*, Y(u^\alpha, z_1)I^{\beta,0}(Y(v^\beta, z_0)x^0, z_2)w^0\rangle|_{z_0=z_3-z_2}.
\end{aligned}$$

Thus, we get the following:

$$\langle t^*, Y(u^\alpha, z_1)I^{\beta,0}(x^\beta, z_2)w^0\rangle = \langle t^*, I^{\beta,\alpha}(x^\beta, z_2)Y(u^\alpha, z_1)w^0\rangle. \tag{4.4.7}$$

Then

$$\begin{aligned}
&\iota_{123}^{-1}\langle t^*, Y(u^\alpha, z_1)I^{\beta,\gamma}(x^\beta, z_2)Y(v^\gamma, z_3)w^0\rangle\\
&= \iota_{132}^{-1}\langle t^*, Y(u^\alpha, z_1)Y(v^\gamma, z_3)I^{\beta,0}(x^\beta, z_2)w^0\rangle\\
&= \iota_{302}^{-1}\langle t^*, Y(Y(u^\alpha, z_0)v^\gamma, z_3)I^{\beta,0}(x^\beta, z_2)w^0\rangle|_{z_0=z_1-z_3}\\
&= \iota_{230}^{-1}\langle t^*, I^{\beta,\alpha+\gamma}(x^\beta, z_2)Y(Y(u^\alpha, z_0)v^\gamma, z_3)w^0\rangle|_{z_0=z_1-z_3}\\
&= \iota_{213}^{-1}\langle t^*, I^{\beta,\alpha+\gamma}(x^\beta, z_2)Y(u^\alpha, z_1)Y(v^\beta, z_3)w^0\rangle
\end{aligned}$$

and hence we arrive at the following commutativity:

$$\langle t^*, Y(u^\alpha, z_1)I^{\beta,\gamma}(x^\beta, z_2)w^\gamma\rangle = \langle t^*, I^{\beta,\alpha+\gamma}(x^\beta, z_2)Y(u^\alpha, z_1)w^\gamma\rangle. \tag{4.4.8}$$

This completes the proof of Lemma 5.1.12. ∎

Remark 4.4.10. Let M be an irreducible V_D-module and W an irreducible V^0-submodule of M. Even if $D_W \neq 0$, we can apply Theorem 4.4.9 to M as follows. We may consider V_D as a D/D_W-graded simple current extension of V_{D_W} as in Remark 4.4.3. Then we can view M as a D/D_W-stable V_D-module. So by replacing D by D/D_W, we can apply Theorem 4.4.9 to M.

4.4.2 Twisted modules

Let σ be an automorphism on V_D such that V^0 is contained in $V_D^{\langle\sigma\rangle}$, the space of σ-invariants of V_D. Let $V_D^{(r)}$ be a subspace $\{a \in V_D \mid \sigma a = e^{2\pi\sqrt{-1}\,r/|\sigma|}a\}$ for each $0 \le r \le$

$|\sigma|-1$, where $|\sigma|$ is order of σ. Then $V_D^{(r)}$ are V^0-submodules of V_D so that there is a partition $D=\bigsqcup_{i=0}^{|\sigma|-1} D^{(i)}$ such that $V_D^{(i)}=\oplus_{\alpha\in D^{(i)}}V^\alpha$. One can easily verify that $D^{(0)}$ is a subgroup of D and each $D^{(i)}$ is a coset of D with respect to $D^{(0)}$. Namely, if $V^0\subset V_D^{\langle\sigma\rangle}$, then σ is identified with an element of D^*, the dual group of D. Conversely, it is clear from definition that D^* is a subgroup of $\mathrm{Aut}(V_D)$. Thus, we have

Lemma 4.4.11. *An automorphism $\sigma\in\mathrm{Aut}(V)$ satisfies $V^0\subset V_D^{\langle\sigma\rangle}$ if and only if $\sigma\in D^*$.*

The lemma above tells us that an automorphism σ is consistent with the D-grading of V_D if and only if σ belongs to D^*. We consider σ-twisted V_D-modules. Let M be an indecomposable admissible σ-twisted V_D-module. By definition, there is a decomposition

$$M=\bigoplus_{i=0}^{|\sigma|-1} M^{(i)}$$

such that $V_D^{(i)}\cdot M^{(j)}\subset M^{(i+j)}$. It is obvious that each $M^{(i)}$ is a V^0-module. Let W be an irreducible V^0-submodule of $M^{(0)}$, and let D_W, $\mathcal{S}_W$, $A_\lambda(D,\mathcal{S}_W)$ and $\mathbb{C}^{\lambda_s}[D_W]$ be as in Section 4.3. By replacing M by $\sum_{\alpha\in D}V^\alpha\cdot W$ if necessary, we may assume that all $V^\alpha\boxtimes_{V^0}W$, $\alpha\in D_W$, are contained in $M^{(0)}$ so that D_W is a subgroup of $D^{(0)}$. Since M is a completely reducible V^0-module, we have the following decomposition:

$$M=\bigoplus_{s\in\mathcal{S}_W} W^s\otimes \mathrm{Hom}_{V^0}(W^s,M),$$

where we set $W^s:=V^s\boxtimes_{V^0}W$ for $s\in\mathcal{S}_W$ by abuse of notation. Set $U^s:=\mathrm{Hom}_{V^0}(W^s,M)$ for $s\in\mathcal{S}_W$. As we did before, we can find a 2-cocycle $\bar\lambda\in H^2(D,U(\mathbb{C}\mathcal{S}_W))$ and a representation of $A_\lambda(D,\mathcal{S}_W)$ on the space $\oplus_{s\in\mathcal{S}_W}U^s$. Thus, by the same argument, we can show the following.

Theorem 4.4.12. *Let $\sigma\in D^*(\subset\mathrm{Aut}(V_D))$. Viewing as a V^0-module, an indecomposable admissible σ-twisted V_D-module M has the shape*

$$M=\bigoplus_{s\in\mathcal{S}_W} W^s\otimes U^s$$

such that the space $\oplus_{s\in\mathcal{S}_W}U^s$ carries a structure of an $A_\lambda(D,\mathcal{S}_W)$-module. In particular, M is a completely reducible V_D-module. If M is irreducible under V_D, then each U^s, $s\in\mathcal{S}_W$, is irreducible under $\mathbb{C}[D_W]\otimes q(s)$, and also $\oplus_{s\in\mathcal{S}_W}U^s$ is irreducible under $A_\lambda(D,\mathcal{S}_W)$. Moreover, for each pair s and $t\in\mathcal{S}_W$, U^s and U^t are determined by the following rule:

$$U^t\simeq \mathrm{Red}^{A_\lambda(D,\mathcal{S}_W)}_{\mathbb{C}^{\lambda_t}[D_W]}\mathrm{Ind}^{A_\lambda(D,\mathcal{S}_W)}_{\mathbb{C}^{\lambda_s}[D_W]}U^s.$$

Hence, all W^s, $s\in\mathcal{S}_W$, have the same multiplicity in M.

Remark 4.4.13. Since $D_W \subset D^{(0)}$, we note that the decomposition above is a refinement of the decomposition $M = \oplus_{i\in\mathbb{Z}/|\sigma|\mathbb{Z}} M^{(i)}$.

By the theorem above, V_D is σ-rational for all $\sigma \in D^*$. More precisely, we can prove that V_D is σ-regular, that is, every σ-twisted V_D-module is completely reducible (cf. [Y2]).

Corollary 4.4.14. *An extension V_D is σ-regular for all $\sigma \in D^*$.*

Proof: Let M be a σ-twisted V_D-module. Take an irreducible V^0-submodule W of M, which is possible because V^0 is regular. Then $\sum_{\alpha\in D} V^\alpha \cdot W$ is a σ-twisted admissible V_D-submodule. As we have shown that V_D is σ-rational, $\sum_{\alpha\in D} V^\alpha \cdot W$ is a completely reducible V_D-module. Thus, M is a sum of irreducible V_D-submodules and hence M is a direct sum of irreducible V_D-submodules. ∎

4.5 Induced modules

Here we also keep the setup of Section 4.2 and 4.3 Let W be an irreducible V^0-module. We define the stabilizer D_W, the orbit space $\mathcal{S}_W$, intertwining operators $I_s^\alpha(\cdot, z)$, where $\alpha \in D$ and $s \in \mathcal{S}_W$, the twisted algebra $A_\lambda(D, \mathcal{S}_W)$ and the twisted group ring $\mathbb{C}^{\lambda_s}[D_W]$ as in Section 4.3. We set $W^s := V^s \boxtimes_{V^0} W$ for $s \in \mathcal{S}_W$, as we did previously. Let $h(s)$ be the top weight of a V^0-module W^s, which is a rational number by Theorem 3.6.6. It follows from definition that the powers of z in an intertwining operator $I_s^\alpha(\cdot, z)$ are contained in $h(\alpha+s) - h(s) + \mathbb{Z}$. We set $\chi(\alpha, s) := h(\alpha+s) - h(s) \in \mathbb{Q}$. The following assertion is crucial for us.

Lemma 4.5.1. *The following hold for any $\alpha, \beta \in D$ and $s \in \mathcal{S}_W$:*
(i) $\chi(\alpha, \beta+s) - \chi(\alpha, s) \in \mathbb{Z}$;   (ii) $\chi(\alpha, \beta+s) + \chi(\beta, s) - \chi(\alpha+\beta, s) \in \mathbb{Z}$.

Proof: By Theorem 3.7.5, we have:

$$\langle \nu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w^s\rangle = \epsilon_s(\alpha,\beta)\langle \nu, I_{s+\alpha}^\beta(x^\beta, z_2) I_s^\alpha(x^\alpha, z_1) w^s\rangle, \tag{4.5.1}$$

$$\langle \nu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w^s\rangle = \lambda_s(\alpha,\beta)\langle \nu, I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha, z_0)x^\beta, z_2) w^s\rangle|_{z_0=z_1-z_2}, \tag{4.5.2}$$

where $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$, $w^s \in W^s$, $\nu \in (W^{s+\alpha+\beta})^*$, $\epsilon_s(\alpha,\beta)$ is a suitable scalar in $\mathbb{C}^*$, and the equals above mean that the left hand side and the right hand side are analytic extensions of each other. Since all $I_s^\alpha(\cdot, z)$ are intertwining operators among modules involving simple currents, we note that by the convergence property in [H1] [H4] the right hand side of (4.5.2) has the form $z_2^{\chi(\alpha+\beta,s)+r} z_0^s f_1(z_0/z_2)|_{z_0=z_1-z_2}$ in the domain $|z_2| > |z_1 - z_2| > 0$, where r and s are some integers and $f_1(x)$ is an analytic function on $|x| < 1$. Therefore, we have

$$\begin{aligned}&(z_1-z_2)^N\langle \mu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w^2\rangle \\ &= \epsilon_s(\alpha,\beta)(z_1-z_2)^N\langle \mu, I_{s+\beta}^\alpha(x^\beta, z_2) I_s^\beta(x^\alpha, z_2) w^2\rangle\end{aligned} \tag{4.5.3}$$

in the domain $|z_1| > |z_1 - z_2| \geq 0$, $|z_2| > |z_1 - z_2| \geq 0$ for sufficiently large N. Since $z^{-\chi(\alpha,s)} I_s^\alpha(x^\alpha, z) w^s$ contains only integral powers of z, both

$$z_1^{-\chi(\alpha,\beta+s)} z_2^{-\chi(\beta,s)} \langle \nu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w^s \rangle$$

and

$$z_1^{-\chi(\alpha,s)} z_2^{-\chi(\beta,s+\alpha)} \langle \nu, I_{s+\alpha}^\alpha(x^\beta, z_2) I_{s+\alpha}^\beta(x^\beta, z_2) w^s \rangle$$

contain only integral powers of z_1 and z_2. Thus, by (4.5.1), (4.5.3) and the convergence property in [H1] [H4], we obtain the following equality of the meromorphic functions:

$$\begin{aligned}
&(z_1 - z_2)^N \, \iota_{12}^{-1} z_1^{-\chi(\alpha,s+\beta)} z_2^{-\chi(\beta,s)} \langle \nu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w^s \rangle \\
&= (z_1 - z_2)^N \epsilon_s(\alpha, \beta) \iota_{21}^{-1} z_1^{-\chi(\alpha,s)} z_2^{-\chi(\beta,s+\alpha)} \cdot z_1^{\chi(\alpha,s)-\chi(\alpha,s+\beta)} z_2^{\chi(\beta,s+\alpha)-\chi(\beta,s)} \\
&\qquad \times \langle \nu, I_{s+\alpha}^\beta(x^\beta, z_2) I_s^\alpha(x^\alpha, z_1) w^s \rangle.
\end{aligned}$$

Since the equality above holds for any choices of $\log z_1$ and $\log z_2$ in the definitions of $z_1^r = e^{r \log z_1}$ and $z_2^r = e^{r \log z_2}$ (cf. [H1] [H4]), we have $\chi(\alpha, s) - \chi(\alpha, s + \beta) \in \mathbb{Z}$ and $\chi(\beta, s + \alpha) - \chi(\beta, s) \in \mathbb{Z}$. This proves (i). The proof of (ii) is similar. By (4.5.2) and the convergence property in [H1] [H4], we obtain the following equality of the meromorphic functions:

$$\begin{aligned}
&\lambda_s(\alpha, \beta)^{-1} \iota_{12}^{-1} z_1^{-\chi(\alpha,s+\beta)} z_2^{-\chi(\beta,s)} \langle \nu, I_{s+\beta}^\alpha(x^\alpha, z_1) I_s^\beta(x^\beta, z_2) w^s \rangle \\
&= \iota_{20}^{-1} z_2^{-\chi(\alpha+\beta,s)} \langle \nu, I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha, z_0) x^\beta, z_2) w^s \rangle \cdot (z_2 + z_0)^{-\chi(\alpha,s+\beta)} z_2^{\chi(\alpha+\beta,s)-\chi(\beta,s)} |_{z_0 = z_1 - z_2}.
\end{aligned}$$

Again the equality holds for any choices of $\log(z_1 - z_2)$ and $\log z_2$ in the definitions of $(z_1 - z_2)^r = e^{r \log(z_1 - z_2)}$ and $z_2^r = e^{r \log z_2}$ (cf. [H1] [H4]). Since $(z_2 + z_0)^r = z_2^r (1 + z_0/z_2)^r$ and $(1 + x)^r = \sum_{i \geq 0} \binom{r}{i} x^i$ is analytic in the domain $|x| < 1$, we see that $\chi(\alpha + \beta, s) - \chi(\alpha, s + \beta) - \chi(\beta, s) \in \mathbb{Z}$. This completes the proof of (ii). ∎

By the lemma above, we find that $\chi(\alpha, s) + \mathbb{Z}$ is independent of $s \in \mathcal{S}_W$. So we may set $\chi(\alpha) := \chi(\alpha, s)$ for $\alpha \in D$. Then by (ii) of Lemma 4.5.1 we find that $\chi(\cdot)$ satisfies the homomorphism condition $\chi(\alpha + \beta) + \mathbb{Z} = \chi(\alpha) + \chi(\beta) + \mathbb{Z}$. Since $\chi(\alpha) \in \mathbb{Q}$, there exists an $n \in \mathbb{N}$ such that χ defines a group homomorphism from D/D_W to $\mathbb{Z}/n\mathbb{Z} = \{j/n + \mathbb{Z} \mid 0 \leq j \leq n - 1\}$. It is clear that χ naturally defines an element $\hat{\chi}$ of D^* by $\hat{\chi}(\alpha) := e^{-2\pi\sqrt{-1}\,\chi(\alpha)}$. Thus χ gives rise to an element of $\mathrm{Aut}(V_D)$. In the following, we will construct irreducible $\hat{\chi}$-twisted V_D-modules which contain W as V^0-submodules.

Take an $s \in S_W$. Let φ be an irreducible representation of $\mathbb{C}^{\lambda_s}[D_W]$ on a space U. For each $t \in S_W$, set

$$U^t := \mathrm{Red}_{\mathbb{C}^{\lambda_t}[D_W]}^{A_\lambda(D,S_W)} \mathrm{Ind}_{\mathbb{C}^{\lambda_s}[D_W]}^{A_\lambda(D,S_W)} U.$$

Then each U^t, $t \in \mathcal{S}_W$, is a $\mathbb{C}^{\lambda_t}[D_W]$-module and a direct sum $\oplus_{s \in \mathcal{S}_W} U^s$ naturally (and uniquely) carries a structure of an irreducible $A_\lambda(D, S_W)$-module. Set

$$\mathrm{Ind}_{V^0}^{V_D}(W, \varphi) := \bigoplus_{s \in S_W} W^s \otimes U^s$$

and define the vertex operator $\hat{Y}(\cdot,z)$ of V_D on $\mathrm{Ind}_{V^0}^{V_D}(W,\varphi)$ by

$$\hat{Y}(x^\alpha,z)w^t\otimes\mu^t := I_t^\alpha(x^\alpha,z)w^t\otimes\{e^\alpha\otimes q(t)\cdot\mu^t\}$$

for $x^\alpha\in V^\alpha$ and $w^t\otimes\mu^t\in W^t\otimes U^t$. We prove

Theorem 4.5.2. $(\mathrm{Ind}_{V^0}^{V_D}(W,\varphi),\hat{Y}(\cdot,z))$ *is an irreducible* $\hat{\chi}$*-twisted* V_D*-module.*

Proof: Since the powers of z in $\hat{Y}(x^\alpha,z)$ are contained in $\chi(\alpha)+\mathbb{Z}$, we only need to show the commutativity and the $\hat{\chi}$-twisted associativity of vertex operators. We use a technique of generalized rational functions developed in [DL]. Let $x^\alpha\in V^\alpha$, $x^\beta\in V^\beta$, $w^s\otimes\mu^s\in W^s\otimes U^s$ and $\nu\in\mathrm{Ind}_{V^0}^{V_D}(W,\varphi)^*$. We note that $z^{-\chi(\alpha)}I_s^\alpha(x^\alpha,z)w^s\in W^{s+\alpha}((z))$. For sufficiently large $N\in\mathbb{N}$, we have

$$\begin{aligned}
&(z_1-z_2)^N\iota_{12}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle\nu,\hat{Y}(x^\alpha,z_1)\hat{Y}(x^\beta,z_2)w^s\otimes\mu^s\rangle\\
&=(z_1-z_2)^N\iota_{12}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle\nu,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\\
&\qquad\otimes\{e^\alpha\otimes q(s+\beta)\cdot(e^\beta\otimes q(s)\cdot\mu^s)\}\rangle\\
&=(z_1-z_2)^N\iota_{123}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle\nu,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)I_s^0(\mathbb{1},z_3)w^s\\
&\qquad\otimes\{\lambda_s(\alpha,\beta)^{-1}e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle\\
&=\iota_{345}^{-1}(z_3+z_4)^{-\chi(\alpha)}(z_3+z_5)^{-\chi(\beta)}\\
&\qquad\cdot\langle\nu,\lambda_s(\alpha,\beta)I_s^{\alpha+\beta}((z_4-z_5)^NY_{V_D}(x^\alpha,z_4)Y_{V_D}(x^\beta,z_5)\mathbb{1},z_3)w^s\\
&\qquad\otimes\{\lambda_s(\alpha,\beta)^{-1}e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle|_{z_4=z_1-z_3,\,z_5=z_2-z_3}\\
&=\iota_{354}^{-1}(z_3+z_4)^{-\chi(\alpha)}(z_3+z_5)^{-\chi(\beta)}\langle\nu,I_s^{\alpha+\beta}((z_4-z_5)^NY_{V_D}(x^\beta,z_5)Y_{V_D}(x^\alpha,z_4)\mathbb{1},z_3)w^s\\
&\qquad\otimes\{e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle|_{z_4=z_1-z_3,\,z_5=z_2-z_3}\\
&=(z_1-z_2)^N\iota_{123}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle\nu,\lambda_s(\beta,\alpha)^{-1}I_{s+\alpha}^\beta(x^\beta,z_2)I_s^\alpha(x^\alpha,z_1)I_s^0(\mathbb{1},z_3)w^s\\
&\qquad\otimes\{(\lambda_s(\beta,\alpha)e^\beta\otimes q(s+\alpha)*e^\alpha\otimes q(s))\cdot\mu^s\}\rangle\\
&=(z_1-z_2)^N\iota_{12}^{-1}z_1^{-\chi(\alpha)}z_2^{-\chi(\beta)}\langle\nu,\hat{Y}(x^\beta,z_2)\hat{Y}(x^\alpha,z_1)w^s\otimes\mu^s\rangle.
\end{aligned}$$

Therefore, we get the commutativity. Similarly, we have

$$\begin{aligned}
&\iota_{12}^{-1}z_1^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}\langle\nu,\hat{Y}(x^\alpha,z_1)\hat{Y}(x^\beta,z_2)w^s\otimes\mu^s\rangle\\
&=\iota_{12}^{-1}z_1^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}\langle\nu,I_{s+\beta}^\alpha(x^\alpha,z_1)I_s^\beta(x^\beta,z_2)w^s\otimes\{e^\alpha\otimes q(s+\beta)\cdot(e^\beta\otimes q(s)\cdot\mu^s)\}\rangle\\
&=\iota_{20}^{-1}\langle\nu,\lambda_s(\alpha,\beta)(z_2+z_0)^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}I_s^{\alpha+\beta}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w^s\\
&\qquad\otimes\{\lambda_s(\alpha,\beta)^{-1}e^{\alpha+\beta}\otimes q(s)\cdot\mu^s\}\rangle|_{z_0=z_1-z_2}\\
&=\iota_{20}^{-1}\langle\nu,(z_2+z_0)^{-\chi(\alpha)+N}z_2^{-\chi(\beta)}\hat{Y}(Y_{V_D}(x^\alpha,z_0)x^\beta,z_2)w^s\otimes\mu^s\rangle|_{z_0=z_1-z_2}.
\end{aligned}$$

Hence, we obtain the associativity. ∎

Suppose that a simple VOA V and a finite group G acting on V is given. Then the G-invariants V^G of V, called the G-orbifold of V, is also a simple VOA by [DM1]. It is an important problem to classify the module category of V^G in the orbifold conformal field theory. It was conjectured in [DVVV] that every irreducible V^G-module appears in a g-twisted V-module for some $g \in G$. In our case, V^0 is exactly the D^*-invariants of the extension V_D. By Theorem 4.5.2, we see that the conjecture is true for a pair (V_D, D^*).

Theorem 4.5.3. *Let V^0 be a rational, C_2-cofinite and CFT-type VOA, and D a finite abelian group. Assume that $V_D = \oplus_{\alpha\in D} V^\alpha$ be a D-graded simple current extension of V^0. Then every irreducible V^0-module W is contained in an irreducible σ-twisted V_D-module for some $\sigma \in D^*$. Moreover, σ is uniquely determined by W.*

4.6 Extension property

At the last of this chapter, we present a useful theorem by which we can make a simple current extension larger.

Theorem 4.6.1. *(The extension property) Let $V^{(0,0)}$ be a simple rational C_2-cofinite VOA of CFT-type, and let D_1, D_2 be finite abelian groups. Assume that we have a set of inequivalent irreducible simple current $V^{(0,0)}$-modules $\{V^{(\alpha,\beta)} \mid (\alpha,\beta) \in D_1 \oplus D_2\}$ with $D_1 \oplus D_2$-graded fusion rules $V^{(\alpha_1,\beta_1)} \boxtimes_{V^{(0,0)}} V^{(\alpha_2,\beta_2)} = V^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}$ for any $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in D_1 \oplus D_2$. Further assume that all $V^{(\alpha,\beta)}$, $(\alpha,\beta) \in D_1 \oplus D_2$, have integral top weights and we have D_1- and D_2-graded simple current extensions $V_{D_1} = \oplus_{\alpha\in D_1} V^{(\alpha,0)}$ and $V_{D_2} = \oplus_{\beta\in D_2} V^{(0,\beta)}$. Then $V_{D_1\oplus D_2} := \oplus_{(\alpha,\beta)\in D_1\oplus D_2} V^{(\alpha,\beta)}$ possesses a unique structure of a simple vertex operator algebra as a $D_1 \oplus D_2$-graded simple current extension of $V^{(0,0)}$.*

Remark 4.6.2. If $D_2 = \mathbb{Z}_2 = \{0, 1\}$ and the $\mathbb{Z}_2$-graded space $V_{D_2} = V^{(0,0)} \oplus V^{(0,1)}$ is a simple vertex operator superalgebra, then the following proof with suitable modifications shows that $V_{D_1\oplus D_2} = \oplus_{(\alpha,\beta)\in D_1\oplus D_2} V^{(\alpha,\beta)}$ is a simple vertex operator superalgebra with even part $\oplus_{\alpha\in D_1} V^{(\alpha,0)}$ and odd part $\oplus_{\beta\in D_1} V^{(\beta,1)}$.

Proof: First, we note that $V_{D_1\oplus\beta} := \mathrm{Ind}_{V^{(0,0)}}^{V_{D_1}} V^{(0,\beta)} = \oplus_{\alpha\in D_1} V^{(\alpha,\beta)}$ has a unique irreducible V_{D_1}-module structure for any $\beta \in D_2$ by Theorem 4.5.2.

Claim. $V_{D_1\oplus\beta} \boxtimes_{V_{D_1}} V_{D_1\oplus\gamma} = V_{D_1\oplus(\beta+\gamma)}$ for all $\beta,\gamma \in D_2$.

Proof of Claim. By the assumption that V_{D_2} is a simple vertex operator algebra, the restriction of vertex operator map $Y_{V_{D_2}}(\cdot, z)$ on $V^{(0,\beta)} \otimes V^{(0,\gamma)}$ gives a $V^{(0,0)}$-intertwining operator of type $V^{(0,\beta)} \times V^{(0,\gamma)} \to V^{(0,\beta+\gamma)}$. Since all of $V_{D_1\oplus\beta}$, $V_{D_1\oplus\gamma}$ and $V_{D_1\oplus(\beta+\gamma)}$ are D_1-stable irreducible V_{D_1}-modules, we can use Theorem 4.4.9 and hence obtain the V_{D_1}-intertwining operators of type $V_{D_1\oplus\beta} \times V_{D_1\oplus\gamma} \to V_{D_1\oplus(\beta+\gamma)}$ which is the lifting of the vertex operator maps on V_{D_2}. This completes the proof of Claim.

By the claim above, we can take the non-trivial V_{D_1}-intertwining operator $I_{\beta,\gamma}(\cdot,z)$ of type $V_{D_1\oplus\beta}\times V_{D_1\oplus\gamma}\to V_{D_1\oplus(\beta+\gamma)}$ which is the lifting of the vertex operator map on V_{D_2} for each $\beta,\gamma\in D_2$. By its construction it is normalized such that

$$I_{0,\gamma}(\mathbb{1},z)x^\gamma = x^\gamma,\quad I_{\beta,0}(x^\beta,z)x^0 = e^{zL(-1)}I_{0,\beta}(x^0,z)x^\beta \tag{4.6.1}$$

for all $\beta,\gamma\in D_2$, $x^\beta\in V_{D_1\oplus\beta}$ and $x^\gamma\in V_{D_1\oplus\gamma}$. Now define a vertex operator map $\tilde{Y}(\cdot,z): V_{D_1\oplus D_2}\times V_{D_1\oplus D_2}\to V_{D_1\oplus D_2}$ by

$$\tilde{Y}(x^\alpha,z)x^\beta := I_{\alpha,\beta}(x^\alpha,z)x^\beta$$

for $x^\alpha\in V_{D_1\oplus\alpha}$, $x^\beta\in V_{D_1\oplus\beta}$. We claim that $\tilde{Y}(\cdot,z)$ defines a vertex operator algebra structure on $V_{D_1\oplus D_2}$. By our normalization (4.6.1), $\tilde{Y}(\cdot,z)$ satisfies the axioms for the vacuum vector, and since $I_{\alpha,\beta}(\cdot,z)$ satisfies the $L(-1)$-derivation property, $\tilde{Y}(\cdot,z)$ also satisfies the $L(-1)$-derivation property. Therefore, we only need to establish the mutually commutativity for $\tilde{Y}(\cdot,z)$. In the following, we make it a rule that $x^{(\alpha,\beta)}$ always denotes an arbitrary element in $V^{(\alpha,\beta)}$ for any $(\alpha,\beta)\in D_1\oplus D_2$. By Theorem 3.7.5 (cf. [H1] [H4]), there exists a non-zero scalar $\lambda_{\alpha,\beta,\gamma,\delta}$ such that

$$(z_1-z_2)^N\tilde{Y}(x^{(0,\beta)},z_1)\tilde{Y}(x^{(0,\gamma)},z_2)x^{(\alpha,\delta)} = \lambda_{\alpha,\beta,\gamma,\delta}(z_1-z_2)^N\tilde{Y}(x^{(0,\gamma)},z_2)\tilde{Y}(x^{(0,\beta)},z_1)x^{(\alpha,\delta)}.$$

On the other hand, since V_{D_2} forms a vertex operator algebra, $\tilde{Y}(x^{(0,\beta)},z)$ and $\tilde{Y}(x^{(0,\gamma)},z)$ are mutually local fields on V_{D_2}. Since $\tilde{Y}(x^{(\alpha,0)},z)$ and $\tilde{Y}(x^{(0,\beta)},z)$ are mutually local fields on $V_{D_1\oplus D_2}$, there is a positive integer $N\gg 0$ such that

$$\begin{aligned}
&(z_1-z_2)^N(z_2-z_3)^N(z_3-z_1)^N\tilde{Y}(x^{(0,\beta)},z_1)\tilde{Y}(x^{(0,\gamma)},z_2)\tilde{Y}(x^{(\alpha,0)},z_3)x^{(0,\delta)}\\
&= (z_1-z_2)^N(z_2-z_3)^N(z_3-z_1)^N\tilde{Y}(x^{(\alpha,0)},z_3)\tilde{Y}(x^{(0,\beta)},z_1)\tilde{Y}(x^{(0,\gamma)},z_2)x^{(0,\delta)}\\
&= (z_1-z_2)^N(z_2-z_3)^N(z_3-z_1)^N\tilde{Y}(x^{(\alpha,0)},z_3)\tilde{Y}(x^{(0,\gamma)},z_2)\tilde{Y}(x^{(0,\beta)},z_1)x^{(0,\delta)}\\
&= (z_1-z_2)^N(z_2-z_3)^N(z_3-z_1)^N\tilde{Y}(x^{(0,\gamma)},z_2)\tilde{Y}(x^{(0,\beta)},z_1)\tilde{Y}(x^{(\alpha,0)},z_3)x^{(0,\delta)}.
\end{aligned}$$

Therefore, $\lambda_{\alpha,\beta,\gamma,\delta}=1^*$ and hence $\tilde{Y}(x^{(0,\beta)},z)$ and $\tilde{Y}(x^{(0,\gamma)},z)$ are mutually local fields on $V_{D_1\oplus D_2}$. Now recall the n-th normal ordered product $\circ_n$ defined in Section 3.2. By our definition of $\tilde{Y}(\cdot,z)$, we have

$$\tilde{Y}(x^{(\alpha,0)}_{(n)}x^{(0,\beta)},z) = \tilde{Y}(x^{(\alpha,0)},z)\circ_n\tilde{Y}(x^{(0,\beta)},z).$$

Then by Dong's lemma, all $\tilde{Y}(x^{(\alpha,\beta)},z)$, $\alpha\in D_1$, $\beta\in D_2$, are mutually local fields on $V_{D_1\oplus D_2}$. Therefore, $(V_{D_1\oplus D_2},\tilde{Y}(\cdot,z))$ carries a structure of a vertex operator algebra. It is clear that $V_{D_1\oplus D_2}$ is simple. Hence, $V_{D_1\oplus D_2}$ is a $D_1\oplus D_2$-graded simple current extension of $V^{(0,0)}$. ∎

*In the case where $D_2=\mathbb{Z}_2$ and V_{D_2} is an SVOA as in Remark 4.6.2, the scalar $\lambda_{\alpha,1,1,\delta}$ will be -1.

Chapter 5

Examples of Simple Current Extensions

We present examples of simple current extensions.

5.1 A theory of semisimple primary vectors

In this section we review a theory of semisimple primary vectors introduced by Li [Li0] [Li4] [Li5]. Semisimple primary vectors have nice properties so that there are many applications, for example, simple current extensions of vertex operator algebra [Li0] [Li5] [Li8] [DLM5], abelian coset construction [Li9] and theta functions defined on vertex operator algebras [Y2].

Definition 5.1.1. A vector $h \in V$ is called a *semisimple primary vector* if it satisfies the followings.

(i) $L(n)h = \delta_{n,0}h$ for $n \geq 0$,

(ii) $h_{(n)}h = \delta_{n,1}\gamma\mathbb{1}$ for $n \geq 0$ with $\gamma \in \mathbb{Q}$,

(iii) $h_{(0)}$ acts on V semisimply.

In addition, if $h_{(0)}$ acts on V with integral eigenvalues, then h is called *integral* and if $h_{(0)}$ acts on V with rational eigenvalues, then h is called *rational*.

For a semisimple primary vector h, we can associate the *internal automorphism* $\sigma(h) := e^{-2\pi\sqrt{-1}\,h_{(0)}}$. Since $h_{(0)}$ is a derivative operator, i.e., $[h_{(0)}, a_{(n)}] = (h_{(0)}a)_{(n)}$ for any $a \in V$, $n \in \mathbb{Z}$, $\sigma(h)$ defines an element in $\mathrm{Aut}(V)$. The automorphism $\sigma(h)$ is identical if and only if h is integral. Using semisimple primary vectors, we can transform structures of a module. Define

$$E^{\pm}(h,z) := \exp\left(\sum_{n=1}^{\infty} \frac{h_{(\pm n)}}{n} z^{\mp n}\right), \tag{5.1.1}$$

$$\Delta(h,z) := z^{h_{(0)}} E^{+}(-h,-z). \tag{5.1.2}$$

Lemma 5.1.2. *(Lemma 2.15, 3.7, 3.8 in [Li0]) Let W^i, $i=1,2,3$, be V-modules on which $h_{(0)}$ acts semisimply and let $I(\cdot,z)$ be an intertwining operator of type $W^1 \times W^2 \to W^3$. Then the following identities hold:*

$$\begin{aligned}
&\Delta(h,z_2)I(u^1,z_0) = I(\Delta(h,z_2+z_0)u^1,z_0)\Delta(h,z_2),\\
&I(E^-(h,z_1)u,z_2) = E^-(h,z_1+z_2)E^-(-h,z_2)I(u,z_2)\\
&\qquad\qquad\qquad \times z_2^{-h_{(0)}}E^+(h,z_2)(z_2+z_1)^{h_{(0)}}E^+(-h,z_2+z_1),\\
&E^-(-h,z_1)I(u,z_2)E^-(h,z_1) = I(\Delta(h,z_2-z_1)\Delta(-h,z_2)u,z_2).
\end{aligned}$$

Proposition 5.1.3. *([Li0] [Li2]) Let h be a semisimple primary vector and $(M, Y_M(\cdot,z))$ a V-module. Then $(M, Y_M(\Delta(h,z)\cdot,z))$ is a $\sigma(h)$-twisted V-module. Moreover, it is irreducible if and only if M is irreducible. We will denote it simply by $\tilde{M}$.*

By the proposition above, there exist a canonical linear isomorphism $\phi : M \to \tilde{M}$ such that $Y_M(a,z)\phi = \phi Y_{\tilde{M}}(\Delta(h,z)a,z)$ for all $a \in V$.

Proposition 5.1.4. *Let h be an integral semisimple primary vector. Let W^i, $i=1,2,3$, be irreducible V-modules on which $h_{(0)}$ acts semisimply and $I(\cdot,z)$ an intertwining operator of type $W^1\times W^2 \to W^3$. Take canonical linear isomorphisms $\phi_i : W^i \to \tilde{W^i}$, $i=2,3$, such that $Y_{W^i}(a,z)\phi_i = \phi Y_{\tilde{W^i}}(\Delta(h,z)a,z)$ for all $a \in V$. Then $\phi_3 I(\Delta(h,z)\cdot,z)\phi_2^{-1}$ provides an intertwining operator of type $W^1 \times \tilde{W}^2 \to \tilde{W}^3$.*

Since $\Delta(h,z)\Delta(-h,z) = \Delta(0,z) = 1$, the Delta operator $\Delta(h,z)$ defines a linear isomorphism between $\binom{W^3}{W^1\ W^2}_V$ and $\binom{\tilde{W^3}}{W^1\ \tilde{W^2}}_V$. By abuse of notation, we denote such an isomorphism by $\Delta(h,z)$. As a corollary of Proposition 5.1.4, We note that $\tilde{V}$ gives a simple current V-module.

Corollary 5.1.5. *Assume that h is an integral semisimple primary vector. Then $\tilde{V} = (V, Y_V(\Delta(h,z)\cdot,z))$ is a simple current V-module.*

Proof: It is shown in Lemma 2.4.3 that $V \times M = M$ for all V-module M. Then by Proposition 5.1.4 we have $\tilde{V} \times M = \tilde{M}$. ∎

It is possible to construct a simple current extension of V by $\tilde{V}$ under the assumption below.

Assumption 5.1.6. We assume that $\tilde{\tilde{V}}$ as a V-module is isomorphic to V and the constant γ (which is determined by $h_{(1)}h = \gamma\mathbb{1}$) is an integer.

Let $I(\cdot, z)$ be an intertwining operator of type $W^1 \times W^2 \to W^3$ and $u^i \in W^i$, $i = 1, 2$. Recall the transpose operator ${}^tI(u^2, z)u^1 := e^{zL(-1)}I(u^1, e^{\pi\sqrt{-1}}z)u^2$ which is an intertwining operator of type $W^2 \times W^1 \to W^3$.

Remark 5.1.7. There are two ways to define the transpose of $I(\cdot, z)$ according to the choice of square roots of unity, namely $e^{zL_{-1}}I(u^2, e^{\pm\pi\sqrt{-1}}z)u^1$. However, if all W^i, $i = 1, 2, 3$, are irreducible, then these two intertwining operators differ from the others only by scalar multiples. Thus there is no essential difference between them.

Let $\pi : V \to \tilde{\tilde{V}}$ be a V-isomorphism. Then we can obtain (non-zero) intertwining operators of type $Y^{10}(\cdot, z) : \tilde{V} \times V \to \tilde{V}$ and $Y^{11}(\cdot, z) : \tilde{V} \times \tilde{V} \to V$ by the following symmetries.

$$\binom{V}{V\ V} \ni Y_V(\cdot, z) \xrightarrow{\Delta(h,z)} \binom{\tilde{V}}{V\ \tilde{V}} \xrightarrow{\text{transpose}} Y^{10}(\cdot, z) \in \binom{\tilde{V}}{\tilde{V}\ V}$$
$$\xrightarrow{\Delta(h,z)} \binom{\tilde{\tilde{V}}}{\tilde{V}\ \tilde{V}} \xrightarrow{\pi^{-1}} Y^{11}(\cdot, z) \in \binom{V}{\tilde{V}\ \tilde{V}}.$$

Set $Y^{00}(\cdot, z) = Y_V(\cdot, z) \in \binom{V}{V\ V}$, $Y^{01}(\cdot, z) = Y_{\tilde{V}}(\cdot, z) \in \binom{\tilde{V}}{V\ \tilde{V}}$, $V^0 := V$, and $V^1 := \tilde{V}$. Define a vertex operator $\bar{Y}(\cdot, z)$ on $V^0 \oplus V^1$ by $\bar{Y}(a, z)b := Y^{ij}(a, z)b$ for $a \in V^i$ and $b \in V^j$. Then $(V^0 \oplus V^1, \bar{Y}(\cdot, z), \mathbb{1}, \omega)$ carries a structure of a VOA if γ is even and an SVOA if γ is odd.

Theorem 5.1.8. *(Theorem 3.9 in [Li0]) $(V \oplus \tilde{V}, \bar{Y}(\cdot, z), \mathbb{1}, \omega)$ is a VOA if γ is even and is an SVOA if γ is odd.*

The assumption $\tilde{\tilde{V}} \simeq V$ always induces a V-isomorphism $\tilde{\tilde{W}} \simeq W$ for every irreducible V-module W.

Lemma 5.1.9. *Let W be an irreducible V-module. Then $\tilde{\tilde{W}}$ is isomorphic to W as a V-module. Furthermore, the operator $h_{(0)}$ acts on each irreducible V-module semisimply with eigenvalues in either $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$.*

Proof: By the following isomorphisms of the space of V-intertwining operators, we obtain a non-trivial V-intertwining operator $J(\cdot, z)$ of type $V \times W \to \tilde{\tilde{W}}$.

$$\binom{W}{V\ W} \simeq \binom{W}{W\ V} \simeq \binom{\tilde{\tilde{W}}}{W\ \tilde{\tilde{V}}} \simeq \binom{\tilde{\tilde{W}}}{W\ V} \simeq \binom{\tilde{\tilde{W}}}{V\ W}.$$

Then one can verify that $J(\mathbb{1}, z)$ gives a non-trivial V-isomorphism between W and $\tilde{\tilde{W}}$. Hence, W and $\tilde{\tilde{W}}$ are isomorphic if W is irreducible. Now assume that W is irreducible.

Since the operator $h_{(0)}$ preserves each $L(0)$-weight subspace of W, there is an eigen vector $0 \neq w \in W$ such that $h_{(0)}w = \alpha w$ with $\alpha \in \mathbb{C}$. Since $h_{(0)}$ acts on V semisimply with integral eigenvalues, it acts on $W = V \cdot w$ semisimply with eigenvalues in $\alpha + \mathbb{Z}$. Take the canonical linear isomorphism $\phi : W \to \tilde{\tilde{W}}$ such that $Y_{\tilde{\tilde{W}}}(a,z)\phi = \phi Y_W(\Delta(h,z)a,z)$. By a direct calculation, we find that $L(0)\phi = \phi(L(0) + 2h_{(0)} + 2\gamma)$. Since W and $\tilde{\tilde{W}}$ are isomorphic, they have the same top weight. Hence, $2\alpha + 2\gamma \in \mathbb{Z}$ and we reach $\alpha \in \frac{1}{2}\mathbb{Z}$. ∎

By the lemma above, $W \oplus \tilde{W}$ is invariant under $V \oplus \tilde{V}$ in the fusion algebra for V. So it is natural for us to expect that we can find a $V \oplus \tilde{V}$-module structure in $W \oplus \tilde{W}$. Let $\pi_W : W \to \tilde{\tilde{W}}$ be a V-isomorphism. Define V-intertwining operators as follows:

$$\begin{aligned} Y^{00}(a,z)w &:= Y_W(a,z)w \qquad \text{for } a \in V, w \in W, \\ Y^{01}(a,z)\tilde{w} &:= Y_{\tilde{W}}(a,z)\tilde{w} \qquad \text{for } a \in V, \tilde{w} \in \tilde{W}. \end{aligned}$$

Then define V-intertwining operators $Y^{10}(\cdot,z) \in \binom{\tilde{W}}{\tilde{V}\ W}$ and $Y^{11}(\cdot,z) \in \binom{W}{\tilde{V}\ \tilde{W}}$ by the following isomorphisms:

$$\binom{W}{V\ W} \ni Y^{00}(\cdot,z) \xrightarrow{\text{transpose}} \binom{W}{W\ V} \xrightarrow{\Delta(h,z)} \binom{\tilde{W}}{W\ \tilde{V}} \xrightarrow{\text{transpose}} Y^{10}(\cdot,z) \in \binom{\tilde{W}}{\tilde{V}\ W}$$
$$\xrightarrow{\Delta(h,z)} \binom{\tilde{\tilde{W}}}{\tilde{V}\ \tilde{W}} \xrightarrow{\pi_W^{-1}} Y^{11}(\cdot,z) \in \binom{W}{\tilde{V}\ \tilde{W}}.$$

Then introduce a vertex operator $\bar{Y}(\cdot,z)$ on $W \oplus \tilde{W}$ by $\bar{Y}(\cdot,z) := (Y^{00} \oplus Y^{01} \oplus Y^{10} \oplus Y^{11})(\cdot,z)$.

Theorem 5.1.10. *(Theorem 3.13 in [Li0]) Let W be an irreducible V-module. Let $w \in W$ be a vector such that $h_{(0)}w = \alpha w$ with $\alpha \in \frac{1}{2}\mathbb{Z}$. Then*

(i) If γ is even and $\alpha \in \mathbb{Z}$, then $(W \oplus \tilde{W}, \bar{Y}(\cdot,z))$ is an untwisted $V \oplus \tilde{V}$-module.

(ii) If γ is even and $\alpha \in \frac{1}{2} + \mathbb{Z}$, then $(W \oplus \tilde{W}, \bar{Y}(\cdot,z))$ is a $\mathbb{Z}_2$-twisted $V \oplus \tilde{V}$-module.

(iii) If γ is odd and $\alpha \in \mathbb{Z}$, then $(W \oplus \tilde{W}, \bar{Y}(\cdot,z))$ is a $\mathbb{Z}_2$-graded $V \oplus \tilde{V}$-module.

(iv) If γ is odd and $\alpha \in \frac{1}{2} + \mathbb{Z}$, then $(W \oplus \tilde{W}, \bar{Y}(\cdot,z))$ is a $\mathbb{Z}_2$-twisted $V \oplus \tilde{V}$-module.

Remark 5.1.11. By the theorem above, we note that in the study of simple current extensions by a simple module, $\mathbb{Z}_2$-twisted modules for vertex operator superalgebras naturally appear.

Lifting of intertwining operators. Let W^i, $i = 1,2,3$, be irreducible V-modules on which $h_{(0)}$ acts with eigenvalues in $\alpha_i + \mathbb{Z}$, $\alpha_i \in \frac{1}{2}\mathbb{Z}$, respectively. Let $\phi_0 : V \to \tilde{V}$, $\phi_i : W^i \to \tilde{W}^i$ and $\phi_i' : \tilde{W}^i \to \tilde{\tilde{W}}^i$ be canonical linear isomorphisms such that $Y_{\tilde{V}}(a,z)\phi_0 =$

$\phi_0 Y_V(\Delta(h,z)a,z)$, $Y_{\tilde{W}^i}(a,z)\phi_i = \phi_i Y_{W^i}(\Delta(h,z)a,z)$ and $Y_{\tilde{\tilde{W}}^i}(a,z)\phi_i' = \phi_i' Y_{\tilde{W}^i}(\Delta(h,z)a,z)$ for all $a \in V$. Let $\pi_i : W^i \to \tilde{\tilde{W}}^i$ be V-isomorphisms. Note that π_i are not determined uniquely since non-zero scalar multiplications are allowed. We will choose suitable ones later. Let $I^{00}(\cdot,z)$ be a non-zero V-intertwining operator of type $W^1 \times W^2 \to W^3$. We assume that $\dim \binom{W^3}{W^1\ W^2}_V = 1$. We can get $I^{01}(\cdot,z) \in \binom{\tilde{W}^3}{W^1\ \tilde{W}^2}$ by $\Delta(h,z) : \binom{W^3}{W^1\ W^2} \ni I^{00}(\cdot,z) \mapsto I^{01}(\cdot,z) \in \binom{\tilde{W}^3}{W^1\ \tilde{W}^2}$. On the other hand, we obtain

$$\binom{W^3}{W^1\ W^2} \ni I^{00}(\cdot,z) \xrightarrow{\text{transpose}} \binom{W^3}{W^2\ W^1} \xrightarrow{\Delta(h,z)} \binom{\tilde{W}^3}{W^2\ \tilde{W}^1} \xrightarrow{\text{transpose}}$$
$$\to I^{10}(\cdot,z) \in \binom{\tilde{W}^3}{\tilde{W}^1\ W^2} \xrightarrow{\Delta(h,z)} \binom{\tilde{\tilde{W}}^3}{\tilde{W}^1\ \tilde{W}^2} \xrightarrow{\pi_3^{-1}} I^{11}(\cdot,z) \in \binom{W^3}{\tilde{W}^1\ \tilde{W}^2}$$

We claim that a linear operator

$$\bar{I}(\cdot,z) := \left(I^{00} \oplus I^{01} \oplus I^{10} \oplus I^{11}\right)(\cdot,z) : \left(W^1 \oplus \tilde{W}^1\right) \otimes \left(W^2 \oplus \tilde{W}^2\right) \to \left(W^3 \oplus \tilde{W}^3\right)$$

is a $V \oplus \tilde{V}$-intertwining operator if we choose suitable multiples of π_i's. By considering the following isomorphisms:

$$\binom{W^3}{W^1\ W^2} \xrightarrow{\text{transpose}} \binom{W^3}{W^2\ W^1} \xrightarrow{\Delta(2h,z)} \binom{\tilde{\tilde{W}}^3}{W^2\ \tilde{\tilde{W}}^1} \xrightarrow{\text{transpose}} \binom{\tilde{\tilde{W}}^3}{\tilde{\tilde{W}}^1\ W^2} \simeq \binom{W^3}{W^1\ W^2},$$

we can find the following relation with some $\lambda \in \mathbb{C}$.

$$I^{00}(\phi_1^{-1}\phi_1'^{-1}\pi_1 u^1, z)u^2 = \lambda E^-(-2h,z)\phi_3\phi_3'^{-1}\pi_3 I^{00}(u^1,z)\Delta(-2h,-z)u^2.$$

By exchanging π_1 by $\lambda^{-1}\pi_1$ with π_3 fixed, we may assume that $\lambda = 1$. Similarly, since both $I^{00}(u^1,z)$ and $\pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(2h,z)u^1,z)\phi_2^{-1}\phi_2'^{-1}\pi_2$ are V-intertwining operators of type $W^1 \times W^2 \to W^3$, we may assume that they are equal after a suitable scalar multiplication on π_2. We should note that the $V \oplus \tilde{V}$-module structures on $W^i \oplus \tilde{W}^i$ do not depend on the choice of π_i's. We introduce a parity function $\epsilon(\gamma, w^1) \in \mathbb{Z}_2$ for $w^1 \in W^1 \cup \tilde{W}^1$ by $p(\gamma, w^1) = 1$ if γ is odd and $w^1 \in \tilde{W}^1$ and $p(\gamma, w^1) = 0$ for the other cases.

Theorem 5.1.12. *Assume that* $\binom{W^3}{W^1\ W^2}_V = \mathbb{C}I^{00}(\cdot,z)$. *Then the V-intertwining operator* $\bar{I}(\cdot,z) = (I^{00} \oplus I^{01} \oplus I^{10} \oplus I^{11})(\cdot,z)$ *satisfies the following (generalized) Jacobi identity:*

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\frac{z_1 - z_2}{z_0}\right)^{-\alpha_1} Y(\phi_0 a, z_1)\bar{I}(w^1, z_2)w^2 \\
&-(-1)^{\epsilon(\gamma,w^1)}\zeta^{\alpha_1} z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\left(\frac{z_2 - z_1}{z_0}\right)^{-\alpha_1} \bar{I}(w^1, z_2)Y(\phi_0 a, z_1)w^2 \qquad (5.1.3)\\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} \bar{I}(Y(\phi_0 a, z_0)w^1, z_2)w^2,
\end{aligned}$$

where $a \in V$, $w^i \in W^i \cup \tilde{W}^i$, $i = 1, 2$, *and* $\zeta = e^{\pi\sqrt{-1}}$ *is the root of unity used in the transpose-operation. Therefore,* $\bar{I}(\cdot, z)$ *provides a* $(V \oplus \tilde{V})$*-intertwining operator of type* $(W^1 \oplus \tilde{W}^1) \times (W^2 \oplus \tilde{W}^2) \to (W^3 \oplus \tilde{W}^3)$.

Proof: The proof is too long to write so that we give it in Appendix. ∎

Remark 5.1.13. We note that if both α_1 and α_2 are integers and γ is odd, then the intertwining operator $\bar{I}(\cdot, z)$ above is an intertwining operator among untwisted modules for the SVOA $V \oplus \tilde{V}$, whereas if α_1 is integral and α_2 is half-integral, then $\bar{I}(\cdot, z)$ is a $\mathbb{Z}_2$-twisted intertwining operator.

5.2 Lattice VOAs as SCE of free bosonic VOAs

5.2.1 Fusion rules for free bosonic VOA

Let $\mathfrak{h}$ be a finite dimensional linear space with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Then we can construct the free bosonic VOA $M_{\mathfrak{h}}(1, 0)$ as in Section 2.6.1. It is shown in Section 2.6.1 that the set of inequivalent irreducible $M_{\mathfrak{h}}(1, 0)$-modules is given by $\{M_{\mathfrak{h}}(1, \alpha) \mid \alpha \in \mathfrak{h}\}$. The fusion algebra for $M_{\mathfrak{h}}(1, 0)$ has the following structure:

$$M_{\mathfrak{h}}(1, \alpha) \times M_{\mathfrak{h}}(1, \beta) = M_{\mathfrak{h}}(1, \alpha + \beta).$$

Namely, it is isomorphic to the group algebra $\mathbb{C}[\mathfrak{h}]$ associated to the additive group $\mathfrak{h}$. We give a proof of the fusion rule above by using the idea of semisimple primary vectors.

In the following, we identify $\alpha \in \mathfrak{h}$ with $\alpha(-1)\mathbb{1} \in M_{\mathfrak{h}}(1, 0)$. It is easy to check that $\alpha(-1)\mathbb{1} \in M_{\mathfrak{h}}(1, 0)$ is a semisimple primary vector and so we can consider the delta operator $\Delta(\alpha, z)$ associated to $\alpha \in \mathfrak{h}$. Consider the $M_{\mathfrak{h}}(1, 0)$-module $M_{\mathfrak{h}}(1, \beta)^{(\alpha)} := (M_{\mathfrak{h}}(1, \beta), Y_{M_{\mathfrak{h}}(1,\beta)}(\Delta(\alpha, z)\,\cdot, z))$.

Lemma 5.2.1. *As an* $M_{\mathfrak{h}}(1, 0)$*-module,* $M_{\mathfrak{h}}(1, \beta)^{(\alpha)}$ *is isomorphic to* $M_{\mathfrak{h}}(1, \alpha + \beta)$.

Proof: Since $M_{\mathfrak{h}}(1, \beta)$ is an irreducible $M_{\mathfrak{h}}(1, 0)$-module, so is $M_{\mathfrak{h}}(1, \beta)^{(\alpha)}$. By definition, we can take a linear isomorphism $\phi : M_{\mathfrak{h}}(1, \beta) \to M_{\mathfrak{h}}(1, \beta)^{(\alpha)}$ such that

$$Y_{M_{\mathfrak{h}}(1,\beta)^{(\alpha)}}(a, z)\phi = \phi Y_{M_{\mathfrak{h}}(1,\beta)}(\Delta(\alpha, z)a, z)$$

for any $a \in V$. By a direct computation, we find the following relation:

$$h(n)\phi = \phi(h(n) + \delta_{n,0}\langle \alpha, h \rangle).$$

Therefore, $v \in M_{\mathfrak{h}}(1, \beta)$ is a highest weight vector if and only if $\phi v \in M_{\mathfrak{h}}(1, \beta)^{(\alpha)}$ is a highest weight vector with highest weight $\alpha + \beta$. Since $M_{\mathfrak{h}}(1, \beta)^{(\alpha)}$ is irreducible, $M_{\mathfrak{h}}(1, \beta)^{(\alpha)} \simeq M_{\mathfrak{h}}(1, \gamma)$ for some $\gamma \in \mathfrak{h}$, and hence we have $M_{\mathfrak{h}}(1, \beta)^{(\alpha)} \simeq M_{\mathfrak{h}}(1, \alpha + \beta)$. ∎

Since $M_{\mathfrak{h}}(1,0)$ is a simple current module, we note that all $M_{\mathfrak{h}}(1,\alpha)$, $\alpha \in \mathfrak{h}$, are also simple currents because they are realized as deformations of the simple current $M_{\mathfrak{h}}(1,0)$ by using the Delta operators. By this fact, one can verity the following:

Proposition 5.2.2. *We have the fusion rule* $M_{\mathfrak{h}}(1,\alpha) \times M_{\mathfrak{h}}(1,\beta) = M_{\mathfrak{h}}(1,\alpha+\beta)$ *for any* $\alpha, \beta \in \mathfrak{h}$.

Proof: Since $\binom{M_{\mathfrak{h}}(1,\beta)}{M_{\mathfrak{h}}(1,0)\ M_{\mathfrak{h}}(1,\alpha)}_{M_{\mathfrak{h}}(1,0)} = \delta_{\alpha,\beta}\mathbb{C}Y_{M_{\mathfrak{h}}(1,\alpha)}(\cdot,z)$ by Lemma 2.4.3, we have

$$\begin{aligned}
&\begin{pmatrix} & M_{\mathfrak{h}}(1,\gamma) & \\ M_{\mathfrak{h}}(1,\alpha) & & M_{\mathfrak{h}}(1,\beta)\end{pmatrix}_{M_{\mathfrak{h}}(1,0)} \simeq \begin{pmatrix} & M_{\mathfrak{h}}(1,\gamma)^{(-\beta)} & \\ M_{\mathfrak{h}}(1,\alpha) & & M_{\mathfrak{h}}(1,\beta)^{(-\beta)}\end{pmatrix}_{M_{\mathfrak{h}}(1,0)} \\
&\simeq \begin{pmatrix} & M_{\mathfrak{h}}(1,\gamma-\beta) & \\ M_{\mathfrak{h}}(1,\alpha) & & M_{\mathfrak{h}}(1,0)\end{pmatrix}_{M_{\mathfrak{h}}(1,0)} \simeq \begin{pmatrix} & M_{\mathfrak{h}}(1,\gamma-\beta) & \\ M_{\mathfrak{h}}(1,0) & & M_{\mathfrak{h}}(1,\alpha)\end{pmatrix}_{M_{\mathfrak{h}}(1,0)} \\
&\simeq \delta_{\alpha,\gamma-\beta}\mathbb{C}Y_{M_{\mathfrak{h}}(1,\alpha)}(\cdot,z).
\end{aligned}$$

Therefore, we have the desired identity $M_{\mathfrak{h}}(1,\alpha) \times M_{\mathfrak{h}}(1,\beta) = M_{\mathfrak{h}}(1,\alpha+\beta)$. ∎

5.2.2 Construction of lattice VOAs

Let L be a rational lattice in $\mathfrak{h}$ such that $\dim \mathfrak{h} = \mathrm{rank}(L)$. Let $\mathbb{C}[L] = \oplus_{\alpha\in L}\mathbb{C}e^{\alpha}$ be the group algebra associated to the additive group L and set

$$V_L := \mathbb{C}[L] \underset{\mathbb{C}}{\otimes} M_{\mathfrak{h}}(1,0) = \bigoplus_{\alpha\in L} \mathbb{C}e^{\alpha} \underset{\mathbb{C}}{\otimes} M_{\mathfrak{h}}(1,0).$$

Then define

$$Y_0(e^0 \otimes a, z)\cdot e^{\alpha}\otimes v := e^{\alpha}\otimes Y_{M_{\mathfrak{h}}(1,0)}(\Delta(\alpha,z)a,z)v$$

for $e^0\otimes a \in e^0 \otimes M_{\mathfrak{h}}(1,0)$ and $e^{\alpha}\otimes v \in V_L$. Then $(V_L, Y_0(\cdot,z))$ is an $M_{\mathfrak{h}}(1,0)$-module which is isomorphic to $\oplus_{\alpha\in L}M_{\mathfrak{h}}(1,\alpha)$. For $\alpha \in L$, we define a linear endomorphism $\psi_{\alpha} \in V_L$ by

$$\psi_{\alpha}\cdot e^{\beta}\otimes v := e^{\alpha+\beta}\otimes v.$$

Namely, $\psi_{\alpha} = \mathrm{ad}(e^{\alpha})\otimes \mathrm{id}_{M_{\mathfrak{h}}(1,0)}$. Then $\psi_0 = \mathrm{id}_{V_L}$ and $\psi_{\alpha}\psi_{\beta} = \psi_{\alpha+\beta}$ and so ψ gives rise to a representation of L on V_L. We introduce a (generalized) vertex operator algebra structure on V_L. For $x^{\alpha} = e^{\alpha}\otimes a$ and $x^{\beta} = e^{\beta}\otimes b$, define

$$\begin{aligned}
\tilde{Y}_{V_L}(x^{\alpha},z)x^{\beta} &:= \psi_{\alpha+\beta}E^-(-\alpha,z)Y_0(\psi_{-\alpha}\Delta(\beta,z)x^{\alpha},z)\Delta(\alpha,-z)\psi_{-\beta}x^{\beta} \\
&= e^{\alpha+\beta}\otimes z^{\langle\alpha,\beta\rangle}E^-(-\alpha,z)Y_{M_{\mathfrak{h}}(1,0)}(\Delta(\beta,z)a,z)E^+(-\alpha,z)(-z)^{\alpha_{(0)}}b \\
&\in e^{\alpha+\beta}\otimes M_{\mathfrak{h}}(1,0)\{\{z\}\},
\end{aligned}$$

where $E^{\pm}(\alpha,z)$ are defined as in (5.1.1) and $(-z)^{\alpha_{(0)}} = (e^{\pi\sqrt{-1}}z)^{\alpha_{(0)}}$.

Remark 5.2.3. We note that the map above is exactly the $M_{\mathfrak{h}}(1,0)$-intertwining operator of type $M_{\mathfrak{h}}(1,\alpha) \times M_{\mathfrak{h}}(1,\beta) \to M_{\mathfrak{h}}(1,\alpha+\beta)$ which we constructed in Proposition 5.2.2.

By a similar computation as in the proof of Theorem 5.1.12, we can show the following Jacobi identity:

Theorem 5.2.4. *([DLM5, Theorem 3.5]) Let $x^\alpha \in e^\alpha \otimes M_{\mathfrak{h}}(1,0)$, $\alpha \in L$. Then the following identity holds:*

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\frac{z_1-z_2}{z_0}\right)^{-\langle\alpha,\beta\rangle}\tilde{Y}_{V_L}(x^\alpha,z_1)\tilde{Y}_{V_L}(x^\beta,z_2)x^\gamma\\
&-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{-\langle\alpha,\beta\rangle}\tilde{Y}_{V_L}(x^\beta,z_2)\tilde{Y}_{V_L}(x^\alpha,z_1)x^\gamma\\
&=z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\left(\frac{z_2+z_0}{z_1}\right)^{-\langle\alpha,\gamma\rangle}\tilde{Y}_{V_L}(\tilde{Y}_{V_L}(x^\alpha,z_0)x^\beta,z_2)x^\gamma.
\end{aligned}$$

In particular, $(V_L,\tilde{Y}_{V_L}(\cdot,z))$ forms a generalized vertex algebra in the sense of [DL].

Corollary 5.2.5. *Assume that L equipped with $\langle\cdot,\cdot\rangle$ is an integral lattice. Then we have*

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\tilde{Y}_{V_L}(x^\alpha,z_1)\tilde{Y}_{V_L}(x^\beta,z_2)\\
&-(-1)^{\langle\alpha,\beta\rangle}z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\tilde{Y}_{V_L}(x^\beta,z_2)\tilde{Y}_{V_L}(x^\alpha,z_1)\\
&=z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)\tilde{Y}_{V_L}(\tilde{Y}_{V_L}(x^\alpha,z_0)x^\beta,z_2)
\end{aligned}$$

for $x^\alpha \in e^\alpha \otimes M_{\mathfrak{h}}(1,0)$, $x^\beta \in e^\beta \otimes M_{\mathfrak{h}}(1,0)$.

Assume that L is integral. Set $L^0 := \{\alpha \in L \mid \langle\alpha,\alpha\rangle \in 2\mathbb{Z}\}$ and $L^1 := L \setminus L^0$. Then $V_L = V_{L^0} \oplus V_{L^1}$ is a $\mathbb{Z}_2$-graded algebra by the corollary above. To obtain vertex superalgebra structure on V_L, we shall need a 2-cocycle on L. Let $\{\alpha^1,\ldots,\alpha^{\mathrm{rank}(L)}\}$ be a $\mathbb{Z}$-basis for L. Define ε to be the (uniquely determined) $\{\pm1\}$-valued multiplicative function on $L \times L$ such that

$$\varepsilon(\alpha^i,\alpha^j) := \begin{cases} (-1)^{\langle\alpha^i,\alpha^j\rangle+\langle\alpha^i,\alpha^i\rangle\langle\alpha^j,\alpha^j\rangle} & \text{if } i>j,\\ (-1)^{(\langle\alpha^i,\alpha^i\rangle+\langle\alpha^i,\alpha^i\rangle^2)/2} & \text{if } i=j,\\ 1 & \text{if } i<j, \end{cases}$$

(cf. [FLM] [DL]). Note that $\varepsilon(\alpha,\alpha) = (-1)^{(\langle\alpha,\alpha\rangle+\langle\alpha,\alpha\rangle^2)/2}$ and by the bimultiplicability we have $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle+\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}$ for $\alpha,\beta \in L$. Then define a vertex operator map $\hat{Y}_{V_L}(\cdot,z)$ on V_L by

$$Y_{V_L}(x^\alpha,z)x^\beta := \varepsilon(\alpha,\beta)\tilde{Y}_{V_L}(x^\alpha,z)x^\beta$$

for $x^\alpha \in e^\alpha \otimes M_{\mathfrak{h}}(1,0)$, $x^\beta \in e^\beta \otimes M_{\mathfrak{h}}(1,0)$. Then by Corollary 5.2.5 we have:

Theorem 5.2.6. *Let L be an integral lattice in $\mathfrak{h}$ such that $\mathbb{C} \otimes L \simeq \mathfrak{h}$. As an L-graded simple current extension of $M_{\mathfrak{h}}(1,0)$, $V_L = \oplus_{\alpha \in L} M_{\mathfrak{h}}(1,\alpha)$ equipped with $Y_{V_L}(\cdot, z)$ carries a structure of a vertex superalgebra with the even part V_{L^0} and the odd part V_{L^1}. Moreover, if L^0 is positive definite, then V_L is a vertex operator superalgebra.*

Now consider modules for a lattice VOA V_L associated to positive definite even lattice L. By Theorem 5.2.4 we can introduce an irreducible V_L-module structure on $V_{L+\lambda}$ for each coset $\lambda + L \in L^\circ/L$, and it is shown in [D1] that every irreducible V_L-module is isomorphic to some coset module $V_{L+\lambda}$. Here we give another construction of $V_{L+\lambda}$. As $\mathfrak{h} \subset M_{\mathfrak{h}}(1,0)$ by identification, we can construct a Delta operator $\Delta(\lambda, z)$ for each $\lambda \in L^\circ$.

Proposition 5.2.7. *([Li0]) As a V_L-module, $(V_L, Y_{V_L}(\Delta(\lambda,z)\,\cdot, z))$ is isomorphic to $V_{L+\lambda}$.*

Proof: For simplicity, we denote $(V_L, Y_{V_L}(\Delta(\lambda,z)\,\cdot, z))$ by $\tilde{V}_L$. By a direct computation, we have

$$Y_{V_L}(\Delta(\lambda,z)h, z) = Y_{V_L}(h,z) + z^{-1}\langle \lambda, h\rangle$$

for $h \in \mathfrak{h}$. Thus $\tilde{V}_L$ is a completely reducible $M_{\mathfrak{h}}(1,0)$-module and the set of $\mathfrak{h}$-weights of $\tilde{V}_L$ is exactly $L + \lambda$. Thus $\tilde{V}_L \simeq V_{L+\lambda}$. ∎

Therefore, all V_L-modules are simple currents. The following is a simple corollary.

Theorem 5.2.8. *We have the fusion rule $V_{L+\lambda} \times V_{L+\mu} = V_{L+\lambda+\mu}$ for any $\lambda, \mu \in L^\circ$. Therefore, the fusion algebra for V_L is isomorphic to the group algebra $\mathbb{C}[L^\circ/L]$.*

5.3 $\mathbb{Z}_2$-graded SCE of affine VOAs

In this section we consider an application of Theorem 5.1.8 to the affine VOAs. Here we consider a relatively simple case, the case of $\hat{sl}_2(\mathbb{C})$; the other cases are similar. Let $\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$ be the Lie algebra $sl_2(\mathbb{C})$ with the standard Lie brackets $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$. We normalize the invariant bilinear form on $\mathfrak{g}$ such that $\langle h, h\rangle = 2$. Let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra of type $A_1^{(1)}$. For any non-negative integer ℓ, we have an affine VOA $L_{\mathfrak{g}}(\ell,0)$ as in Section 2.6.3. It is shown in [FZ] that $L_{\mathfrak{g}}(\ell,0)$ is rational and irreducible $L_{\mathfrak{g}}(\ell,0)$-modules are $\{L_{\mathfrak{g}}(\ell,j) \mid j = 0,1,\dots,\ell\}$. The fusion algebra for $L_{\mathfrak{g}}(\ell,0)$ has the following structure:

$$L_{\mathfrak{g}}(\ell,i) \times L_{\mathfrak{g}}(\ell,j) = \sum_{k=\max\{0,i+j-\ell\}}^{\min\{i,j\}} L_{\mathfrak{g}}(\ell, i+j-2k). \tag{5.3.1}$$

We note that $L_{\mathfrak{g}}(\ell,0)$ and $L_{\mathfrak{g}}(\ell,\ell)$ are all the simple current $L_{\mathfrak{g}}(\ell,0)$-modules.

Remark 5.3.1. For $0 \le j \le \ell$, set $\chi_j(\tau, z) := \mathrm{tr}_{L_{\mathfrak{g}}(\ell,j)} e^{-\pi\sqrt{-1}\, z h_{(0)}} q_\tau^{L(0)-c/24}$. Then the following character formula is well-known (cf. [Wak], for example):

$$\chi_j(\tau, z) = \frac{\theta_{j+1,\ell+2}(\tau, z) - \theta_{-j-1,\ell+2}(\tau, z)}{\theta_{1,2}(\tau, z) - \theta_{-1,2}(\tau, z)},$$

where $\theta_{n,m}(\tau, z) := \displaystyle\sum_{j\in\mathbb{Z}+\frac{n}{2m}} e^{2\pi\sqrt{-1}\, mzj} q_\tau^{mj^2}$. Using the formula above, the following modular transformation is obtained:

$$\chi_j(\frac{-1}{\tau}, \tau z) = \frac{1}{2\sqrt{-1}}\sqrt{\frac{2}{\ell+2}}\, e^{\frac{1}{2}\pi\sqrt{-1}\,\ell\frac{z^2}{\tau}} \sum_{k=0}^{\ell} \left\{ e^{\pi\sqrt{-1}\,\frac{(j+1)(k+1)}{\ell+2}} - e^{-\pi\sqrt{-1}\,\frac{(j+1)(k+1)}{\ell+2}} \right\} \chi_k(\tau, z).$$

Since $\mathbb{C}h$ is the Cartan subalgebra of $\mathfrak{g}$, $h_{(0)}$ acts on $L_{\mathfrak{g}}(\ell, 0)$ semisimply with eigenvalues in $2\mathbb{Z}$. Therefore, the set of integral semisimple primary vectors in $\mathbb{C}h$ is $\frac{1}{2}\mathbb{Z}h$.

Proposition 5.3.2. *We have isomorphisms* $(L_{\mathfrak{g}}(\ell, j), Y(\Delta(\frac{1}{2}h, z)\,\cdot, z)) \simeq L_{\mathfrak{g}}(\ell, \ell - j)$ *as* $L_{\mathfrak{g}}(\ell, 0)$*-modules.*

Proof: For simplicity, let us denote $(L_{\mathfrak{g}}(\ell, j), Y(\Delta(\frac{1}{2}h, z)\,\cdot, z))$ by $\tilde{L}_{\mathfrak{g}}(\ell, j)$. Then we can take a canonical linear isomorphism $\psi : L_{\mathfrak{g}}(\ell, 0) \to \tilde{L}_{\mathfrak{g}}(\ell, 0)$ such that $Y(a, z)\psi = \psi Y(\Delta(\frac{1}{2}h, z)a, z)$ for any $a \in L_{\mathfrak{g}}(\ell, 0)$. Then by definition we have

$$h_{(n)}\psi = \psi(h_{(n)} + \delta_{n,0}\ell), \quad e_{(n)}\psi = \psi e_{(n+1)}, \quad f_{(n)}\psi = \psi f_{(n-1)}.$$

Thus $\psi\mathbb{1}$ is a highest weight vector with highest weight ℓ and $\tilde{L}_{\mathfrak{g}}(\ell, 0) \simeq L_{\mathfrak{g}}(\ell, \ell)$. Now by the fusion rules (5.3.1) we have $\tilde{L}_{\mathfrak{g}}(\ell, j) \simeq L_{\mathfrak{g}}(\ell, \ell - j)$. ∎

Remark 5.3.3. By $\tilde{\tilde{L}}_{\mathfrak{g}}(\ell, 0) \simeq L_{\mathfrak{g}}(\ell, 0)$, we note that $(L_{\mathfrak{g}}(\ell, j), Y(\Delta(h, z)\,\cdot, z)) \simeq L_{\mathfrak{g}}(\ell, j)$.

Now by Theorem 5.1.8, we can construct a $\mathbb{Z}_2$-graded extension $L_{\mathfrak{g}}(\ell, 0) \oplus L_{\mathfrak{g}}(\ell, \ell)$ if $\langle \frac{1}{2}h, \frac{1}{2}h\rangle = \frac{1}{2}\ell$ is an integer.

Theorem 5.3.4. *([Li5]) Let ℓ be an even integer.*
(1) If $\ell/2$ is even, then the simple current extension $L_{\mathfrak{g}}(\ell, 0) \oplus L_{\mathfrak{g}}(\ell, \ell)$ is a simple $\mathbb{Z}_2$-graded vertex operator algebra.
(2) If $\ell/2$ is odd, then the simple current extension $L_{\mathfrak{g}}(\ell, 0) \oplus L_{\mathfrak{g}}(\ell, \ell)$ is a simple vertex operator superalgebra.

Remark 5.3.5. If ℓ is odd, then $L_{\mathfrak{g}}(\ell, 0) \oplus L_{\mathfrak{g}}(\ell, \ell)$ does not form a vertex operator algebra nor superalgebra. This is due to the failure of the locality. However, it is shown in [Li8] that by adding some bosonic fields we can define a $\mathbb{Z}_2$-graded simple current extension of $L_{\mathfrak{g}}(\ell, 0)$ even if ℓ is odd.

Lattice construction of affine VOAs. Before we end this section, we give an explicit construction of some affine VOAs. Let L be a root lattice of type A, D or E. Then we can construct the lattice VOA V_L as a simple current extension of the free bosonic VOA $M_{\mathbb{C}L}(1,0)$. The weight one subspace of V_L is

$$(V_L)_1 \simeq \mathbb{C}L \oplus \bigoplus_{\substack{\alpha\in L \\ \langle\alpha,\alpha\rangle=2}} \mathbb{C}e^{\alpha}$$

and forms a Lie algebra under the 0-th product in V_L. Since $L(1)(V_L)_1 = 0$, V_L possesses unique invariant bilinear form up to normalization and so the Lie algebra $(V_L)_1$ has a non-degenerate symmetric invariant bilinear form.

Theorem 5.3.6. *([FLM]) The Lie algebra $(V_L)_1$ is a simple Lie algebra with a non-degenerate invariant bilinear form. It's Cartan subalgebra is $\mathbb{C}L$. Therefore, if L is a root lattice of type A_n, D_n or E_n, then $(V_L)_1$ is a simple Lie algebra of type A_n, D_n or E_n, respectively.*

Denote the simple Lie algebra $(V_L)_1$ by $\mathfrak{g}_L$. The vertex operators of elements of $\mathfrak{g}_L$ provide a level 1 representation of the affine Lie algebra $\hat{\mathfrak{g}}_L$ on V_L. Therefore, $\mathfrak{g}_L$ generates a level 1 affine VOA $L_{\mathfrak{g}_L}(1,0)$ inside V_L. Since V_L is generated by its weight one subspace, we have $V_L \simeq L_{\mathfrak{g}_L}(1,0)$. Now consider a tensor product algebra $V_L^{\otimes \ell} \simeq V_{L^{\oplus\ell}}$ of ℓ copies of V_L. For $x \in \mathfrak{g}_L$, define its diagonal component in $V_L^{\otimes\ell}$ by

$$\tilde{x} := x \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes x \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes x \in (V_L^{\otimes\ell})_1.$$

Then $\tilde{\mathfrak{g}}_L := \{\tilde{x} \in (V_L^{\otimes\ell})_1 \mid x \in \mathfrak{g}_L\}$ forms a simple Lie algebra isomorphic to $\mathfrak{g}_L$ and their vertex operators define a level ℓ representation of the affine Lie algebra $\hat{\mathfrak{g}}_L$ on $V_L^{\otimes\ell}$. Hence, $\tilde{\mathfrak{g}}_L$ generates a level ℓ affine VOA $L_{\mathfrak{g}_L}(\ell,0)$ in $V_L^{\otimes\ell}$.

Theorem 5.3.7. *([DL]) The sub VOA of $V_L^{\otimes\ell}$ generated by the diagonal components $\tilde{x}$, $x \in \mathfrak{g}_L$, is isomorphic to $L_{\mathfrak{g}_L}(\ell,0)$. Therefore, we can realize the affine VOA $L_{\mathfrak{g}_L}(\ell,0)$ as a subalgebra of the lattice VOA $V_{L^{\oplus\ell}}$.*

Remark 5.3.8. By the theorem above, we can also construct the $\mathbb{Z}_2$-graded extensions of the affine VOA associated to $\hat{sl}_2(\mathbb{C})$ by using the lattice $A_1^{\oplus\ell}$ (cf. [LLY]).

5.4 $\mathbb{Z}_2$-graded SCE of the unitary Virasoro VOAs

Recall from Section 2.6.4 the unitary series of the Virasoro VOAs $L_{\mathrm{Vir}}(c_m,0)$ with $c_m = 1-6/(m+2)(m+3)$, $m \in \mathbb{N}$. It is shown in [DMZ] [Wan] that $L_{\mathrm{Vir}}(c_m,0)$ is rational and all the irreducible $L_{\mathrm{Vir}}(c_m,0)$-modules are provided by $L_{\mathrm{Vir}}(c_m,h_{r,s}^{(m)})$ with

$$h_{r,s}^{(m)} = \frac{\{r(m+3)-s(m+2)\}^2-1}{4(m+2)(m+3)}, \quad 1 \le r \le m+1, \quad 1 \le s \le m+2. \qquad (5.4.1)$$

Note that $h^{(m)}_{r,s} = h^{(m)}_{m+3-r,m+2-s}$.

Remark 5.4.1. The q-character of $L_{\mathrm{Vir}}(c_m, h^{(m)}_{r,s})$ is well-known (cf. [KR]) and given as follows:

$$\mathrm{ch}_{L_{\mathrm{Vir}}(c_m,h^{(m)}_{r,s})}(\tau) = \sum_{k\in\mathbb{Z}}(q^{b(k)} - q^{a(k)})/\prod_{i=1}^{\infty}(1-q^i),$$

where

$$a(k) := \frac{\{2(m+2)(m+3)k + (m+3)r + (m+2)s\}^2 - 1}{4(m+2)(m+3)},$$
$$b(k) := \frac{\{2(m+2)(m+3)k + (m+3)r - (m+2)s\}^2 - 1}{4(m+2)(m+3)}.$$

Set $\chi^{(m)}_{r,s}(\tau) := q^{-c_m/24}\mathrm{ch}_{L_{\mathrm{Vir}}(c_m,h^{(m)}_{r,s})}(\tau)$ for $1 \le s \le r \le m+1$. Then following modular transformation law holds:

$$\chi^{(m)}_{r,s}(-1/\tau) = \sqrt{\frac{8}{(m+2)(m+3)}} \sum_{1\le j\le i\le m+1} (-1)^{(r+s)(i+j)} \sin\frac{\pi ri}{m+2} \sin\frac{\pi sj}{m+3}\chi^{(m)}_{i,j}(\tau).$$

The fusion rules are computed in [Wan] and given as follows:

$$L_{\mathrm{Vir}}(c_m, h^{(m)}_{r_1,s_1}) \times L_{\mathrm{Vir}}(c_m, h^{(m)}_{r_2,s_2}) = \sum_{i\in I,\, j\in J} L_{\mathrm{Vir}}(c_m, h^{(m)}_{|r_1-r_2|+2i-1,|s_1-s_2|+2j-1}), \tag{5.4.2}$$

where

$$I = \{1, 2, \dots, \min\{r_1, r_2, m+2-r_1, m+2-r_2\}\},$$
$$J = \{1, 2, \dots, \min\{s_1, s_2, m+3-s_1, m+3-s_2\}\}.$$

By the formula above, we find that only $L_{\mathrm{Vir}}(c_m, h^{(m)}_{1,1})$ and $L_{\mathrm{Vir}}(c_m, h^{(m)}_{m+1,1})$ are simple currents. We also note that $h^{(m)}_{1,1} = 0$ is the minimal value and $h^{(m)}_{m+1,1} = m(m+1)/4$ is the maximal value among $h^{(m)}_{r,s}$, $1 \le s \le r \le m+1$. The fusion rules between $L_{\mathrm{Vir}}(c_m, 0)$ and $L_{\mathrm{Vir}}(c_m, h^{(m)}_{m+1,1})$ have a $\mathbb{Z}_2$-symmetry, so we can expect that $L_{\mathrm{Vir}}(c_m, 0) \oplus L_{\mathrm{Vir}}(c_m, h^{(m)}_{m+1,1})$ forms a $\mathbb{Z}_2$-graded simple current extension. We prove that this is true by using Theorem 5.3.4.

Let $\mathfrak{g}$ and $L_{\mathfrak{g}}(\ell, j)$, $0 \le j \le \ell$, be as in the previous section. The weight one subspace $L_{\mathfrak{g}}(\ell, 0)_1$ of $L_{\mathfrak{g}}(\ell, 0)$ is $\mathbb{C}e_{(-1)}\mathbb{1} \oplus \mathbb{C}h_{(-1)}\mathbb{1} \oplus \mathbb{C}f_{(-1)}\mathbb{1}$ and forms a simple Lie algebra isomorphic to $\mathfrak{g}$ under the 0-th product in $L_{\mathfrak{g}}(\ell, 0)$. In the following, we identify $L_{\mathfrak{g}}(\ell, 0)_1$ with $\mathfrak{g}$. Let m be a positive integer. Let h^1, e^1, f^1 be the generators of $\mathfrak{g}$ in $L_{\mathfrak{g}}(1, 0)$ and let h^m, e^m, f^m be the generators of $\mathfrak{g}$ in $L_{\mathfrak{g}}(m, 0)$. Then $h^{m+1} := h^1 \otimes \mathbb{1} + \mathbb{1} \otimes h^m$, $e^{m+1} := e^1 \otimes \mathbb{1} + \mathbb{1} \otimes e^m$ and $f^{m+1} := f^1 \otimes \mathbb{1} + \mathbb{1} \otimes f^m$ generate a sub VOA isomorphic to $L_{\mathfrak{g}}(m+1, 0)$ in the tensor product $L_{\mathfrak{g}}(1, 0) \otimes L_{\mathfrak{g}}(m, 0)$. Denote by Ω^1, Ω^m and Ω^{m+1} the corresponding Virasoro vector of $L_{\mathfrak{g}}(1, 0)$, $L_{\mathfrak{g}}(m, 0)$ and $L_{\mathfrak{g}}(m+1, 0)$. Then it is shown in [GKO] [DL] and [KR] that $\omega^m := \Omega^1 \otimes \mathbb{1} + \mathbb{1} \otimes \Omega^m - \Omega^{m+1}$ is also a Virasoro vector with central charge $c_m = 1 - 6/(m+2)(m+3)$ and the subalgebra generated by ω^m is a simple Virasoro VOA $L_{\mathrm{Vir}}(c_m, 0)$. Since a decomposition $\omega^m + \Omega^{m+1}$ is orthogonal, $L_{\mathfrak{g}}(1,0) \otimes L_{\mathfrak{g}}(m,0)$ contains a sub VOA $L_{\mathrm{Vir}}(c_m,0) \otimes L_{\mathfrak{g}}(m+1,0)$. This is the famous GKO-construction of the unitary Virasoro VOAs [GKO]. In [GKO], the following decomposition law is also established:

$$L_{\mathfrak{g}}(1,\epsilon) \otimes L_{\mathfrak{g}}(m,j) = \bigoplus_{\substack{0 \le s \le m+1 \\ s \equiv j+\epsilon \bmod 2}} L_{\mathrm{Vir}}(c_m, h^{(m)}_{j+1,s+1}) \otimes L_{\mathfrak{g}}(m+1,s), \tag{5.4.3}$$

where $\epsilon = 0,1$ and $0 \le j \le m$. We note that all $L_{\mathrm{Vir}}(c_m, h^{(m)}_{r,s})$, $1 \le s \le r \le m+1$, appear in the above decompositions.

Remark 5.4.2. We note that $\mathrm{Com}_{L_{\mathfrak{g}}(1,0)\otimes L_{\mathfrak{g}}(m,0)}(L_{\mathfrak{g}}(m,0)) = L_{\mathrm{Vir}}(c_m,0)$.

Now set

$$U(m) := \begin{cases} L_{\mathfrak{g}}(1,0) \otimes L_{\mathfrak{g}}(m,0) \oplus L_{\mathfrak{g}}(1,0) \otimes L_{\mathfrak{g}}(m,m) & \text{if } m \text{ is even,} \\ L_{\mathfrak{g}}(1,0) \otimes L_{\mathfrak{g}}(m,0) \oplus L_{\mathfrak{g}}(1,1) \otimes L_{\mathfrak{g}}(m,m) & \text{if } m \text{ is odd.} \end{cases}$$

Then $U(m)$ is a $\mathbb{Z}_2$-graded simple current extension of $L_{\mathfrak{g}}(1,0) \otimes L_{\mathfrak{g}}(m,0)$ and it is a simple VOA if $m \equiv 0,3 \mod 4$ and is a simple SVOA if $m \equiv 1,2 \mod 4$ by Theorem 5.1.8. The commutant subalgebra of $L_{\mathfrak{g}}(m+1,0)$ is

$$\mathrm{Com}_{U(m)}(L_{\mathfrak{g}}(m+1,0)) = L_{\mathrm{Vir}}(c_m,0) \oplus L_{\mathrm{Vir}}(c_m, h^{(m)}_{m+1,1})$$

and hence we obtain the desired simple current extension.

Theorem 5.4.3. *([LLY]) The simple current extension $L_{\mathrm{Vir}}(c_m,0) \oplus L_{\mathrm{Vir}}(c_m, h^{(m)}_{m+1,1})$ is a simple $\mathbb{Z}_2$-graded vertex operator algebra if $m \equiv 0$ or $3 \mod 4$ and is a simple vertex operator superalgebra if $m \equiv 1$ or $2 \mod 4$.*

In [LLY], all irreducible $L_{\mathrm{Vir}}(c_m,0) \oplus L_{\mathrm{Vir}}(c_m, h^{(m)}_{m+1,1})$-modules are classified and all the fusion rules for the extensions are also computed in the case where the extensions are again vertex operator algebras. Here we give some of their results. Below we denote $L_{\mathrm{Vir}}(c,h)$ by $L(c,h)$ for simplicity.

$\star$ **The case $m = 3$: $L(4/5,0) \oplus L(4/5,3)$.**

Theorem 5.4.4. *([KMY] [LLY]) A VOA $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ is rational and every irreducible module is isomorphic to one of the following:*

$$W(0) := L(\tfrac{4}{5},0) \oplus L(\tfrac{4}{5},3), \quad W(\tfrac{2}{3})^{\pm} := L(\tfrac{4}{5},\tfrac{2}{3})^{\pm},$$
$$W(\tfrac{2}{5}) := L(\tfrac{4}{5},\tfrac{2}{5}) \oplus L(\tfrac{4}{5},\tfrac{7}{5}), \quad W(\tfrac{1}{15})^{\pm} := L(\tfrac{4}{5},\tfrac{1}{15})^{\pm},$$

where $W(h)^-$ is the σ-conjugate module of $W(h)^+$. The dual modules are as follows: $(W(h)^{\pm})^* \simeq W(h)^{\mp}$ if $h = \frac{2}{3}$ or $\frac{1}{15}$ and $W(h)^* \simeq W(h)$ for the others.

Remark 5.4.5. We may exchange the sign $\pm$ since there is no canonical way to determine the type $+$ and $-$ for the modules $W(h)^+$ and $W(h)^-$. However, if we determine a sign of one module, then the following fusion rules automatically determine all the signs.

The fusion algebra for $W(0)$ has a natural $\mathbb{Z}_3$-symmetry. For convenience, we use the following $\mathbb{Z}_3$-graded names.

$$A^0 := W(0), \quad A^1 := W(\tfrac{2}{3})^+, \quad A^2 := W(\tfrac{2}{3})^-,$$
$$B^0 := W(\tfrac{2}{5}), \quad B^1 := W(\tfrac{1}{15})^+, \quad B^2 := W(\tfrac{1}{15})^-.$$

Theorem 5.4.6. *([M2] [LLY]) The fusion rules for irreducible $W(0)$-modules are given as*

$$A^i \times A^j = A^{i+j}, \quad A^i \times B^j = B^{i+j}, \quad B^i \times B^j = A^{i+j} + B^{i+j},$$

where $i, j \in \mathbb{Z}_3$. Therefore, the fusion algebra for $W(0)$ has a natural $\mathbb{Z}_3$-symmetry.

$\star$ **The case** $m = 4$ **:** $L(6/7, 0) \oplus L(6/7, 5)$**.**

Theorem 5.4.7. *([LY] [LLY]) A VOA $L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$ is rational and all its irreducible modules are the following:*

$$N(0) := L(\tfrac{6}{7}, 0) \oplus L(\tfrac{6}{7}, 5), \quad N(\tfrac{1}{7}) := L(\tfrac{6}{7}, \tfrac{1}{7}) \oplus L(\tfrac{6}{7}, \tfrac{22}{7}), \quad N(\tfrac{5}{7}) := L(\tfrac{6}{7}, \tfrac{5}{7}) \oplus L(\tfrac{6}{7}, \tfrac{12}{7}),$$
$$N(\tfrac{4}{3})^{\pm} := L(\tfrac{6}{7}, \tfrac{4}{3})^{\pm}, \qquad N(\tfrac{1}{21})^{\pm} := L(\tfrac{6}{7}, \tfrac{1}{21})^{\pm}, \qquad N(\tfrac{10}{21})^{\pm} := L(\tfrac{6}{7}, \tfrac{10}{21})^{\pm},$$

where $N(h)^-$ is the σ-conjugate module of $N(h)^+$. Also, the dual modules are as follows: $(N(h)^{\pm})^* \simeq N(h)^{\mp}$ if $h = \frac{4}{3}, \frac{1}{21}$ or $\frac{10}{21}$ and $N(h)^* \simeq N(h)$ for the others.

The fusion algebra for $N(0)$ is also determined in [LY] and [LLY]. To state the fusion rules, we assign $\mathbb{Z}_3$-graded names to irreducible modules (cf. [LY]). Define

$$C^0 := N(0), \quad C^1 := N(\tfrac{4}{3})^+, \quad C^2 := N(\tfrac{4}{3})^-,$$
$$D^0 := N(\tfrac{1}{7}), \quad D^1 := N(\tfrac{10}{21})^+, \quad D^2 := N(\tfrac{10}{21})^-,$$
$$E^0 := N(\tfrac{5}{7}), \quad E^1 := N(\tfrac{1}{21})^+, \quad E^2 := N(\tfrac{1}{21})^-.$$

Theorem 5.4.8. *([LY] [LLY]) The fusion rules for irreducible $N(0)$-modules are given as*

$$\begin{aligned} C^i \times C^j &= C^{i+j}, & D^i \times D^j &= C^{i+j} + E^{i+j}, \\ C^i \times D^j &= D^{i+j}, & D^i \times E^j &= D^{i+j} + E^{i+j}, \\ C^i \times E^j &= E^{i+j}, & E^i \times E^j &= C^{i+j} + D^{i+j} + E^{i+j}, \end{aligned}$$

where $i, j \in \mathbb{Z}_3$. Therefore, the fusion algebra for $N(0)$ has a natural $\mathbb{Z}_3$-symmetry.

Chapter 6

The Moonshine VOA I: Frenkel-Lepowsky-Muerman Construction

In this section we consider the famous moonshine vertex operator algebra $V^\natural$ constructed by Frenkel-Lepowsky-Muerman [FLM]. The moonshine VOA is constructed as a so-called $\mathbb{Z}_2$-orbifold construction from the lattice VOA V_Λ associated to the Leech lattice Λ. The original proof of the existence of a structure of a vertex operator algebra on $V^\natural$ in [FLM] uses many group theoretic results and so seems to be very complicated. After [FLM], Huang suggested a simple proof of the existence of vertex operator algebra structure on $V^\natural$ by using a theory of fusion products in [H3]. In each case, $V^\natural$ is constructed as a $\mathbb{Z}_2$-graded simple current extension of a $\mathbb{Z}_2$-orbifold subalgebra of V_Λ.

6.1 $\mathbb{Z}_2$-orbifold theory of lattice VOAs

Let $(L, \langle\cdot,\cdot\rangle)$ be a positive definite even lattice.

6.1.1 Central extension

Let us review the central extension of L for a while. Let $\mathbb{Z}_2 = \langle \kappa \mid \kappa^2 = 1\rangle$ be a group of order 2 and consider the following central extension:

$$1 \to \mathbb{Z}_2 \hookrightarrow \hat{L} \xrightarrow{\pi} L \to 1. \tag{6.1.1}$$

Let $e : L \ni \alpha \mapsto e^\alpha \in \hat{L}$ be a section, that is, a map such that $\pi(e^\alpha) = \alpha$. We may choose e to satisfy $e^0 = 1_{\hat{L}}$. Then $\hat{L} = \{k^s e^\alpha \mid \alpha \in L,\ s = 0, 1\}$ and we can find a 2-cocycle $\varepsilon : L \times L \to \mathbb{Z}_2$ such that $e^\alpha \cdot e^\beta = \varepsilon(\alpha,\beta) e^{\alpha+\beta}$. The cocycle ε depends on the choice of the section e, but it is known that it is unique up to 2-coboundary. More precisely, it is known that the set of equivalence classes of central extensions (6.1.1) is in one-to-one correspondence with the second cohomology group $H^2(L,\mathbb{Z}_2)$ (cf. [FLM]).

One can verify that the commutator map $c(\alpha,\beta) := \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)^{-1} \in \mathbb{Z}_2$ is symmetric bimultiplicative map. It is shown in [FLM] that the central extension (6.1.1) is uniquely determined by the commutator map $c(\cdot,\cdot)$ up to equivalence. In the construction of the lattice VOA V_L, we need the twisted relation $e^\alpha \cdot e^\beta = \kappa^{\langle\alpha,\beta\rangle} e^\beta \cdot e^\alpha$ for all $\alpha,\beta \in L$, or equivalently a central extension (6.1.1) determined by the commutator map $c(\alpha,\beta) = \kappa^{\langle\alpha,\beta\rangle}$. We can give such a central extension explicitly. Let $A = \{\alpha^1,\dots,\alpha^{\mathrm{rank}(L)}\}$ be a $\mathbb{Z}$-basis of L. Then define the bimultiplicative map $\epsilon : L\times L \to \mathbb{Z}_2$ by

$$\epsilon(\alpha^i,\alpha^j) = \kappa^{\langle\alpha_i,\alpha_j\rangle} \quad \text{if} \quad i>j \quad \text{and} \quad 1 \quad \text{otherwise}.$$

Then ϵ defines a 2-cocycle in $Z^2(L,\mathbb{Z}_2)$ and satisfies the relation $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha)^{-1} = \kappa^{\langle\alpha,\beta\rangle}$ for all $\alpha,\beta\in L$. Hence, we obtain a central extension of type (6.1.1) with the desired commutator relation. In the following we fix the such above central extension $\hat{L}$. Let $\chi : \langle\kappa\rangle \to \mathbb{C}^*$ be the faithful character defined by $\chi(\kappa) = -1$. Denote by $\mathbb{C}_\chi$ the one-dimensional space $\mathbb{C}$ viewed as a $\langle\kappa\rangle$-module on which $\langle\kappa\rangle$ acts according to χ and denote by $\mathbb{C}\{L\}$ the induced $\hat{L}$-module $\mathbb{C}[\hat{L}]\otimes_{\mathbb{C}[\langle\kappa\rangle]}\mathbb{C}_\chi$. Then $\mathbb{C}\{L\} = \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha\in L\}$ and linearly isomorphic to $\mathbb{C}[L]$. Since $\langle\kappa\rangle$ is a central subgroup, $\mathbb{C}\{L\}$ is an $\hat{L}$-bimodule and hence becomes a twisted group algebra.

Let $\theta : L \ni \alpha \mapsto -\alpha \in L$ be the (-1)-isometry. Then θ naturally acts on $\hat{L}$ by $\theta : e^\alpha \mapsto e^{-\alpha}$ and $\kappa\mapsto\kappa$ by our construction of $\hat{L}$. Let $K = \{\kappa^{\langle\alpha,\alpha\rangle/2}(e^\alpha)^2 \mid a\in L\}$. Then $\pi K = 2L$ and $\theta(e^\alpha)\cdot(e^{-\alpha})\in K$ for all $\alpha\in L$. Also set $R = \{\alpha\in L \mid \langle\alpha,L\rangle\subset 2\mathbb{Z}\}$. Then $2L\subset R$ and the pull-back $\hat{R}$ of R in $\hat{L}$ is the center of $\hat{L}$ and K is a subgroup of $\hat{R}$. In particular, K is a normal subgroup of $\hat{L}$.

Proposition 6.1.1. *([FLM, Proposition 7.4.8]) There are exactly $|R/2L|$ central character $\chi : \hat{R}/K \to \mathbb{C}^*$ of $\hat{L}/K$ such that $\chi(\kappa K) = -1$. For each such χ, there is a unique (up to equivalence) irreducible $\hat{L}/K$-module T_χ with central character χ, and every irreducible $\hat{L}/K$-module on which κK acts as -1 is equivalent to one of these. In particular, viewing T_χ as an $\hat{L}$-module, θe^α on $T_\chi = e^\alpha$ on T_χ for $\alpha\in L$.*

6.1.2 $\mathbb{Z}_2$-orbifold of lattice VOAs

Let V_L be the lattice VOA associated to L. As a linear space, V_L is isomorphic to $M_{\mathbb{C}L}(1,0)\otimes\mathbb{C}\{L\}$, where $\mathbb{C}\{L\}$ described in the previous subsection. Take a $\mathbb{Z}$-basis $A = \{\alpha^1,\dots,\alpha^{\mathrm{rank}(L)}\}$ of L. Then V_L has a linear basis

$$\{\beta_1(-n_1)\cdots\beta_r(-n_r)e^\gamma \mid \beta_i\in A,\ \gamma\in L,\ n_1\geq\cdots\geq n_r\geq 1,\ r\geq 0\}.$$

Let $\theta : L \ni \alpha \mapsto -\alpha \in L$ be the (-1)-isometry on L. Let θ act on V_L as follows:

$$\theta : \beta_1(-n_1)\cdots\beta_r(-n_r)e^{\gamma} \mapsto (-1)^r \cdot \beta_1(-n_1)\cdots\beta_r(-n_r)e^{-\gamma}.$$

Then θ above is not only a linear automorphism on V_L but also preserves the VOA structure of V_L, namely, $\theta \in \mathrm{Aut}(V_L)$. Clearly $\theta^2 = 1$ on V_L. Thus $V_L = V_L^+ \oplus V_L^-$ where $V_L^{\pm}$ are the eigenspace of θ with eigenvalue ± 1. The subalgebra V_L^+ is often called the $\mathbb{Z}_2$-*orbifold* or *charge conjugate orbifold* of V_L. By the quantum Galois theory, V_L^+ is a simple VOA and V_L^- is an irreducible V_L^+-module. In the study of V_L^+ we naturally meet the θ-twisted representations of V_L.

6.1.3 $\mathbb{Z}_2$-twisted representations

Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. Consider the twisted affine Lie algebra $\hat{\mathfrak{h}}_T = \mathbb{C}t^{1/2}[t,t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}c$ with commutator relations

$$[t^m \otimes h, t^n \otimes k] = \delta_{m+n,0} m\langle h,k\rangle c, \quad [\hat{\mathfrak{h}}_T, c] = 0.$$

We often denote $t^n \otimes h$ by $h(n)$. As in the untwisted case, we have a triangular decomposition $\hat{\mathfrak{h}}_T = \hat{\mathfrak{h}}_T^+ \oplus \hat{\mathfrak{h}}_T^0 \oplus \hat{\mathfrak{h}}_T^-$ with $\hat{\mathfrak{h}}_T^{\pm} = \oplus_{n\geq 0} t^{\pm(n+\frac{1}{2})} \otimes \mathfrak{h}$ and $\hat{\mathfrak{h}}^0 = \mathbb{C}c$, and so we can define a highest weight $\hat{\mathfrak{h}}_T$-module $M_{\mathfrak{h}}(1)$. Let $\mathbb{C}$ a one-dimensional $\hat{\mathfrak{h}}_T^0$-module such that $c1 = 1$. Viewing $\mathbb{C}$ as a trivial $\hat{\mathfrak{h}}_T^+$-module, we have the induced module

$$M_{\mathfrak{h},T}(1) := \mathrm{Ind}_{\mathfrak{U}(\hat{\mathfrak{h}}_T^0 + \hat{\mathfrak{h}}_T^+)}^{\mathfrak{U}(\hat{\mathfrak{h}}_T)} \mathbb{C} = \mathfrak{U}(\hat{\mathfrak{h}}_T) \underset{\mathfrak{U}(\hat{\mathfrak{h}}_T^0 + \hat{\mathfrak{h}}_T^+)}{\otimes} 1.$$

Let T_χ be an irreducible $\hat{L}/K$-module as in Proposition 6.1.1. Define a twisted space

$$V_L^{T_\chi} := M_{\mathfrak{h},T}(1) \underset{\mathbb{C}}{\otimes} T_\chi.$$

Define the operator $Z_\theta(e^\alpha, z)$, $\alpha \in L$, of $\mathrm{End}(V_L^{T_\chi})[[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]]$ as

$$Z_\theta(e^\alpha, z) := 2^{-\langle \alpha,\alpha\rangle} \exp\left(\sum_{n\in \frac{1}{2}+\mathbb{N}} \frac{\alpha(-n)}{-n} z^n \right) \exp\left(\sum_{n\in\frac{1}{2}+\mathbb{N}} \frac{\alpha(n)}{n} z^{-n} \right) \otimes e^\alpha z^{-\langle\alpha,\alpha\rangle/2}.$$

For $v = \alpha_1(-n_1)\cdots\alpha_r(-n_r)e^\beta \in V_L$, we define

$$W_\theta(v,z) := {}^\circ_\circ \left(\frac{1}{(n_1-1)!} \partial_z \alpha_1(z) \right) \cdots \left(\frac{1}{(n_r-1)!} \partial_z \alpha_r(z) \right) Z_\theta(e^\beta, z) {}^\circ_\circ,$$

where $\alpha_i(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} \alpha_i(n) z^{-n-1}$ and the normal ordering ${}^\circ_\circ A {}^\circ_\circ$ is inductively defined by following rule:

$${}^\circ_\circ \alpha(m) X {}^\circ_\circ = \begin{cases} \alpha(m) \cdot {}^\circ_\circ X {}^\circ_\circ & \text{if } m < 0, \\ {}^\circ_\circ X {}^\circ_\circ \cdot \alpha(m) & \text{if } m > 0 \end{cases}$$

for $\alpha \in \mathfrak{h}$ and $X \in \mathrm{End}(V_L^{T_\chi})$. Let $c_{m,n}$ be the complex numbers determined by the formula

$$-\log\left(\frac{(1+x)^{1/2}+(1+y)^{1/2}}{2}\right) = \sum_{m,n\geq 0} c_{m,n}x^m y^n,$$

and then set

$$\Delta_z := \sum_{m,n\in\mathbb{N}} \sum_{i=1}^{\mathrm{rank}(L)} c_{m,n}\beta^i(m)\beta^i(n)z^{-m-n}$$

where $\{\beta^1, \ldots, \beta^{\mathrm{rank}(L)}\}$ is an orthonormal basis of $\mathfrak{h}$. Finally we define the θ-twisted vertex operator associated to $a \in V_L$ to be

$$Y_{V_L^{T_\chi}}(a,z) := W_\theta(e^{\Delta_z}a, z).$$

Then we have the θ-twisted representation of V_L.

Theorem 6.1.2. *([FLM]) The pair $(V_L^{T_\chi}, Y_{V_L^{T_\chi}}(\cdot, z))$ is an irreducible θ-twisted V_L-module.*

And the following theorem is established in [D2].

Theorem 6.1.3. *([D2]) The set of inequivalent irreducible θ-twisted V_L-modules is given by $\{(V_L^{T_\chi}, Y_{V_L^{T_\chi}}(\cdot, z)\}$ where T_χ runs over the irreducible $\hat{L}/K$-modules described in Proposition 6.1.1.*

6.2 Leech lattice VOA

In this section we construct the Leech lattice VOA V_Λ and its unique irreducible θ-twisted module V_Λ^T explicitly.

Let $\Omega = \{1, 2, \ldots, 24\}$ be a set of 24 elements and $\mathcal{C} \subset \mathcal{P}(\Omega)$ (the power set of Ω) be the binary Golay code. Then $\mathcal{C}$ is a vector space over $\mathbb{F}_2$ of dimension 12 under symmetric difference. We shall fix the following basis for $\mathcal{C}$:

$$C_i = \{i, 1+i, 2+i, 3+i, 4+i, 7+i, 10+i, 12+i\},\ i = 1, \ldots, 11, \quad C_{12} = \Omega.$$

Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space with a basis $\{\alpha_i \mid i = 1, \ldots, 24\}$, and consider the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ such that $\langle\alpha_i, \alpha_j\rangle = 2\delta_{i,j}$. For $C \subset \Omega$ set $\alpha_C := \sum_{i\in C}\alpha_i$. Then Leech lattice Λ can be realized as follows:

$$\Lambda := \sum_{C\in\mathcal{C}} \mathbb{Z}\frac{1}{2}\alpha_C + \sum_{i\in\Omega}\mathbb{Z}(\frac{1}{4}\alpha_\Omega - \alpha_i).$$

Theorem 6.2.1. *The Leech lattice Λ is a positive definite even unimodular lattice with no element of norm 2. It is unique up to isometry.*

By Dong [D1], every irreducible V_Λ-module is isomorphic to V_Λ itself. Thus we have:

Corollary 6.2.2. *The lattice VOA V_Λ is a holomorphic VOA.*

Now consider θ-twisted representation of V_Λ. In the construction of V_Λ, we have to define a 2-cocycle. We define it as follows. The lattice Λ has a basis $\{\beta_1,\dots,\beta_{24}\}$ where $\beta_1 = \frac{1}{4}\alpha_\Omega - \alpha_1$, $\beta_i = \alpha_{i-1} - \alpha_i$ for $i = 2,\dots,12$ and $\beta_{12+j} = \frac{1}{2}\alpha_{C_i}$ for $j = 1,\dots,12$. Define a bimultiplicative form ϵ on Λ by $\epsilon(\beta_i,\beta_j) = (-1)^{\langle\beta_i,\beta_j\rangle}$ if $i > j$, 1 otherwise. Then $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{\langle\alpha,\beta\rangle}$ and $\epsilon(\alpha,\alpha) = (-1)^{\langle\alpha,\alpha\rangle} = 1$ for $\alpha,\beta \in \Lambda$. Let Q be the sublattice of Λ generated by $\{\gamma_1,\dots,\gamma_{24}\}$ where $\gamma_1 = 2\alpha_1$, $\gamma_i = \alpha_{i-1} - \alpha_i$ for $2 \le i \le 23$ and $\gamma_{24} = \frac{1}{2}\alpha_\Omega$. Then one can check that $\epsilon(\gamma_i,\gamma_j) = 1$ for $1 \le i,j \le 24$. Namely, the 2-cocycle vanishes on Q.

Let $\hat\Lambda$ be the central extension of Λ by the cyclic group $\langle\pm 1\rangle$: $1 \to \langle\pm 1\rangle \to \hat\Lambda \to \Lambda \to 1$ which is determined by the 2-cocycle ϵ. Then the pull-back $\hat Q$ of Q in $\hat\Lambda$ is isomorphic to a direct product $\langle\pm\rangle \times Q$ and is a maximal abelian subgroup of $\hat\Lambda$. Let $e : \Lambda \to \hat\Lambda$ be a section determined by ϵ. Then $e^\alpha \cdot e^\beta = e^{\alpha+\beta}$ for $\alpha,\beta \in Q$. The (-1)-isometry θ on Λ acts on $\hat\Lambda$ by $e^\alpha \mapsto e^{-\alpha}$ and $-1 \mapsto -1$. We have $\theta(e^\alpha)\cdot(e^\alpha)^{-1} = (-1)^{\langle\alpha,\alpha\rangle/2}(e^\alpha)^2$ and so if we set $K = \{\theta(x)\cdot x^{-1} \mid x \in \hat\Lambda\}$ then $K \cap \langle\pm 1\rangle = 1$ and $K \subset \hat Q$. Define a character $\psi : \hat Q \to \langle\pm 1\rangle$ by $\psi(Q) = 1$ and $\psi(-1) = -1$. Let $\mathbb{C}_\psi$ be the one-dimensional module for $\hat Q$ defined by the character ψ. Then define the induced $\hat\Lambda$-module $T = \mathbb{C}[\hat\Lambda] \otimes_{\mathbb{C}[\hat Q]} \mathbb{C}_\psi$.

Theorem 6.2.3. *([FLM]) The $\mathbb{C}[\hat\Lambda]$-module T is the unique irreducible module for the quotient group $\hat\Lambda/K$ on which $-K$ acts as a scalar -1. In particular, the associated θ-twisted V_Λ-module V_Λ^T is the unique irreducible θ-twisted representation of V_Λ.*

Remark 6.2.4. It is not difficult to show that $\dim T = 2^{12}$.

The twisted space V_Λ^T is linearly isomorphic to $M_{\mathbb{C}\Lambda,T}(1) \otimes_{\mathbb{C}} T$ by definition. Now define the action of θ on T as -1. Since θ naturally acts on $M_{\mathbb{C}\Lambda,T}(1)$, by letting θ act on $M_{\mathbb{C}\Lambda,T}(1) \otimes T$ as $\theta\otimes\theta$ we have an involutive action of θ on V_Λ^T. Then we have a decomposition $V_\Lambda^T = (V_\Lambda^T)^+ \oplus (V_\Lambda^T)^-$ where $(V_\Lambda^T)^\pm$ are eigenspace for θ with eigenvalue ± 1. By the definition of the twisted vertex operator map on V_L^T, we can show that $L(0)$ acts on T as a scalar $3/2$. So we have $\mathbb{N}$-graded decompositions $(V_\Lambda^T)^+ = \oplus_{n\ge 2}(V_\Lambda)_n^T$ and $(V_\Lambda^T)^- = \oplus_{n\ge 0}(V_\Lambda^T)^-_{n+3/2}$. In particular, both $(V_\Lambda^T)^\pm$ are irreducible V_Λ^+-modules.

At now, we have constructed four inequivalent irreducible V_Λ^+-modules, namely, $V_\Lambda^\pm$ and $(V_\Lambda^T)^\pm$. It is shown in [AD] that they are all the irreducible V_Λ^+-modules:

Theorem 6.2.5. *([D3] [AD]) An irreducible V_Λ^+-module is isomorphic to one and only one of V_Λ^+, V_Λ^-, $(V_\Lambda^T)^+$ and $(V_\Lambda)^-$.*

6.3 Moonshine module

The moonshine module is defined to be $V^\natural := V_\Lambda^+ \oplus (V_\Lambda^T)^+$. The moonshine module is a $\mathbb{Z}$-graded space and it is known that its q-character is the $\mathrm{SL}_2(\mathbb{Z})$-invariant j-function $q^{-c/24}\mathrm{ch}_{V^\natural}(\tau) = J(q) = j(q) - 744$. The following result is established in [FLM]:

Theorem 6.3.1. *([FLM]) The $\mathbb{Z}$-graded space $V^\natural$ has a natural vertex operator algebra structure with central charge 24 and its automorphism group is the Monster finite simple sporadic group $\mathbb{M}$.*

The original proof in [FLM] uses many results from the finite group theory, especially Griess' result in [G] and Conway's result [C]. Let us explain their ideas briefly. There is a sub VOA $U^{(00)}$ of V_Λ^+ and its inequivalent irreducible modules $U^{(01)}$, $U^{(10)}$ and $U^{(11)}$ that such that $V_\Lambda^+ = U^{(00)} \oplus U^{(01)}$ and $(V_\Lambda^T)^+ = U^{(10)} \oplus U^{(11)}$. Then they construct an automorphism $\sigma \in \mathrm{Aut}(U^{(00)})$, called *triality*, such that σ permutes $U^{(01)}$, $U^{(10)}$ and $U^{(11)}$. Using σ, we can mix the untwisted space V_Λ^+ and the twisted space $(V_\Lambda^T)^+$ and hence we can introduce an algebraic operation on $(V_\Lambda^T)^+$ by which we define a vertex operator algebra structure on $V_\Lambda^+ \oplus (V_\Lambda^T)^+$.

On the other hand, Huang suggested another proof of the theorem above by using a theory of fusion products which he and Lepowsky devised. Here we present Huang's proof of the existence of a vertex operator algebra structure on $V^\natural$. First, we prepare some facts about V_Λ^+.

Theorem 6.3.2. *([Ab] [ABD] [M10] [Yams]) The $\mathbb{Z}_2$-orbifold V_L^+ is rational C_2-cofinite VOA of CFT-type for any positive definite even lattice L.*

Theorem 6.3.3. *([D3] [ADL]) The fusion algebra for V_Λ^+ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ by the following association:*

$$V_\Lambda^+ \leftrightarrow (0,0), \quad V_\Lambda^- \leftrightarrow (1,0), \quad (V_\Lambda^T)^+ \leftrightarrow (0,1), \quad (V_\Lambda^T)^- \leftrightarrow (1,1).$$

Remark 6.3.4. It is shown in [ADL] that Theorem 6.3.3 is still true if we replace Λ by any positive definite even unimodular lattice.

Now we prove

Theorem 6.3.5. *([H3]) The extension $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$ has a structure of a simple vertex operator algebra with central charge 24 as a $\mathbb{Z}_2$-graded simple current extension of V_Λ^+.*

Proof: We need to define a vertex operator map on $V^\natural$. Since all irreducible V_Λ^+-modules are self-dual, we can use a method in [FHL]. For $a, b \in V_\Lambda^+$ and $u, v \in (V_\Lambda^T)^+$, we define $Y^\natural(a,z)b := Y_{V_\Lambda}(a,z)b$, $Y^\natural(a,z)u := Y_{V_\Lambda^T}(a,z)u$, $Y^\natural(u,z)a := e^{zL(-1)}Y^\natural(a,-z)u$, and $Y^\natural(u,z)v \in V_\Lambda^+((z))$ is defined by

$$\langle a, Y^\natural(u,z)v\rangle_{V_\Lambda^+} = \langle Y_{V_\Lambda^T}(a,-z^{-1})e^{zL(1)}(-z^{-2})^{L(0)}u, e^{z^{-1}L(1)}v\rangle_{(V_\Lambda^T)^+}$$

for all $a \in V_\Lambda^+$. Then it is shown in Theorem 5.6.2 of [FHL] that $Y^\natural(\cdot,z)$ on $(V_\Lambda^T)^+ \otimes (V_\Lambda^T)^+$ is a V_Λ^+-intertwining operator of type $(V_\Lambda^T)^+ \times (V_\Lambda^T)^+ \to V_\Lambda^+$ and the Jacobi identity for $Y^\natural(\cdot,z)$ is satisfied except the case for three module elements in $(V_\Lambda^T)^+$. Therefore, by the theory of local systems, we only need to show the mutually commutativity

$$(z_1-z_2)^N Y^\natural(u,z_1)Y^\natural(v,z_2)w = (z_1-z_2)^N Y^\natural(v,z_2)Y^\natural(u,z_1)w$$

for $u,v,w \in (V_\Lambda^T)^+$ and $N \gg 0$. By Theorem 3.7.5 (cf. [H1] [H4]) there is a non-zero scalar λ such that

$$(z_1-z_2)^N Y^\natural(u,z_1)Y^\natural(v,z_2)w = \lambda(z_1-z_2)^N Y^\natural(v,z_2)Y^\natural(u,z_1)w$$

holds for any three elements $u,v,w \in (V_\Lambda^T)^+$ and suitable $N \gg 0$ since all irreducible V_Λ^+-modules are simple currents. So we should prove that $\lambda = 1$. It is shown in [FLM] that for a suitable element $x \in (V_\Lambda^T)_2^+$ we have $\langle x,x\rangle_{(V_\Lambda^T)^+} = 1$. Thus by Theorem 3.4.6 the V_Λ^+-invariant bilinear form on $(V_\Lambda^T)^+$ is symmetric. Then it is shown in Proposition 5.6.1 of [FHL] that $Y^\natural(\cdot,z)$ satisfies the skew-symmetry. Then the skew-symmetry together with associativity for $Y^\natural(\cdot,z)$ provides the commutativity for $Y^\natural(\cdot,z)$. This completes the proof. ∎

We call the structure $(V^\natural, Y^\natural(\cdot,z))$ *the moonshine vertex operator algebra.*

Corollary 6.3.6. *(cf. [H3]) Let τ be an involution on $V^\natural$ such that $\tau = 1$ on V_Λ^+ and $\tau = -1$ on $(V_\Lambda^T)^+$. Then $V_\Lambda^- \oplus (V_\Lambda^T)^-$ is a unique irreducible τ-twisted $V^\natural$-module.*

Proof: By the fusion rules for V_Λ^+ and Theorem 4.5.3, an irreducible V_Λ^+-module V_Λ^- is uniquely lifted to be a τ-twisted $V^\natural$-module. ∎

Remark 6.3.7. It is known that the involution τ in the theorem above belongs to the 2B-conjugacy class of the Monster.

Remark 6.3.8. In [H3], Huang proved a stronger theorem which includes both Theorem 6.3.5 and Corollary 6.3.6. He proved that the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-graded space $W^\natural := V_\Lambda^+ \oplus V_\Lambda^- \oplus (V_\Lambda^T)^+ \oplus (V_\Lambda^T)^-$ has a structure of an abelian intertwining operator algebra with central charge 24.

Remark 6.3.9. As Huang said in [H3], his approach to prove the vertex operator algebra structure is independent of the triality and the finite group theory. This independence allows us to obtain another proof of the theorem saying that the Monster is the full automorphism group of the moonshine vertex operator algebra based on the theorem saying that the Monster is the full automorphism group of the Griess algebra proved by Griess [G] and Tits [Ti], simplified by Conway [C].

6.4 2A-frame

Let $\mathbb{Z}\gamma$ be a lattice of type $\sqrt{2}A_1$, that is, $\langle \gamma, \gamma\rangle = 4$, and consider a lattice VOA $V_{\mathbb{Z}\gamma}$. In $V_{\mathbb{Z}\gamma}$, we can find a pair of mutually orthogonal conformal vectors with central charge $1/2$ as follows:

$$w^0 = \frac{1}{16}\gamma_{(-1)}^2 \mathbb{1} + \frac{1}{4}(e^{\gamma} + e^{-\gamma}), \quad w^1 = \frac{1}{16}\gamma_{(-1)}^2 \mathbb{1} - \frac{1}{4}(e^{\gamma} + e^{-\gamma}).$$

We also note that $\omega = w^0 + w^1$ is an orthogonal decomposition of the Virasoro vector ω of $V_{\mathbb{Z}\gamma}$ and both w^0 and w^1 are contained in the $\mathbb{Z}_2$-orbifold $V_{\mathbb{Z}\gamma}^+$. Since the invariant bilinear form on $V_{\mathbb{Z}\gamma}^+$ is positive definite, the subalgebras generated by w^i, $i = 0, 1$, are isomorphic to the simple unitary Virasoro VOA $L_{\mathrm{Vir}}(1/2, 0)$. Therefore, $V_{\mathbb{Z}\gamma}^+$ contains a subalgebra isomorphic to $L_{\mathrm{Vir}}(1/2, 0) \otimes L_{\mathrm{Vir}}(1/2, 0)$.

Recall the Leech lattice Λ constructed in Section 6.2. One can easily verify that the sublattice generated by $\{\alpha_{2i-1} \pm \alpha_{2i} \mid 1 \le i \le 12\}$ is a direct sum of 24 copies of $\mathbb{Z}\gamma$ above, or equivalently is isomorphic to a lattice of type $\sqrt{2}A_1^{\oplus 24}$. Therefore, the $\mathbb{Z}_2$-orbifold V_Λ^+ contains a tensor product $(V_{\mathbb{Z}\gamma}^+)^{\otimes 24}$ of 24 copies of $V_{\mathbb{Z}\gamma}^+$ and hence we obtain an embedding of $L_{\mathrm{Vir}}(1/2, 0)^{\otimes 48}$ into the moonshine VOA $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$.

Theorem 6.4.1. *([DMZ]) There is an orthogonal decomposition $\omega^\natural = e^1 + \cdots + e^{48}$, where $\omega^\natural$ is the Virasoro vector of $V^\natural$, such that each of e^i, $1 \le i \le 48$, generates $L_{\mathrm{Vir}}(1/2, 0)$ inside $V^\natural$.*

Remark 6.4.2. This fact has been generalized in [DLMN].

Since [DMZ] the study of $V^\natural$ as a module for the unitary Virasoro VOA $L_{\mathrm{Vir}}(1/2, 0)$ was started by many mathematicians. In particular, Miyamoto succeeded to reconstruct the moonshine VOA $V^\natural$ from the representation theory of $L_{\mathrm{Vir}}(1/2, 0)$ in [M5]. Miyamoto's theory contains many important results on the moonshine VOA and also on the Monster simple group. In the next section, we review Miyamoto's reconstruction of the moonshine VOA.

Chapter 7

The Moonshine VOA II: Miyamoto Construction

In this section we review Miyamoto's reconstruction of the moonshine vertex operator algebra which uses representations of the unitary Virasoro VOA $L_{\mathrm{Vir}}(1/2,0)$. Throughout this section, we denote $L_{\mathrm{Vir}}(c,h)$ simply by $L(c,h)$.

7.1 The Ising model SVOA

In this section we will give an explicit construction of the Ising model SVOA $L(1/2,0)\oplus L(1/2,1/2)$ and its $\mathbb{Z}_2$-twisted modules $L(1/2,1/16)^{\pm}$. This construction is well-known and the most of contents in this section can be found in [KR], [FFR] and [FRW].

7.1.1 Realization of Ising models

Let $\mathcal{A}_\psi$ be the algebra generated by $\{\psi_k \mid k\in\mathbb{Z}+\frac{1}{2}\}$ subject to the defining relations

$$[\psi_m,\psi_n]_+ := \psi_m\psi_n+\psi_n\psi_m=\delta_{m+n,0}, \quad m,n\in\mathbb{Z}+\tfrac{1}{2},$$

and denote a subalgebra of $\mathcal{A}_\psi$ generated by $\{\psi_k \mid k\in\mathbb{Z}+\frac{1}{2}, k>0\}$ by $\mathcal{A}_\psi^+$. Let $\mathbb{C}\mathbb{1}$ be a trivial $\mathcal{A}_\psi^+$-module. Define a canonical induced $\mathcal{A}_\psi$-module M by

$$M := \mathrm{Ind}_{\mathcal{A}_\psi^+}^{\mathcal{A}_\psi}\mathbb{C}\mathbb{1}=\mathcal{A}_\psi\underset{\mathcal{A}_\psi^+}{\otimes}\mathbb{C}\mathbb{1}.$$

We can define a unique symmetric contravariant Hermitian form $\langle\cdot|\cdot\rangle$ on M such that $\langle\mathbb{1}|\mathbb{1}\rangle=1$ and $\langle\psi_n a|b\rangle=\langle a|\psi_{-n}b\rangle$ for all $n\in\mathbb{Z}$.

We can find a representation of the Virasoro algebra on M. Following [KR], set

$$L^M(n) := \frac{1}{2}\sum_{k>-n/2}(n+2k)\psi_{-k}\psi_{n+k}, \quad n\in\mathbb{Z}. \tag{7.1.1}$$

Then $\{L^M(n) \mid n \in \mathbb{Z}\}$ gives a representation of the Virasoro algebra with central charge $1/2$ on M. Since the invariant Hermitian bilinear form on M is clearly Vir-invariant, M is a completely reducible Vir-module. The unitary highest weight representations for the Virasoro algebra with central charge is only $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$ (cf. [KR]), we have the the following decomposition

$$M = L(1/2,0) \oplus L(1/2,1/2)$$

as a Vir-module. The highest weight vectors of $L(1/2,0)$ and $L(1/2,1/2)$ are $\mathbb{1}$ and $\psi_{-\frac{1}{2}}\mathbb{1}$, respectively. It is clear that the decomposition above coincides with the standard $\mathbb{Z}_2$-graded decomposition

$$L(1/2,\delta/2) = \mathrm{Span}_{\mathbb{C}}\left\{\psi_{-n_1}\cdots\psi_{-n_k}\mathbb{1} \;\middle|\; n_1 > \cdots > n_k > 0,\ n_1+\cdots+n_k \in \mathbb{Z}+\delta/2\right\}$$

for $\delta = 0,1$, and we also note that the basis above is an orthonormal basis for $(M, \langle\cdot|\cdot\rangle)$.

Another unitary Vir-module $L(1/2,1/16)$ is realized as follows. Let $\mathcal{A}_\phi$ be the other algebra generated by $\{\phi_n \mid n \in \mathbb{Z}\}$ with defining relation

$$[\phi_m, \phi_n]_+ = \delta_{m+n,0}, \quad m,n \in \mathbb{Z}.$$

Let $\mathcal{A}_\phi^+$ be a subalgebra of $\mathcal{A}_\phi$ generated by $\{\phi_n | n > 0\}$ and let $\mathbb{C}v_0$ be a trivial one-dimensional $\mathcal{A}_\phi^+$-module. Then set $N = \mathrm{Ind}_{\mathcal{A}_\phi^+}^{\mathcal{A}_\phi}\mathbb{C}v_0$ as we did previously. We can introduce a symmetric contravariant Hermitian bilinear form $\langle\cdot|\cdot\rangle$ on N such that $\langle v_0|v_0\rangle = 1$, $\langle v_0|\phi_0 v_0\rangle = \langle \phi_0 v_0|v_0\rangle = 0$ and $\langle \phi_n a|b\rangle = \langle a|\phi_{-n}b\rangle$.

We can find an action of the Virasoro algebra on N. Set

$$L^N(n) := \frac{1}{16}\delta_{n,0} + \frac{1}{2}\sum_{k>-n/2}(n+2k)\phi_{-k}\phi_{n+k}, \quad n \in \mathbb{Z}. \tag{7.1.2}$$

Then $\{L^N(n) \mid n \in \mathbb{Z}\}$ defines an action of the Virasoro algebra with central charge $1/2$ on N. The invariant Hermitian bilinear form on N is clearly Vir-invariant, so N is a direct sum of irreducible unitary highest weight modules for the Virasoro algebra. In N we can find two distinct highest weight vectors v_0 and $\phi_0 v_0$ with highest weight $1/16$ and so N decomposes as follows (cf. [KR]):

$$N = L(1/2,1/16) \oplus L(1/2,1/16).$$

Remark 7.1.1. The irreducible Vir-modules $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$ are called *Ising model*.

7.1.2 SVOA structure on Ising models

We keep the same notation as previous. By its construction, M is generated by $\mathbb{1}$ over $\mathcal{A}_\psi$. Define the generating series

$$\psi(z) := \sum_{n\in\mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1}.$$

Since $[\psi(z), \psi(w)]_+ = z^{-1}\delta(\frac{w}{z})$, $\psi(z)$ is local with itself and it follows from the defining relations of $\mathcal{A}_\psi$ that $\psi(z)$ satisfies $L^M(-1)$-derivation property: $[L^M(-1), \psi(z)] = \partial_z\psi(z)$. Therefore we can consider a subalgebra of a local system on M generated by $\psi(z)$ and $I(z) = \mathrm{id}_M$. By a direct calculation, one sees that

$$\frac{1}{2}\psi(z) \circ_{-2} \psi(z) = \sum_{n\in\mathbb{Z}} L^M(n) z^{-n-2},$$

where $\circ_n$ denotes the n-th normal ordered product defined in Section 3.2. Since we have a basis

$$\psi_{-n_1+\frac{1}{2}}\psi_{-n_2+\frac{1}{2}} \cdots \psi_{-n_k+\frac{1}{2}}\mathbb{1},\ n_1 > n_2 > \cdots > n_k > 0,\ k \geq 0$$

of M, we can define a vertex operator superalgebra structure on M by defining a vertex operator of each base element above. For $k = 0$ we set $Y(\mathbb{1}, z) := \mathrm{id}_M$ and inductively we define $Y_M(\psi_{-n+\frac{1}{2}}a, z) := \psi(z) \circ_n Y(a, z)$. Then by the theory of the local system, we have the following well-known statement.

Theorem 7.1.2. *By the above definition,* $(L(1/2, 0) \oplus L(1/2, 1/2), Y_M(\cdot, z), \mathbb{1}, \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\mathbb{1})$ *has a unique simple SVOA structure with even part* $L(1/2, 0)$ *and odd part* $L(1/2, 1/2)$*.*

Note that the invariant bilinear form on M coincides with the contravariant bilinear form on M since $\psi_n^* = \psi_{-n}$ for all $n \in \mathbb{Z} + \frac{1}{2}$.

Next, we consider $L(1/2, 0) \oplus L(1/2, 1/2)$-module structures on $L(1/2, 1/16)$. There are two highest weight vectors v_0 and $\phi_0 v_0$ in N and each of them generates $L(1/2, 1/16)$ under the Virasoro algebra. Set $v^{\pm}_{\frac{1}{16}} := \phi_0\mathbb{1} \pm \frac{1}{\sqrt{2}}\mathbb{1}$. Then we have $\phi_0 \cdot v^{\pm}_{\frac{1}{16}} = \pm\frac{1}{\sqrt{2}} v^{\pm}_{\frac{1}{16}}$. Both of $v^+_{\frac{1}{16}}$ and $v^-_{\frac{1}{16}}$ are highest weight vectors and each of them generates $L(1/2, 1/16)$ under the Virasoro algebra. Denote the Vir-modules generated over $v^{\pm}_{\frac{1}{16}}$ by $L(1/2, 1/16)^{\pm}$, respectively. Then we have $N = L(1/2, 1/16)^+ \oplus L(1/2, 1/16)^-$. Note that $L(1/2, 1/16)^+$ and $L(1/2, 1/16)^-$ are isomorphic as Vir-modules but they are not isomorphic to each other as $\mathcal{A}_\phi$-modules. Consider the generating series

$$\phi(z) := \sum_{n\in\mathbb{Z}} \phi_n z^{-n-\frac{1}{2}}.$$

By direct calculations one can show that $\phi(z)$ is local with itself and satisfies the derivation property $[L^N(-1), \phi(z)] = \partial_z\phi(z)$. Now consider a local system on N containing $\phi(z)$.

Since the powers of z in $\phi(z)$ lie in $\mathbb{Z}+\frac{1}{2}$, we have to use the twisted normal ordered product in [Li2]. Define a generating series $L(z)$ of operators on N by

$$L(z_2) := \frac{1}{2}\mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^{-2}\left(\frac{z_1-z_0}{z_2}\right)^{\frac{1}{2}}$$
$$\times\left\{z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\phi(z_1)\phi(z_2)+z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\phi(z_2)\phi(z_1)\right\}.$$

Then we have the following by a direct computation.

Lemma 7.1.3. $L(z)=\sum_{n\in\mathbb{Z}} L^N(n)z^{-n-2}$, *where* $L^N(n)$ *are defined by* (7.1.2).

Thanks to the above lemma, we can find a $\mathbb{Z}_2$-twisted $L(1/2,0)\oplus L(1/2,1/2)$-module structure on $L(1/2,1/16)^{\pm}$. We will associate a vertex operator on N for every element in $L(1/2,0)\oplus L(1/2,1/2)$. And then we prove that these vertex operators define a homomorphism of vertex superalgebras. Set $Y_N(\mathbb{1},z):=\mathrm{id}_N$, and define inductively a vertex operator of $\psi_{-n+\frac{1}{2}}a$ on N by

$$Y_N(\psi_{-n+\frac{1}{2}}a,z) := \frac{1}{2}\mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^{-n}\left(\frac{z_1-z_0}{z_2}\right)^{\frac{1}{2}}$$
$$\times\left\{z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\phi(z_1)Y_N(a,z_2)-(-1)^{|a|}z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)Y_N(a,z_2)\phi(z_1)\right\},$$

where $a=\psi_{-n_1+\frac{1}{2}}\cdots\psi_{-n_k+\frac{1}{2}}\mathbb{1}$, $n>n_1>\cdots>n_k>0$, and $|a|$ denotes the parity of a. Then extend linearly on $L(1/2,0)\oplus L(1/2,1/2)$. Let $\mathfrak{A}$ be a $\mathbb{Z}_2$-twisted local system on N in which $\phi(z)$ is contained (cf. [Li2]). It is shown in [Li2, Theorem 3.14] that $\mathfrak{A}$ is a vertex superalgebra under the $\mathbb{Z}_2$-twisted normal ordered product.

Lemma 7.1.4. *The linear map* $L(1/2,0)\oplus L(1/2,1/2)\ni a\mapsto Y_N(a,z)\in\mathfrak{A}$ *defined above gives an vertex superalgebra homomorphism.*

Proof: We should show that $Y_N(a_{(m)}b,z)=Y_N(a,z)\circ_m Y_N(b,z)$ for any $a,b\in L(1/2,0)\oplus L(1/2,1/2)$ and $m\in\mathbb{Z}$, where $\circ_m$ denotes the $\mathbb{Z}_2$-twisted m-th normal ordered product in $\mathfrak{A}$. We may assume that $a=\psi_{-n_1+\frac{1}{2}}\cdots\psi_{-n_k+\frac{1}{2}}\mathbb{1}$, $n_1>\cdots>n_k>0$. We proceed by induction on k. The case $k=0$ is trivial and the case $k=1$ is just the definition. Assume that $k\geq 1$ and $Y_N(a_{(m)}b,z)=Y_N(a,z)\circ_m Y_N(b,z)$ holds for arbitrary

$b \in L(1/2,0) \oplus L(1/2,1/2)$ and $m \in \mathbb{Z}$. Take any $n > n_1$. Then we have

$$\begin{aligned}
&Y_N\left((\psi_{-n+\frac{1}{2}}a)_{(m)}b,z\right)\\
&=\sum_{i=0}^{\infty}(-1)^i\binom{-n}{i}Y_N\left(\psi_{-n-i+\frac{1}{2}}a_{(m+i)}b-(-1)^{|a|-n}a_{(-n+m-i)}\psi_{i+\frac{1}{2}}b,z\right)\\
&=\sum_{i=0}^{\infty}(-1)^i\binom{-n}{i}\Big\{\phi(z)\circ_{-n-i}Y_N(a_{(m+i)}b,z)\\
&\qquad\qquad -(-1)^{|a|-n}Y_N(a,z)\circ_{-n+m-i}Y_N(\psi_{i+\frac{1}{2}}b,z)\Big\}\\
&=\sum_{i=0}^{\infty}(-1)^i\binom{-n}{i}\Big\{\phi(z)\circ_{-n-i}\Big(Y_N(a,z)\circ_{m+i}Y_N(b,z)\Big)\\
&\qquad\qquad -(-1)^{|a|-n}Y_N(a,z)\circ_{-n+m-i}\Big(\phi(z)\circ_i Y_N(b,z)\Big)\Big\}\\
&=\Big(\phi(z)\circ_{-n}Y_N(a,z)\Big)\circ_m Y_N(b,z) \qquad \text{by the iterate formula in } \mathfrak{A}\\
&=Y_N\left(\psi_{-n+\frac{1}{2}}a,z\right)\circ_m Y_N(b,z).
\end{aligned}$$

Therefore, by induction, the mapping $M \ni a \mapsto Y_N(a,z) \in \mathfrak{A}$ defines a vertex superalgebra homomorphism. ∎

Let V be an arbitrary SVOA. By Proposition 3.17 in [Li2], giving a $\mathbb{Z}_2$-twisted V-module structure on N is equivalent to giving a vertex superalgebra homomorphism from V to a local system of $\mathbb{Z}_2$-twisted vertex operators on N. Since both $L(1/2,1/16)^+$ and $L(1/2,1/16)^-$ are stable under the action $Y_N(\cdot\,,z)$ define as above, we arrive at the following conclusion:

Theorem 7.1.5. *The following $\mathbb{Z}_2$-twisted Jacobi identity holds on N:*

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_N(a,z_1)Y_N(b,z_2)-(-1)^{\varepsilon(a,b)}z_0^{-1}\left(\frac{-z_2+z_1}{z_0}\right)Y_N(b,z_2)Y_N(a,z_1)\\
&=z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)\left(\frac{z_2+z_0}{z_1}\right)^{\varepsilon(a,a)/2}Y_N(Y_M(a,z_0)b,z_2),
\end{aligned}$$

where $a,b \in M = L(1/2,0)\oplus L(1/2,1/2)$ and $\varepsilon(\cdot,\cdot)$ denotes the standard parity function. Therefore, the vertex operator map $Y_N(\cdot,z)$ defines inequivalent irreducible $\mathbb{Z}_2$-twisted $L(1/2,0)\oplus L(1/2,1/2)$-module structures on $L(1/2,1/16)^{\pm}$.

Remark 7.1.6. The vertex operator $Y_N(\cdot,z)$ gives $L(1/2,0)$-intertwining operators of type $L(1/2,\delta)\times L(1/2,1/16)\to L(1/2,1/16)$ for $\delta = 0,1/2$. Therefore, Theorem 7.1.5 gives another proof of Proposition 4.1 and 4.2 of [M3]. In particular, by the explicit constructions above, we can perform explicit calculates of all the intertwining operations of any type for the unitary Virasoro VOA $L(1/2,0)$ by using the symmetry of fusion rules.

7.2 Miyamoto involution

Let us consider the fusion rules for the unitary Virasoro VOA $L(c_m,0)$ with $c_m = 1 - 6/(m+2)(m+3)$, $m \in \mathbb{N}$. Recall that the set of inequivalent irreducible $L(c_m,0)$ is $\{L(c_m,h_{r,s}^{(m)}) \mid 1 \le s \le r \le m+1\}$, where $h_{r,s}^{(m)}$ are defined by (5.4.1), and their fusion rules are given by the formula (5.4.2). Since $h_{m+2-r,m+3-s}^{(m)} = h_{r,s}^{(m)}$, we can find the following $\mathbb{Z}_2$-symmetry in the fusion algebra for $L(c_m,0)$.

Lemma 7.2.1. *The following linear map defines an automorphism of the fusion algebra for $L(c_m,0)$:*

$$\tau := \begin{cases} (-1)^{r+1} & \text{on} \quad L(c_m,h_{r,s}^{(m)}) \quad \text{if } m \text{ is even,} \\ (-1)^{s+1} & \text{on} \quad L(c_m,h_{r,s}^{(m)}) \quad \text{if } m \text{ is odd.} \end{cases}$$

Then by Proposition 3.8.9 the above $\mathbb{Z}_2$-symmetry will lift to be an involutive automorphism of a vertex operator algebra:

Theorem 7.2.2. *([M1]) Assume that a vertex operator algebra V contains a sub VOA (U,e) isomorphic to the unitary Virasoro VOA $L(c_m,0)$. Then we have a decomposition $V = \oplus_{1\le s\le r\le m+1} V(h_{r,s}^{(m)})$ with $V(h_{r,s}^{(m)}) := L(c_m,h_{r,s}^{(m)}) \otimes \mathrm{Hom}_U(L(c_m,h_{r,s}^{(m)}),V)$. Define the linear map τ_e which acts on $V(h_{r,s}^{(m)})$ as $(-1)^{r+1}$ if m is even and as $(-1)^{s+1}$ if m is odd. Then τ_e is an automorphism of a vertex operator algebra V.*

Proof: By Proposition 3.8.9 the vertex operator map $Y_V(\cdot,z)$ can be written as a sum of tensor products $I(\cdot,z)\otimes J(\cdot,z)$ where $I(\cdot,z)$ are $L(c_m,0)$-intertwining operators and $J(\cdot,z)$ are complement of $I(\cdot,z)$ in $Y_V(\cdot,z)$. Therefore, the automorphism of the fusion algebra for $L(c_m,0)$ can be extended to be that of the vertex operator algebra structure on V. ∎

Among the unitary series, the first the first unitary Virasoro VOA $L(1/2,0)$ is especially important. It is a rational VOA of CFT-type and has exactly three irreducible modules $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$ (cf. [DMZ] [Wan]), which we have constructed explicitly in the previous subsection. By Proposition 2.6.2 we can also verify that $L(1/2,0)$ is C_2-cofinite. The fusion rules are as follows:

$$\begin{aligned} &L(1/2,1/2)\times L(1/2,1/2) = L(1/2,0),\\ &L(1/2,1/2)\times L(1/2,1/16) = L(1/2,1/16), \\ &L(1/2,1/16)\times L(1/2,1/16) = L(1/2,0)+L(1/2,1/2), \end{aligned} \tag{7.2.1}$$

where $L(1/2,0)$ is of course the identity. One can directly check that the fusion algebra for $L(1/2,0)$ is associative and $L(1/2,0)$ and $L(1/2,1/2)$ are simple currents. Let V be a VOA and (U,e) be a sub VOA of V isomorphic to $L(1/2,0)$. Then the involution defined in Theorem 7.2.2 is as follows:

$$\tau_e := \ 1 \quad \text{on } V(0) \oplus V(1/2) \quad \text{and} \quad -1 \quad \text{on } V(1/16),$$

where $V(h) = L(1/2,h) \otimes \mathrm{Hom}_U(L(1/2,h),V)$ for $h = 0, 1/2, 1/16$. If τ_e is trivial on V, then there is no component $V(1/16)$. Then the linear map

$$\sigma_e := \ 1 \quad \text{on } V(0) \quad \text{and} \quad -1 \quad \text{on } V(1/2),$$

also defines an automorphism on V by the fusion rules (7.2.1). These automorphisms are often called *Miyamoto involutions*.

Remark 7.2.3. It is shown in [M1] that the first Miyamoto involution τ_e belongs to the 2A-conjugacy class of the Monster $\mathbb{M}$ for any sub VOA $(L(1/2,0),e)$ of the moonshine VOA. Moreover, the correspondence $(L(1/2,0),e) \leftrightarrow \tau_e \in \mathbb{M}$ is one to one by [C].

Remark 7.2.4. Let A be the set of all conformal vectors with central charge $1/2$ in V. Then the subgroup E generated by the first Miyamoto involutions $\{\tau_e \mid e \in A\}$ a normal subgroup of $\mathrm{Aut}(V)$. Consider the fixed point subalgebra V^E. Then it is shown in [M1] that the subgroup of $\mathrm{Aut}(V^E)$ generated by the second Miyamoto involutions $\{\sigma_e \mid e \in E\}$ is a 3-transposition group.

7.3 Code VOAs

Definition 7.3.1. A simple vertex operator algebra (V,ω) is called *2A-framed* if there is an orthogonal decomposition $\omega = e^1 + \cdots + e^n$ such that each e^i generates a sub VOA isomorphic to $L(1/2,0)$. The decomposition $\omega = e^1 + \cdots + e^n$ is called a *2A-frame* of V.

Remark 7.3.2. As we have seen in Section 6.4, the Leech lattice VOA V_Λ and the moonshine VOA $V^\natural$ are examples of 2A-framed VOAs.

Let (V,ω) be a 2A-framed VOA with a 2A-frame $\omega = e^1 + \cdots + e^n$. Set $T := \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^n)$, where $\mathrm{Vir}(e^i)$ denotes the sub VOA generated by e^i. Then $T \simeq L(1/2,0)^{\otimes n}$ and V is a direct sum of irreducible T-submodules $\otimes_{i=1}^n L(1/2,h_i)$ with $h_i \in \{0,1/2,1/16\}$. For each irreducible T-module $\otimes_{i=1}^n L(1/2,h_i)$, we associate its 1/16-word $(\alpha_1,\cdots,\alpha_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ by the rule $\alpha_i = 1$ if and only if $h_i = 1/16$. For each $\alpha \in (\mathbb{Z}/2\mathbb{Z})^n$, denote by V^α the sum of all irreducible T-submodules whose 1/16-words are equal to α and define a linear code $S \subset (\mathbb{Z}/2\mathbb{Z})^n$ by $S = \{\alpha \in (\mathbb{Z}/2\mathbb{Z})^n \mid V^\alpha \neq 0\}$. Then we have the 1/16-word decomposition $V = \oplus_{\alpha\in S} V^\alpha$ of V. By the fusion rules (7.2.1), we have an

S-graded structure $V^\alpha \cdot V^\beta \subset V^{\alpha+\beta}$ for all $\alpha, \beta \in S$. Namely, the dual group S^* of an abelian 2-group S acts on V, and we find that this automorphism group coincides with the elementary abelian 2-group generated by the first Miyamoto involutions $\{\tau_{e^i} \mid 1 \le i \le n\}$. Therefore, all V^α, $\alpha \in S$, are irreducible $V^{S^*} = V^0$-module by the quantum Galois theory (cf. [DM1]). Since there is no $L(1/2, 1/16)$-component in V^0, the fixed point subalgebra V^0 has the following shape:

$$V^0 = \bigoplus_{h_i \in \{0,1/2\}} m_{h_1,\dots,h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),$$

where $m_{h_1,\dots,h_n}$ denotes the multiplicity. On V^0 we can define the second Miyamoto involutions σ_{e^i} for $i = 1, \dots, n$. Denote by Q the elementary abelian 2-subgroup of $\mathrm{Aut}(V^0)$ generated by $\{\sigma_{e^i} \mid 1 \le i \le n\}$. Then by the quantum Galois theory we have $(V^0)^Q = T$ and each $m_{h_1,\cdots,h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ is an irreducible T-submodule. Thus $m_{h_1,\cdots,h_n} \in \{0, 1\}$ and we obtain an even linear code $D := \{(2h_1, \cdots, 2h_n) \in (\mathbb{Z}/2\mathbb{Z})^n \mid m_{h_1,\cdots,h_n} \ne 0\}$ such that

$$V^0 = \bigoplus_{\alpha = (\alpha_1, \cdots, \alpha_n) \in D} L(1/2, \alpha_1/2) \otimes \cdots \otimes L(1/2, \alpha_n/2). \tag{7.3.1}$$

The sub VOA V^0 with the linear code D is called a *code VOA*. Since all V^α, $\alpha \in S$, are irreducible modules for V^0, the representation theory of V^0 is important to study a 2A-framed VOA V. We present in the next subsection that there exists a simple VOA of the shape (7.3.1) for every linear even code D.

7.3.1 Construction of code VOAs

Let $(\mathbb{Z}/2\mathbb{Z})^n$ be the linear code of length n. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $(\mathbb{Z}/2\mathbb{Z})^n$ defined as $\langle \alpha, \beta \rangle = \sum_{i=1}^n \alpha_i \beta_i$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$. Consider the following central extension of $(\mathbb{Z}/2\mathbb{Z})^n$ by a group $\{\pm 1\}$ of order 2:

$$1 \to \mathbb{Z}_2 = \{\pm 1\} \to \widehat{(\mathbb{Z}/2\mathbb{Z})}^n \xrightarrow{\pi} (\mathbb{Z}/2\mathbb{Z})^n \to 1. \tag{7.3.2}$$

Take a section $e : (\mathbb{Z}/2\mathbb{Z})^n \ni \alpha \mapsto e^\alpha \in \widehat{(\mathbb{Z}/2\mathbb{Z})}^n$ such that $e^0 = 1$. Then we obtain a 2-cocycle $\epsilon \in Z^2((\mathbb{Z}/2\mathbb{Z})^n, \{\pm 1\})$ such that $e^\alpha \cdot e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}$. It is shown in [FLM] that the central extension (7.3.2) determines the second cohomology class of ϵ uniquely and that the set of inequivalent classes of the central extension (7.3.2) is in one-to-one correspondence with the secondo cohomology group $H^2((\mathbb{Z}/2\mathbb{Z})^n, \{\pm 1\})$. Define the commutator map $c(\alpha, \beta) := \epsilon(\alpha, \beta)\epsilon(\beta, \alpha)$ for $\alpha, \beta \in (\mathbb{Z}/2\mathbb{Z})^n$. We construct an central extension whose commutator map is $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}$. Set $\gamma_1 = (1, 0, \dots, 0)$,

$\gamma_2 = (0, 1, \dots, 0), \dots, \gamma_n = (0, 0, \dots, 1)$. Then $\{\gamma_1, \dots, \gamma_n\}$ is a basis of $(\mathbb{Z}/2\mathbb{Z})^n$. Define a 2-cocycle $\hat{\epsilon} \in Z^2((\mathbb{Z}/2\mathbb{Z})^n, \{\pm 1\})$ bilinearly as follows:

$$\hat{\epsilon}(\gamma_i, \gamma_j) = -1 \quad \text{if} \quad i > j \quad \text{and} \quad 1 \quad \text{otherwise.}$$

Since $\hat{\epsilon}$ is bilinear, clearly $\hat{\epsilon} \in Z^2((\mathbb{Z}/2\mathbb{Z})^n, \{\pm 1\})$, and by a direct calculation one can verify that $\hat{\epsilon}(\alpha, \beta)\hat{\epsilon}(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}$ for any $\alpha, \beta \in (\mathbb{Z}/2\mathbb{Z})^n$. In the following we fix the cocycle $\hat{\epsilon}$ defined as above.

Recall the SVOA $M = L(1/2, 0) \oplus L(1/2, 1/2)$ constructed in Section 7.2.1. For a while we set $M^0 = L(1/2, 0)$, $M^1 = L(1/2, 1/2)$ and we denote the vertex operator map on M by $Y_M(\cdot, z)$ as in Section 7.2.1. Consider a tensor product $U := M^{\otimes n}$ of n copies of M. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ we set $U^\alpha := M^{\alpha_1} \otimes \cdots \otimes M^{\alpha_n} \subset U$. Then $U^0 = (M^0)^{\otimes n} \simeq L(1/2, 0)^{\otimes n}$ and we have a decomposition

$$U = \bigoplus_{\alpha \in (\mathbb{Z}/2\mathbb{Z})^n} U^\alpha.$$

Let $\mathbb{1}^i$ and ω^i be respectively the vacuum vector and the Virasoro vector of i-th M^0 of $U^0 = \otimes_{i=1}^n M^0$. We often identify ω^i with $\mathbb{1}^1 \otimes \cdots \otimes \omega^i \otimes \cdots \otimes \mathbb{1}^n \in U^0$. Set

$$\tilde{Y}_U(x^1 \otimes \cdots \otimes x^n, z) := Y_M(x^1, z) \otimes \cdots \otimes Y_M(x^n, z)$$

for $x^1 \otimes \cdots \otimes x^n \in U$. Since M is an SVOA, we have the following commutator relation:

Lemma 7.3.3. *On U we have $\tilde{Y}_U(x^\alpha, z_1)\tilde{Y}_U(x^\beta, z_2) \sim (-1)^{\langle \alpha, \beta \rangle}\tilde{Y}_U(x^\alpha, z_2)\tilde{Y}_U(x^\beta, z_1)$ for $x^\alpha \in U^\alpha$ and $x^\beta \in U^\beta$, where $A(z_1, z_2) \sim B(z_1, z_2)$ means that there is a positive integer $N \gg 0$ such that $(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N B(z_1, z_2)$.*

Therefore, $(U, \tilde{Y}_U(\cdot, z))$ do not form an SVOA and we have to modify the vertex operator map to obtain a desired relation. Set

$$Y_U(x^\alpha, z)x^\beta := \hat{\epsilon}(\alpha, \beta)\tilde{Y}_U(x^\alpha, z)x^\beta$$

for $x^\alpha \in U^\alpha$ and $x^\beta \in U^\beta$. Then we have

$$Y_U(x^\alpha, z_1)Y_U(x^\beta, z_2) \sim (-1)^{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} Y_U(x^\beta, z_2)Y_U(x^\alpha, z_1) \tag{7.3.3}$$

on U for any $x^\alpha \in U^\alpha$ and $x^\beta \in U^\beta$. Set $E(\mathbb{Z}/2\mathbb{Z})^n := \{\alpha \in (\mathbb{Z}/2\mathbb{Z})^n \mid \langle \alpha, \alpha \rangle = 0\}$, $O(\mathbb{Z}/2\mathbb{Z})^n := \{\alpha \in (\mathbb{Z}/2\mathbb{Z})^n \mid \langle \alpha, \alpha \rangle = 1\}$ and $U^{(0)} := \oplus_{\alpha \in E(\mathbb{Z}/2\mathbb{Z})^n} U^\alpha$, $U^{(1)} := \oplus_{\alpha \in O(\mathbb{Z}/2\mathbb{Z})^n} U^\alpha$. Then by (7.3.3) the vertex operator map $Y_U(\cdot, z)$ satisfies the commutativity and hence we have an SVOA structure on $U = U^{(0)} \oplus U^{(1)}$.

Theorem 7.3.4. *The structure $(U, Y_U(\cdot, z))$ is a simple SVOA with even part $U^{(0)}$ and odd part $U^{(1)}$, where the vacuum vector is $\mathbb{1}^1 \otimes \cdots \otimes \mathbb{1}^n$ and the Virasoro vector is $\omega^1 + \cdots + \omega^n$,* Thus we also have

Corollary 7.3.5. *([M2]) Let D be an even linear subcode of $(\mathbb{Z}/2\mathbb{Z})^n$. For each code word $\alpha = (\alpha_1, \cdots, \alpha_n) \in D$, set $U^\alpha := L(1/2, \alpha_1/2) \otimes \cdots \otimes L(1/2, \alpha_n/2)$. Then there is a unique simple vertex operator structure on $U_D := \oplus_{\alpha\in D} U^\alpha$ as a D-graded simple current extension of $U^0 = L(1/2, 0)^{\otimes n}$.*

Proof: Since we can find U_D as a sub VOA of the vertex operator superalgebra given in Theorem 7.3.4, U_D has a simple vertex operator algebra structure. It is clear that all U^α, $\alpha \in D$, are simple current U^0-modules with fusion rules $U^\alpha \times U^\beta = U^{\alpha+\beta}$. Then by Theorem 4.2.3 the VOA structure on U_D is unique over $\mathbb{C}$. ∎

By the corollary above, for each even linear code D we always have a simple vertex operator algebra with the shape (7.3.1). The VOA U_D constructed as above is called *the code VOA associated to a code D* (cf. [M2]).

7.3.2 Representation of code VOAs

Let D be an even linear subcode of $(\mathbb{Z}/2\mathbb{Z})^n$ and let $U_D = \oplus_{\alpha\in D} U^\alpha$ where $U^\alpha = \otimes_{i=1}^n L(1/2, \alpha_i/2)$ be the associated code VOA constructed as in the previous subsection. Since U_D is a D-graded simple current extension of a rational C_2-cofinite VOA $U^0 = L(1/2, 0)^{\otimes n}$ of CFT-type, U_D is also rational and C_2-cofinite. In this subsection we study irreducible U_D-modules in detail.

Let $(X, Y_X(\cdot, z))$ be an irreducible U_D-module. Then M as a U^0-module is completely reducible and so we can take an irreducible U^0-submodule W of X. Since $U^0 \simeq \otimes_{i=1}^n L(1/2, 0)$, $W \simeq \otimes_{i=1}^n L(1/2, h_i)$ with $h_i \in \{0, 1/2, 1/16\}$. Define the 1/16-word $\alpha(W) = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ of W by $\alpha_i = 1$ if and only if $h_i = 1/16$. Then by the fusion rules (7.2.1) we find that the word $\alpha(W)$ is independent of the choice of an irreducible component W and so we can define the 1/16-word of X by $\tau(X) := \alpha(W)$. Since the powers of z in an $L(1/2, 0)$-intertwining operator of type $L(1/2, 1/2) \times L(1/2, 1/16) \to L(1/2, 1/16)$ are contained in $1/2 + \mathbb{Z}$, we have $\tau(X) \in D^\perp := \{\beta \in (\mathbb{Z}/2\mathbb{Z})^n \mid \langle \beta, D\rangle = 0\}$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ we define $D_\alpha := \{\beta \in D \mid \mathrm{Supp}(\beta) \subset \mathrm{Supp}(\alpha)\}$, where we identify $(\mathbb{Z}/2\mathbb{Z})^n$ with the power set of n-point set $\{1, 2, \ldots, n\}$ and we have set $\mathrm{Supp}(\alpha) := \{i \mid \alpha_i = 1\}$. Then by the fusion rules (7.2.1) we have $D_W = \{\alpha \in D \mid U^\alpha \cdot W \simeq W\} = D_{\tau(M)}$. By Theorem 4.4.7, $U_{D_W} \cdot W$ is an irreducible U_{D_W}-submodule of X and there is a central extension

$$1 \to \mathbb{C}^* \to \hat{D}_W \to D_W \to 1 \tag{7.3.4}$$

such that $U_{D_W} \cdot W$ is linearly isomorphic to

$$U_{D_W} \cdot W \simeq W \otimes P$$

with some irreducible $\mathbb{C}^\lambda[D_W]$-module P, where the 2-cocycle $\lambda \in Z^2(D, \mathbb{C}^*)$ is determined by the central extension (7.3.4). Let us determine the 2-cocycle λ explicitly. We may assume that $\tau(X) = (0^s 1^{n-s})$. Let $Y_M(\cdot, z)$ be the vertex operator map on $M = L(1/2, 0) \oplus L(1/2, 1/2)$ and $Y_N(\cdot, z)$ the vertex operator map on the $\mathbb{Z}_2$-twisted $L(1/2, 0) \oplus L(1/2, 1/2)$-module $L(1/2, 1/16)^+$ which we have constructed in Section 7.2.1. Then the vertex operator map $Y_X(\cdot, z)$ on X can be written as

$$\begin{aligned}&Y_X(a^1 \otimes \cdots \otimes a^n, z) w \otimes v \\ &= Y_M(a^1, z) \otimes \cdots \otimes Y_M(a^s, z) \otimes Y_N(a^{s+1}, z) \otimes \cdots \otimes Y_N(a^n, z) w \otimes \pi(\alpha) v\end{aligned}$$

for $a^1 \otimes \cdots \otimes a^n \in U^\alpha$, $\alpha \in D_W$, and $w \otimes v \in W \otimes P$, where $\pi : \mathbb{C}^\lambda[D_W] \to \mathrm{End}(P)$ is the representation map such that $\pi(\alpha)\pi(\beta) = \lambda(\alpha, \beta)\pi(\alpha + \beta)$ for $\alpha, \beta \in D_W$. We know that $Y_M(\cdot, z)$ and $Y_N(\cdot, z)$ satisfy the following associativity:

$$\begin{aligned}(z_0 + z_2)^{k_1} Y_M(a, z_0 + z_2) Y_M(b, z_2) u^1 &= (z_2 + z_0)^{k_1} Y_M(Y_M(a, z_0) b, z_2) u^1, \\ (z_0 + z_2)^{k_2} Y_N(a, z_0 + z_2) Y_N(b, z_2) u^2 &= (z_2 + z_0)^{k_2} Y_N(Y_M(a, z_0) b, z_2) u^2,\end{aligned}$$

for $a, b, u^1 \in M$, $u^2 \in L(1/2, 1/16)^+$, $k_1 \in \mathbb{N}$ and $k_2 \in \frac{1}{2}\mathbb{N}$. On the other hand, since

$$Y_{U_D}(a^1 \otimes \cdots \otimes a^n, z) b^1 \otimes \cdots \otimes b^n = \hat{\epsilon}(\alpha, \beta) Y_M(a^1, z) b^1 \otimes \cdots Y_M(a^n, z) b^n$$

for $a^1 \otimes \cdots \otimes a^n \in U^\alpha$ and $b^1 \otimes \cdots \otimes b^n \in U^\beta$, $\alpha, \beta \in D$, we have the following relation:

$$\pi(\alpha)\pi(\beta) = \hat{\epsilon}(\alpha, \beta)\pi(\alpha + \beta).$$

Therefore, the cocycle λ is cohomologous to the cocycle $\hat{\epsilon}$ and we can replace the central part of the central extension (7.3.4) by $\mathbb{Z}_2 = \{\pm 1\} \subset \mathbb{C}^*$. Namely, the central extension (7.3.4) is essentially the same as the following one:

$$1 \to \mathbb{Z}_2 \to \hat{D}_W \to D_W \to 1. \tag{7.3.5}$$

Thus we have

Lemma 7.3.6. *The space of multiplicity* $\mathrm{Hom}_{U^0}(W, M)$ *is an irreducible* $\mathbb{C}^{\hat{\epsilon}}[D_W]$*-module.*

Remark 7.3.7. One can also show that the 2-cocycle λ in the definition of the twisted algebra $A_\lambda(D, \mathcal{S}_W)$ (cf. Section 4.3) associated to a pair (D, W) is given by $\hat{\epsilon}$.

Set $\mathbb{C}^{\hat{\epsilon}}[D_W] = \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in D_W\}$ with products $e^\alpha e^\beta = \hat{\epsilon}(\alpha, \beta) e^{\alpha+\beta}$. Let E_W be a maximal self-orthogonal linear subcode of D_W. Then $\hat{\epsilon}$ vanishes on E_W and so the subalgebra $\mathbb{C}^{\hat{\epsilon}}[E_W]$ of $\mathbb{C}^{\hat{\epsilon}}[D_W]$ is isomorphic to the group ring $\mathbb{C}[E_W]$.

Lemma 7.3.8. *Let χ be a linear character on E_W and $\mathbb{C}v_\chi$ a linear representation of E_W affording the character χ. Then $\mathrm{Ind}_{\mathbb{C}[E_W]}^{\mathbb{C}^{\hat{\epsilon}}[D_W]}\mathbb{C}v_\chi$ is an irreducible $\mathbb{C}^{\hat{\epsilon}}[D_W]$-module such that the central part $\mathbb{Z}_2 = \{\pm 1\}$ of* (7.3.5) *acts faithfully. Conversely, every irreducible $\mathbb{C}^{\hat{\epsilon}}[D_W]$-module on which the central part $\mathbb{Z}_2$ of* (7.3.5) *acts faithfully is given by an induced module described as above.*

Proof: A $\mathbb{C}^{\hat{\epsilon}}[D_W]$-module is equivalent to a module for the central extension $\hat{D}_W$ in (7.3.5). Since $\hat{D}_W$ is a central product of an abelian group and an extra-special 2-group, we may assume that $\hat{D}_W$ is an extra-special 2-group. In this case $E_W \subsetneq D_W$ and the preimage $\hat{E}_W$ of E_W in $\hat{D}_W$ is a maximal abelian subgroup of $\hat{D}_W$. Thus the induced module $\mathrm{Ind}_{\mathbb{C}[E_W]}^{\mathbb{C}^{\hat{\epsilon}}[D_W]}\mathbb{C}v_\chi$ is not a linear representation of $\hat{D}_W$. Since every irreducible non-linear module for an extra-special 2-group is induced from a linear character of its maximal abelian subgroup, the assertion holds. ∎

By the lemma above, we can classify all irreducible modules for a code VOA U_D according to Theorem 4.5.2.

Theorem 7.3.9. *([M3]) Let $\gamma \in D^\perp$. Let E be a maximal self-orthogonal subcode of D_γ and χ a linear representation of E. Let φ be an irreducible representation of $\mathbb{C}^{\hat{\epsilon}}[D_\gamma]$ induced from χ. For any irreducible U^0-module W with 1/16-word γ, the induced module $\mathrm{Ind}_{U_{D_\gamma}}^{U_D}(W,\varphi)$ which is given by Theorem 4.5.2 is an irreducible U_D-module. Conversely, every irreducible U_D-module is isomorphic to one of the irreducible modules constructed as above.*

Remark 7.3.10. If $\gamma \in D \setminus D^\perp$, then the module $\mathrm{Ind}_{U_{D_\gamma}}^{U_D}(W,\varphi)$ in the above theorem is an irreducible $\mathbb{Z}_2$-twisted U_D-module by Theorem 4.5.2.

Let us consider the case $\tau(X) = (0^n)$. In this case there is no $L(1/2,1/16)$-component in X and hence X is a direct sum of tensor products of $L(1/2,0)$ and $L(1/2,1/2)$. In this case one can easily see that there is a unique coset $\alpha + D$ of $(\mathbb{Z}/2\mathbb{Z})^n$ such that

$$X = \bigoplus_{\beta=(\beta_1,\dots,\beta_n)\in D+\alpha} L(1/2,\beta_1/2)\otimes\cdots L(1/2,\beta_n/2).$$

Since X has a D-grading consistent with the action of U_D, X is a D-stable irreducible U_D-module. Therefore, the U_D-module structure of X is uniquely determined. Such a U_D-module X is called a *coset module* of U_D and denoted by $U_{D+\alpha}$.

Proposition 7.3.11. *All $U_{D+\alpha}$, $\alpha \in (\mathbb{Z}/2\mathbb{Z})^n$, are simple current U_D-module and the fusion rules $U_{D+\alpha}\times U_{D+\beta} = U_{D+\alpha+\beta}$ hold for all $\alpha,\beta \in (\mathbb{Z}/2\mathbb{Z})^n$.*

Proof: To prove that $U_{D+\alpha}$ is a simple current, it is sufficient to prove that $U_{D+\alpha}\times U_{D+\alpha} = U_D$ by Lemma 4.1.2 because U_D is a rational C_2-cofinite VOA of CFT-type.

Thus we only prove the fusion rule above. By Theorem 4.4.9 we have a non-trivial U_D-intertwining operator of type $U_{D+\alpha} \times U_{D+\beta} \to U_{D+\alpha+\beta}$. On the other hand, let X be a non-trivial irreducible U_D-module such that there is a non-trivial U_D-intertwining operator of type $U_{D+\alpha} \times U_{D+\beta} \to X$. Since $U_{D+\alpha}$ and $U_{D+\beta}$ are D-stable irreducible U_D-modules, the space of intertwining operators $\binom{X}{U_{D+\alpha}\ U_{D+\alpha}}_{U_D}$ is linearly isomorphic to

$$\binom{X}{L(1/2,\alpha_1/2)\otimes\cdots\otimes L(1/2,\alpha_n/2)\quad L(1/2,\beta_1/2)\otimes\cdots\otimes L(1/2,\beta_n/2)}_{U^0}$$

by Theorem 4.4.9. Then by the fusion rules (7.2.1), X must contain $\otimes_{i=1}^n L(1/2,\gamma_i/2)$ with $\gamma = \alpha+\beta$ and so $\tau(X) = (0^n)$. Then the U_D-module structure of X is uniquely determine and hence $X \simeq U_{D+\gamma}$. Thus the assertion holds. ∎

Remark 7.3.12. We note that in the construction of the SVOA $U_{(\mathbb{Z}/2\mathbb{Z})^n}$, we have already constructed a U_D-intertwining operator of type $U_{D+\alpha} \times U_{D+\beta} \to U_{D+\alpha+\beta}$.

7.3.3 The Hamming code VOA

Let H_8 be the $[8,4,4]$-Hamming code:

$$H_8 := \mathrm{Span}_{\mathbb{Z}/2\mathbb{Z}}\{(11111111),(11110000),(11001100),(10101010)\}.$$

It is well-known that H_8 is the unique doubly even self-dual linear code of length 8 up to isomorphism. Let us consider the Hamming code VOA U_{H_8}. Since H_8 is self-dual, the 2-cocycle $\hat{\epsilon}$ vanishes on H_8 so that in the definition of the vertex operator map on U_{H_8} we do not have to use it. In order to reconstruct the moonshine VOA as a 2A-framed VOA, we will need some special properties that the Hamming code VOA U_{H_8} has. Roughly speaking, we can identify $L(1/2,1/16)$ with $L(1/2,0)$ and $L(1/2,1/2)$ by the symmetry of the Hamming code VOA.

Let X be an irreducible U_{H_8}-module whose top weight is in $\frac{1}{2}\mathbb{N}$. Then $\tau(X) = (0^8)$ or (1^8) and if $\tau(X) = (0^8)$ then $X \simeq U_{H_8+\gamma}$ for some $\gamma \in (\mathbb{Z}/2\mathbb{Z})^8$. If $\tau(X) = (1^8)$, then $(H_8)_{(1^8)} = H_8$ and so there is a linear character χ on H_8 such that

$$X \simeq \mathrm{Ind}_{U^0}^{U_{H_8}}(L(1/2,1/16)^{\otimes 8},\chi) = L(1/2,1/16)^{\otimes 8} \underset{\mathbb{C}}{\otimes} v_\chi, \tag{7.3.6}$$

where $\mathbb{C}v_\chi$ is a linear representation of H_8 affording the character χ, by Theorem 7.3.9. Since the dual group H_8^* of H_8 is naturally isomorphic to $(\mathbb{Z}/2\mathbb{Z})^8/H_8$, we can find a unique coset $\delta_\chi + H_8 \in (\mathbb{Z}/2\mathbb{Z})^8/H_8$ such that $\chi(\alpha) = \langle \delta_\chi, \alpha\rangle$ for all $\alpha \in H_8$. So in the following we regard χ as an element in $(\mathbb{Z}/2\mathbb{Z})^8$. Set $H(1/16,\chi) = L(1/2,1/16)^{\otimes 8} \otimes_{\mathbb{C}} v_\chi$ for $\chi \in (\mathbb{Z}/2\mathbb{Z})^8$. Then $H(1/16,\chi_1) \simeq H(1/16,\chi_2)$ as U_{H_8}-modules if and only if $\chi_1 - \chi_2 \in$

H_8 and the set of inequivalent irreducible U_{H_8}-modules whose top weights are contained in $\frac{1}{2}\mathbb{N}$ is given by

$$\{U_{H_8+\gamma},\ H(1/16,\chi) \mid \gamma + H_8, \chi + H_8 \in (\mathbb{Z}/2\mathbb{Z})^8/H_8\}.$$

Surprisingly, we can identify a non-simple current $L(1/2,0)^{\otimes 8}$-module $L(1/2,1/16)^{\otimes 8}$ with a coset module as follows:

Proposition 7.3.13. *([M4]) For each $H(1/16,\chi)$ with $\chi \in (\mathbb{Z}/2\mathbb{Z})^8$, there is an automorphism $\sigma \in \mathrm{Aut}(U_{H_8})$ such that the σ-conjugate module $H(1/16,\chi)^\sigma \simeq U_{H_8+\gamma}$ for some $\gamma \in (\mathbb{Z}/2\mathbb{Z})^8$ with $\langle \gamma, \gamma \rangle = 1$. In particular, $H(1/16,\chi)$ is a simple current U_{H_8}-module.*

Proof: This is the one of the main results in [M4]. Here we give a brief explanation. It is shown in [M4] that there are exactly three 2A-frames in U_{H_8}, and these frames are conjugate under the second Miyamoto involutions associated to the 2A-frames in U_{H_8}. Thus by permuting 2A-frames we can find the desired isomorphism. For more details, see [M4]. ∎

Corollary 7.3.14. *([M4]) As a $\mathbb{Z}_2$-graded simple current extension of U_{H_8}, there is a unique simple SVOA structure on $U_{H_8} \oplus H(1/16,\chi)$ for all $\chi \in (\mathbb{Z}/2\mathbb{Z})^8$.*

Proof: We can take an irreducible U_{H_8}-module $U_{H_8+\gamma}$ with $\langle \gamma, \gamma \rangle = 1$ such that there is an automorphism $\sigma \in \mathrm{Aut}(U_{H_8})$ such that the conjugate module $(U_{H_8+\gamma})^\sigma$ is isomorphic to $H(1/16,\chi)$ by Proposition 7.3.13. Then $U_{H_8} \oplus U_{H_8+\gamma}$ and $U_{H_8} \oplus H(1/16,\chi)$ form equivalent $\mathbb{Z}_2$-graded simple current extensions of U_{H_8}. Since $H_8 \cup (H_8+\gamma)$ is an odd code, $U_{H_8} \oplus U_{H_8+\gamma}$ is a simple SVOA. Then so is $U_{H_8} \oplus H(1/16,\chi)$. ∎

As an application of Proposition 7.3.13, the following fusion rules are established in [M4]:

Theorem 7.3.15. *([M4]) We have the following fusion rules:*

$$\begin{aligned}
&U_{H_8+\alpha} \times U_{H_8+\beta} = U_{H_8+\alpha+\beta},\\
&U_{H_8+\alpha} \times H(1/16,\beta) = H(1/16,\beta+\alpha),\\
&H(1/16,\alpha) \times H(1/16,\beta) = U_{H_8+\alpha+\beta},
\end{aligned}$$

where $\alpha, \beta \in (\mathbb{Z}/2\mathbb{Z})^8$.

Thanks to Corollary 7.3.14 and Theorem 7.3.15, if an even linear code D contains many subcodes isomorphic to the Hamming code H_8, then we can construct simple current extensions of the code VOA U_D by using Theorem 4.6.1.

7.4 2A-framed VOAs

Let (V, ω) be a 2A-framed VOA with a 2A-frame $\omega = e^1 + \cdots + e^n$. Then V as a $T = \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^n)$-module is completely reducible and we have an even linear code $S \subset (\mathbb{Z}/2\mathbb{Z})^n$ and an S-graded decomposition $V = \oplus_{\alpha \in S} V^\alpha$ where V^α is a sum of all irreducible non-trivial T-modules whose 1/16-words are all equal to $\alpha \in S$. By definition V^0 has no component isomorphic to $L(1/2, 1/16)$ and hence there is an even linear code $D \subset (\mathbb{Z}/2\mathbb{Z})^n$ such that V^0 is isomorphic to the code VOA U_D. Since we have assumed that V is simple, all V^α, $\alpha \in S$, are irreducible U_D-modules, and by the fusion rules (7.2.1), we have an S-graded structure $V^\alpha \cdot V^\beta = V^{\alpha+\beta}$ for $\alpha, \beta \in S$. Thus we can view V as an S-graded extension of $V^0 = U_D$. We also note that $D \subset S^\perp$ as all V^α, $\alpha \in S$, are untwisted U_D-modules. Summarizing, for a 2A-framed VOA V, we can obtain a pair of codes (D, S) such that $V = \oplus_{\alpha \in S} V^\alpha$, $V^0 \simeq U_D$ and $D \subset S^\perp$. We call a pair (D, S) the *structure codes* of V.

Theorem 7.4.1. *([DGH]) Every 2A-framed VOA is rational C_2-cofinite and of CFT-type.*

Proof: Since V is a module for $T = \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^n)$, it is clear that V is C_2-cofinite and of CFT-type. So we only need to prove that V is rational. Let (D, S) be the structure codes of V. Take a V-module M. Then M is also a U_D-module. Let W be an irreducible U_D-module and $\tau(W) \in D^\perp$ the 1/16-word of W. Then by Lemma 4.4.1 and Remark 4.4.2 all $V^\alpha \cdot W$, $\alpha \in S$, are irreducible $V^0 = U_D$-modules. Moreover, by the fusion rules (7.2.1), the 1/16-word of $V^\alpha \cdot W$ is $\alpha + \tau(W)$. Therefore, $V^\alpha \cdot W \not\simeq V^\beta \cdot W$ whenever $\alpha \neq \beta$ and we have an S-graded decomposition $V \cdot W = \oplus_{\alpha \in S} V^\alpha \cdot W$. This implies that $V \cdot W$ is an irreducible V-module for any irreducible V^0-submodule W of M. Hence, M is a completely reducible V-module. ∎

Theorem 7.4.2. *([DGH] [M5]) Let V be a 2A-framed VOA with structure codes (D, S). If $D = S^\perp$, then V is holomorphic, i.e., any irreducible V-module is isomorphic to V.*

Proof: We have shown that V is rational. Let M be an irreducible V-module. Then we have to prove $M \simeq V$. Let W be an irreducible $V^0 = U_D$-module. Then the 1/16-word $\tau(W)$ is in $D^\perp$. Since $D^\perp = S$, $V^{\tau(W)} \neq 0$ and the 1/16-word of $V^{\tau(W)} \cdot W$ is 0. Namely, $V^{\tau(W)} \cdot W$ as a U_D-module is isomorphic to a coset module $U_{D+\gamma}$. Since $U_{D+\gamma}$ is a simple current U_D-module, we have fusion rules $U_{D+\gamma} \times V^\beta = (V^{\beta+\tau(W)} \cdot W)$ for all $\beta \in S$. One can easily see that the powers of z in an U_D-intertwining operator of type $U_{D+\gamma} \times V^\beta \to (V^{\beta+\tau(W)} \cdot W)$ are contained in $\langle \gamma, \beta \rangle / 2 + \mathbb{Z}$ by the fusion rules (7.2.1). Thus $\langle \gamma, \beta \rangle = 0$ modulo 2 and hence $\gamma \in S^\perp = D$. Namely, $D + \gamma = D$ and hence M contains U_D as a U_D-submodule. Since $U_D \times V^\alpha = V^\alpha$, M as a U_D-module is isomorphic to $\oplus_{\alpha \in S} V^\alpha = V$. Let $\psi : V \to M$ be a U_D-isomorphism. Then $\psi(\mathbb{1}_V)$ is not zero and satisfies $L(-1)\psi(\mathbb{1}_V) = 0$. Therefore, $\psi(\mathbb{1}_V)$ is a vacuum-like vector of M and hence M as a V-module is isomorphic to V by Lemma 3.4.3. ∎

7.4.1 Construction of 2A-framed VOAs

In this subsection we construct certain 2A-framed VOAs. Here we assume the following:

Hypothesis I.

(1) (D, S) is a pair of even linear even codes of $(\mathbb{Z}/2\mathbb{Z})^n$ such that

(1-i) $D \subset S^\perp$,

(1-ii) for each $\alpha \in S$, there is a subcode $E^\alpha \subset D$ such that E^α is a direct sum of the Hamming code

(2) $V^0 = U_D$ is the code VOA associated to the code D.

(3) $\{V^\alpha \mid \alpha \in S\}$ is a set of irreducible V^0-modules such that

(3-i) $\tau(V^\alpha) = \alpha$ for all $\alpha \in S$,

(3-ii) all V^α, $\alpha \in S$, have integral top weights,

(3-iii) the fusion product $V^\alpha \boxtimes_{V^0} V^\beta$ contains at least one $V^{\alpha+\beta}$. That is, there is a non-trivial V^0-intertwining operator of type $V^\alpha \times V^\beta \to V^{\alpha+\beta}$ for any $\alpha, \beta \in S$.

Under Hypothesis I we will prove that $V := \oplus_{\alpha\in S} V^\alpha$ has a structure of an S-graded simple current extension of V^0. Before we begin the proof, we prepare some lemmas.

Lemma 7.4.3. *Under Hypothesis I, all V^α, $\alpha \in S$, are simple current V^0-modules and we have the fusion rules $V^\alpha \times V^\beta = V^{\alpha+\beta}$ of V^0-modules for all $\alpha, \beta \in S$.*

Proof: Suppose the fusion rule $V^\alpha \times V^\alpha = V^0$ of V^0-modules holds. Then by Lemma 4.1.2, V^α is a simple current V^0-module because $V^0 = U_D$ is a rational C_2-cofinite VOA of CFT-type. Then by Hypothesis I (3-iii) we have the desired fusion rule $V^\alpha \times V^\beta = V^{\alpha+\beta}$. Therefore, we only prove the fusion rule $V^\alpha \times V^\alpha = V^0$ for each $\alpha \in S$. By Hypothesis I (1-i), D contains a subcode E^α which is isomorphic to a direct sum of H_8 and $\mathrm{Supp}(E^\alpha) = \mathrm{Supp}(\alpha)$. So we may assume that $\alpha = (1^{8s}0^t)$ with $8s + t = n$. Then U_D contains a sub VOA

$$L := U_{E^\alpha} \otimes L(1/2, 0)^{\otimes t} \simeq (U_{H_8})^{\otimes s} \otimes L(1/2, 0)^{\otimes t}$$

and V^α as a $U_{E^\alpha} \otimes L(1/2, 0)^{\otimes t}$-module contains an irreducible submodule X isomorphic to

$$H(1/16, \chi_1) \otimes \cdots H(1/16, \chi_s) \otimes L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_t)$$

with $\chi_i \in (\mathbb{Z}/2\mathbb{Z})^8$, $1 \le i \le s$, and $h_j \in \{0, 1/2\}$, $1 \le j \le t$. Let $D = \sqcup_{i=0}^k (E^\alpha + \beta_i)$ be a coset decomposition. We write $\beta_i = \gamma_i + \delta_i$ such that $\mathrm{Supp}(\gamma_i) \subset \mathrm{Supp}(\alpha)$ and

$\mathrm{Supp}(\delta_i) \cap \mathrm{Supp}(\alpha) = \emptyset$. Then $U_{E^\alpha+\beta_i}$ is isomorphic to

$$U_{E^\alpha+\gamma_i} \otimes L(1/2, (\delta_i)_{8s+1}/2) \otimes \cdots \otimes L(1/2, (\delta_i)_n/2)$$

as an L-module and $U_D = \oplus_{i=1}^k U_{E^\alpha+\beta_i}$ is a D/E^α-graded simple current extension of L. Then by the fusion rules (7.2.1) and Theorem 7.3.15, $(U_{E^\alpha+\beta_i}) \boxtimes_L X$ is an irreducible L-module and $(U_{E^\alpha+\beta_i}) \boxtimes_L X \not\simeq (U_{E^\alpha+\beta_j}) \boxtimes X$ as $U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}$-module unless $i = j$. Therefore, V^α as an L-module is isomorphic to $V^\alpha = \oplus_{i=1}^k (V_{E^\alpha+\beta_i}) \boxtimes_L X$. Namely, V^α is a D/E^α-stable U_D-module. Then by Theorem 4.4.9 together with fusion rules (7.2.1) and those in Theorem 7.3.15, we have a fusion rule $V^\alpha \times V^\alpha = V^0$ of U_D-modules which is a lifting of the fusion rule $X \times X = L$ of L-modules. ∎

Lemma 7.4.4. *Under Hypothesis I, the space $V^0 \oplus V^\alpha$ forms a simple VOA as a $\mathbb{Z}_2$-graded simple current extension of V^0 for each $\alpha \in S \setminus 0$.*

Proof: Here we use the same notation as in the proof of Lemma 7.4.3. By the coset decomposition $D = \sqcup_{i=1}^k (E^\alpha + \beta_i)$, $V^0 = U_D = \oplus_{i=1}^k U_{E^\alpha+\beta_i}$ is a D/E^α-graded simple current extension of $L = U_{E^\alpha} \otimes L(1/2,0)^{\otimes t} \simeq (U_{H_8})^{\otimes s} \otimes L(1/2,0)^{\otimes t}$. By the fusion rule $X \times X = L$ of L-modules, X is a simple current L-module. Then by the associativity of fusion products (cf. Theorem 3.7.6), an irreducible L-module $(U_{E^\alpha+\beta_i}) \boxtimes_L X$ is also a simple current. Thus we obtain the set of inequivalent simple current L-modules

$$\mathcal{S} = \{\, U_{E^\alpha+\beta_i},\ (U_{E^\alpha+\beta_j}) \boxtimes_L X \mid 1 \le i,j \le k \,\}$$

with the following $((D/E^\alpha) \oplus \mathbb{Z}_2)$-graded fusion rules:

$$\begin{aligned}
&U_{E^\alpha+\beta_i} \times U_{E^\alpha+\beta_j} = U_{E^\alpha+\beta_i+\beta_j},\\
&U_{E^\alpha+\beta_i} \times (U_{E^\alpha+\beta_j} \boxtimes_L X) = (U_{E^\alpha+\beta_i+\beta_j}) \boxtimes_L X,\\
&(U_{E^\alpha+\beta_i} \boxtimes_L X) \times (U_{E^\alpha+\beta_j}) \boxtimes_L X = U_{E^\alpha+\beta_i+\beta_j}.
\end{aligned}$$

Since $U_D = \oplus_{i=1}^k U_{E^\alpha+\beta_i}$ has a structure of a D/E^α-graded simple current extension of L and $L \oplus X$ has a structure of a $\mathbb{Z}_2$-graded simple current extension of L by Corollary 7.3.14, we can apply Theorem 4.6.1 to $\mathcal{S}$ and hence we obtain a $((D/E^\alpha) \oplus \mathbb{Z}_2)$-graded simple current extension

$$\left\{\oplus_{i=1}^k U_{E^\alpha+\beta_i}\right\} \bigoplus \left\{\oplus_{i=1}^k (U_{E^\alpha+\beta_i}) \boxtimes_L X\right\}$$

of L. Since $V^0 = \oplus_{i=1}^k U_{E^\alpha+\beta_i}$ and $V^\alpha = \oplus_{i=1}^k (U_{E^\alpha+\beta_i}) \boxtimes_L X$, the $\mathbb{Z}_2$-graded space $V^0 \oplus V^\alpha$ carries a simple VOA structure which is the desired $\mathbb{Z}_2$-graded simple current extension of V^0. ∎

Now we can prove

Theorem 7.4.5. *([M4]) Under Hypothesis I, the space $V = \oplus_{\alpha \in S} V^\alpha$ has a unique structure of a simple VOA as an S-graded simple current extension of V^0. In particular, there exists a 2A-framed VOA whose structure codes are (D, S).*

Proof: Let $\{\alpha_1, \ldots, \alpha_r\}$ be a linear basis of S and set $S^i := \mathrm{Span}_{\mathbb{Z}/2\mathbb{Z}}\{\alpha_1, \ldots, \alpha_i\}$ for $1 \le i \le r$. We proceed by induction on r. The case $r = 0$ is trivial and the case $r = 1$ is given by Lemma 7.4.4. Now assume that $\oplus_{\beta \in S^i} V^\beta$ has a structure of a simple VOA for $1 \le i \le r-1$. Then the set

$$\mathcal{T} = \{ V^\beta,\ V^{\beta+\alpha_{i+1}} \mid \beta \in S^i \}$$

consists of inequivalent simple current V^0-modules with $(S^i \oplus \mathbb{Z}_2) = S^{i+1}$-graded fusion rules:

$$V^{\beta_1} \times V^{\beta_2} = V^{\beta_1+\beta_2}, \quad V^{\beta_1} \times V^{\beta_2+\alpha_{i+1}} = V^{\beta_1+\beta_2+\alpha_{i+1}}, \quad V^{\beta_1+\alpha_{i+1}} \times V^{\beta_2+\alpha_{i+1}} = V^{\beta_1+\beta_2}$$

where $\beta_1, \beta_2 \in S^i$. By inductive assumption, $\oplus_{\beta \in S^i} V^\beta$ is an S^i-graded simple current extension of V^0, and by Lemma 7.4.4, a direct sum $V^0 \oplus V^{\alpha_{i+1}}$ becomes a $\mathbb{Z}_2$-graded simple current extension of V^0. Therefore, we can apply Theorem 4.6.1 to $\mathcal{T}$ to obtain the S^{i+1}-graded simple current extension $\oplus_{\beta \in S^{i+1}} V^\alpha$ of V^0. Repeating this procedure, we finally obtain $S^r = S$-graded simple current extension $V = \oplus_{\alpha \in S} V^\alpha$ of $V^0 = U_D$. ∎

Remark 7.4.6. In [M4], Miyamoto assumed stronger conditions than those in Hypothesis I. In particular, he assumed that the structure codes (D, S) are of length $8k$ for some positive integer k. Our refinement enable us to construct 2A-framed VOAs with structure codes of any length as long as Hypothesis I is satisfied.

7.4.2 Transformation of structure codes

In the rest of this section we consider a transformation of structure codes of a 2A-framed VOA. One can easily verify the following.

Lemma 7.4.7. *Let V^i, $i = 1, 2$, be 2A-framed VOAs with structure codes (D^i, S^i), $i = 1, 2$, respectively. Then $V^1 \otimes V^2$ is also a 2A-framed VOA with structure codes $(D^1 \oplus D^1, S^1 \oplus S^2)$.*

We will need the following proposition to construct the moonshine VOA.

Proposition 7.4.8. *Suppose that $V = \oplus_{\alpha \in S} V^\alpha$ is a 2A-framed VOA with structure codes (D, S) such that (D, S) and $\{V^\alpha \mid \alpha \in S\}$ satisfy Hypothesis I. Let C be an even code such that $D \subset C \subset S^\perp$. Then the induced module $\mathrm{Ind}_{U_D}^{U_C} V^\alpha$ given by Theorem 4.5.2 is uniquely determined and is an irreducible untwisted U_C-module for all $\alpha \in S$. Moreover, a pair (C, S) and the set $\{\mathrm{Ind}_{U_D}^{U_C} V^\alpha \mid \alpha \in S\}$ also satisfy Hypothesis I.*

Proof: Since $D \subset C$, the code VOA U_C is a D/C-graded simple current extension of $V^0 = U_D$. Since $C \subset S^\perp$, we find that the powers of z in U_D-intertwining operators of type $U_{D+\beta} \times V^\alpha \to U_{D+\beta} \boxtimes_{U_D} V^\alpha$ are contained in $\mathbb{Z}$ for all $\beta \in C$. Therefore, a U_D-module V^α is lifted to be an untwisted U_C-module for each $\alpha \in S$ by Theorem 4.5.2. We show that the U_C-module induced from a U_D-module V^α is unique up to isomorphism. By considering a permutation on the coordinate set, we may assume that $\alpha = (1^{8s}0^t)$ with $8s + t = n$. Then U_D and U_C contain a sub VOA $U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}$ and V^α has a $U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}$-submodule of the form

$$X^\alpha = H(1/16, \chi_{\alpha,1}) \otimes \cdots \otimes H(1/16, \chi_{\alpha,s}) \otimes L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_t)$$

with $\chi_{\alpha,i} \in (\mathbb{Z}/2\mathbb{Z})^8$, $1 \le i \le s$ and $h_j \in \{0, 1/2\}$, $1 \le j \le t$. Let $C = \sqcup_{i=1}^m (E^\alpha + \gamma_i)$ be a coset decomposition of E^α such that $D = \sqcup_{i=1}^k (E^\alpha + \gamma_i)$, $k \le m$, is a coset decomposition of D . Then $U_C = \oplus_{i=1}^m U_{E^\alpha+\gamma_i}$ and $U_D = \oplus_{i=1}^k U_{E^\alpha+\gamma_i}$ are C/E^α-graded and D/E^α-graded simple current extensions of $U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}$, respectively. Since X^α is a simple current $U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}$-module and $U_{E^\alpha+\beta_i} \cdot X^\alpha \not\simeq U_{E^\alpha+\beta_j} \cdot X^\alpha$ as $U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}$-modules if $i \ne j$ by (7.2.1) and Theorem 7.3.15, one sees that the induced U_C-module

$$\mathrm{Ind}_{U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}}^{U_C} X^\alpha = \oplus_{i=1}^m (U_{E^\alpha+\gamma_i}) \cdot X^\alpha. \tag{7.4.1}$$

is uniquely determined and gives an untwisted irreducible U_C-module. On the other hand, we see that

$$V^\alpha = \mathrm{Ind}_{U_{E^\alpha} \otimes L(1/2,0)^{\otimes t}}^{U_D} X^\alpha = \oplus_{i=1}^k (U_{E^\alpha+\gamma_i}) \cdot X^\alpha.$$

Therefore, the induced module $\mathrm{Ind}_{U_D}^{U_C} V^\alpha$ is unique and given by (7.4.1). We also note that $\mathrm{Ind}_{U_D}^{U_C} V^\alpha$ is a C/D-stable U_C-module. Therefore, we have a lifting fusion rule $\mathrm{Ind}_{U_D}^{U_C} V^\alpha \times \mathrm{Ind}_{U_D}^{U_C} V^\beta = \mathrm{Ind}_{U_D}^{U_C} V^{\alpha+\beta}$ of U_C-modules from the fusion rule $V^\alpha \times V^\beta = V^{\alpha+\beta}$ of U_D-modules for $\alpha, \beta \in S$ by Theorem 4.4.9. Thus we have a pair of codes (C, S) and a set of inequivalent simple current $U_C = \mathrm{Ind}_{U_D}^{U_D} V^0$-modules $\{\mathrm{Ind}_{U_D}^{U_C} V^\alpha \mid \alpha \in S\}$ satisfying Hypothesis I. ∎

As a corollary, we have:

Theorem 7.4.9. *Let $V = \oplus_{\alpha \in S} V^\alpha$ be a 2A-framed VOA with structure codes (D, S).*
(i) For a sub code S' of S, $\oplus_{\alpha \in S'} V^\alpha$ is a 2A-framed VOA with structure codes (D, S').
(ii) Suppose that a pair (D, S) and the set $\{V^\alpha \mid \alpha \in S\}$ satisfy Hypothesis I. Then for a subcode C with $D \subset C \subset S^\perp$,

$$\mathrm{Ind}_D^C V := \bigoplus_{s \in S} \mathrm{Ind}_{U_D}^{U_C} V^\alpha$$

has a structure of a 2A-framed VOA with structure codes (C, S).

7.5 The moonshine VOA as a 2A-framed VOA

In this subsection we review Miyamoto's reconstruction of the moonshine vertex operator algebra by using Theorem 7.4.5.

7.5.1 E_8-lattice VOA

Let V_{E_8} the lattice VOA associated to the root lattice of type E_8. It is shown in [GH] [M5] that V_{E_8} is a 2A-framed VOA and there are exactly five inequivalent 2A-frames in V_{E_8}. We recall one of them. Set

$$S_{E_8} := \mathrm{Span}_{\mathbb{Z}/2\mathbb{Z}}\{(1^{16}),\ (1^8 0^8),\ (1^4 0^4 1^4 0^4),\ (\{1100\}^4),\ (\{10\}^8)\}$$

and $D_{E_8} := (S_{E_8})^{\perp}$. Then it is known that S_{E_8} is a Reed-Müller code $RM(4,1)$ and D_{E_8} is a Reed-Müller code $RM(4,2)$.

Theorem 7.5.1. *([M5]) The lattice VOA V_{E_8} is a 2A-framed VOA with structure codes (D_{E_8}, S_{E_8}).*

Write $V_{E_8} = \oplus_{\alpha\in S_{E_8}} V_{E_8}^{\alpha}$. Then $V_{E_8}^0 \simeq U_{D_{E_8}}$ and all $V_{E_8}^{\alpha}$, $\alpha \in S_{E_8}$, are irreducible $V_{E_8}^0$-modules.

Proposition 7.5.2. *([M5]) The pair (D_{E_8}, S_{E_8}) and the set $\{V_{E_8}^{\alpha} \mid \alpha \in S_{E_8}\}$ satisfy Hypothesis I.*

Proof: It is not difficult to see that a pair (D_{E_8}, S_{E_8}) satisfies the condition (1) of Hypothesis I. Since $V_{E_8} = \oplus_{\alpha\in S_{E_8}} V_{E_8}^{\alpha}$ is an S_{E_8}-graded (simple current) extension of $V_{E_8}^0 = U_{D_{E_8}}$, the conditions (2) and (3) of Hypothesis I are also satisfied. ∎

7.5.2 Construction of the moonshine VOA

We begin the Miyamoto's construction. Set $D(0) = D_{E_8} \oplus D_{E_8} \oplus D_{E_8}$ and $S(0) = S_{E_8} \oplus S_{E_8} \oplus S_{E_8}$. Consider a tensor product

$$V(0) := V_{E_8} \otimes V_{E_8} \otimes V_{E_8}$$

of three copies of V_{E_8}. Since $V_{E_8} = \oplus_{\alpha\in S_{E_8}} V_{E_8}^{\alpha}$ is a 2A-framed VOA with structure codes (D_{E_8}, S_{E_8}) by Theorem 7.5.1, $V(0)$ is a 2A-framed VOA with structure codes $(D(0), S(0))$. It is clear from Theorem 7.5.2 that the pair $(D(0), S(0))$ and the set $\{V(0)^{\alpha} \mid \alpha \in S(0)\}$ satisfy Hypothesis I. Set

$$S(1) := \{(\alpha, \alpha, \alpha) \in (\mathbb{Z}/2\mathbb{Z})^{48} \mid \alpha \in S_{E_8}\} \subset S(0).$$

and

$$V(1) := \bigoplus_{\alpha \in S(1)} V(0)^\alpha = \bigoplus_{\beta \in S_{E_8}} V_{E_8}^\beta \otimes V_{E_8}^\beta \otimes V_{E_8}^\beta.$$

Then $V(1)$ is a 2A-framed sub VOA of $V(0)$ with structure codes $(D(0), S(1))$. It is clear that the pair $(D(0), S(1))$ and the set $\{V(1)^\alpha = V(0)^\alpha \mid \alpha \in S(1)\}$ satisfy Hypothesis I. Set $\xi = (10^{15}) \in (\mathbb{Z}/2\mathbb{Z})^{16}$ and

$$\begin{aligned} &Q = \{(0^{48}),\ (\xi,\xi,0),\ (\xi,0,\xi),\ (0,\xi,\xi)\} \subset (\mathbb{Z}/2\mathbb{Z})^{48}, \\ &D(1) := D(0) + Q \subset (\mathbb{Z}/2\mathbb{Z})^{48}. \end{aligned}$$

Then $D(1)$ is an even code and satisfies $D(1) \subset S(1)^\perp$. Then by Proposition 7.4.8 the induced module

$$\mathrm{Ind}_{U_{D(0)}}^{U_{D(1)}} V_{E_8}^\alpha \otimes V_{E_8}^\alpha \otimes V_{E_8}^\alpha$$

is uniquely determined and is an irreducible untwisted $U_{D(1)}$-module for all $\alpha \in S_{E_8}$. Set $W_{E_8}^\alpha := U_{D_{E_8}+\xi} \boxtimes_{U_{D_{E_8}}} V_{E_8}^\alpha$ for $0 \neq \alpha \in S_{E_8}$. Then $W_{E_8}^\alpha \not\simeq V_{E_8}^\alpha$ as a $U_{D_{E_8}}$-module and we have the following decomposition as a $U_{D(0)} = U_{D_{E_8}} \otimes U_{D_{E_8}} \otimes U_{D_{E_8}}$-module:

$$\begin{aligned} \mathrm{Ind}_{U_{D(0)}}^{U_{D(1)}} V_{E_8}^\alpha \otimes V_{E_8}^\alpha \otimes V_{E_8}^\alpha \quad &= \quad V_{E_8}^\alpha \otimes V_{E_8}^\alpha \otimes V_{E_8}^\alpha \bigoplus V_{E_8}^\alpha \otimes W_{E_8}^\alpha \otimes W_{E_8}^\alpha \\ &\qquad \bigoplus W_{E_8}^\alpha \otimes V_{E_8}^\alpha \otimes W_{E_8}^\alpha \bigoplus W_{E_8}^\alpha \otimes W_{E_8}^\alpha \otimes V_{E_8}^\alpha. \end{aligned}$$

Since the fusion products for $U_{D_{E_8}}$-modules is associative and commutative, we have the following fusion rules for $U_{D_{E_8}}$-modules:

$$\begin{aligned} V_{E_8}^\alpha \times W_{E_8}^\beta \quad &= V_{E_8}^\alpha \times (U_{D_{E_8}+\xi} \times V_{E_8}^\beta) \\ &= (V_{E_8}^\alpha \times V_{E_8}^\beta) \times U_{D_{E_8}+\xi} \\ &= V_{E_8}^{\alpha+\beta} \times U_{D_{E_8}+\xi} \\ &= W_{E_8}^{\alpha+\beta}, \\ W_{E_8}^\beta \times W_{E_8}^\gamma \quad &= (U_{D_{E_8}+\xi} \times V_{E_8}^\beta) \times (U_{D_{E_8}+\xi} \times V_{E_8}^\gamma) \\ &= (U_{D_{E_8}+\xi} \times U_{D_{E_8}+\xi}) \times (V_{E_8}^\beta \times V_{E_8}^\gamma) \\ &= V_{E_8}^{\beta+\gamma}, \end{aligned} \tag{7.5.1}$$

where $\alpha, \beta, \gamma \in S_{E_8}$. Now set

$$\begin{aligned} S^\natural \quad &:= \mathrm{Span}_{\mathbb{Z}/2\mathbb{Z}} \{(1^{16}0^{16}0^{16}),\ (0^{16}1^{16}0^{16}),\ (0^{16}0^{16}1^{16}),\ (\alpha,\alpha,\alpha) \in (\mathbb{Z}/2\mathbb{Z})^{48} \mid \alpha \in S_{E_8}\} \\ &= \{(\alpha,\alpha,\alpha),\ (\alpha^c,\alpha,\alpha),\ (\alpha,\alpha^c,\alpha),\ (\alpha,\alpha,\alpha^c) \in (\mathbb{Z}/2\mathbb{Z})^{48} \mid \alpha \in S_{E_8}\} \end{aligned}$$

and $D^\natural := (S^\natural)^\perp$, where we have used α^c to denote a code word $(1^n) + \alpha$ of $(\mathbb{Z}/2\mathbb{Z})^n$ for $\alpha \in (\mathbb{Z}/2\mathbb{Z})^n$. Since $\dim_{\mathbb{Z}/2\mathbb{Z}} S^\natural = 7$, we have $\dim_{\mathbb{Z}/2\mathbb{Z}} D^\natural = 41$ and it is easy to see

$$D^\natural = \{(\beta_1,\beta_2,\beta_3) \in \left((\mathbb{Z}/2\mathbb{Z})^{16}\right)^{\oplus 3} \mid \beta_1+\beta_2+\beta_3 \in D_{E_8},\ \beta_1,\beta_2,\beta_3 \text{ are even}\}.$$

Since $S^\natural \subset S(0)$ by definition, $D(0) \subset D^\natural$ and the pairs $(D(0), S^\natural)$ and $(D^\natural, S^\natural)$ satisfy the condition (1) of Hypothesis I. Now we define $V(2)^\beta$, $\beta \in S^\natural$, as follows:

$$\begin{aligned}
V(2)^{(\alpha,\alpha,\alpha)} &:= V_{E_8}^\alpha \otimes V_{E_8}^\alpha \otimes V_{E_8}^\alpha,\\
V(2)^{(\alpha,\alpha,\alpha^c)} &:= W_{E_8}^\alpha \otimes W_{E_8}^\alpha \otimes V_{E_8}^\alpha,\\
V(2)^{(\alpha,\alpha^c,\alpha)} &:= W_{E_8}^\alpha \otimes V_{E_8}^\alpha \otimes W_{E_8}^\alpha,\\
V(2)^{(\alpha^c,\alpha,\alpha)} &:= V_{E_8}^\alpha \otimes W_{E_8}^\alpha \otimes W_{E_8}^\alpha,
\end{aligned}$$

where $\alpha \in S_{E_8}$. Since the top weight of $W_{E_8}^\alpha$, $\alpha \in S_{E_8} \setminus \{0\}$, is contained in $\frac{1}{2}\mathbb{N}$ by the fusion rules (7.2.1), the top weight of $V(2)^\beta$, $\beta \in S^\natural$, is contained in $\mathbb{N}$. And by the fusion rules (7.5.1), the set $\{V(2)^\alpha \mid \alpha \in S^\natural\}$ of inequivalent irreducible $V(2)^0 = U_{D(0)}$-modules satisfies the condition (3) of Hypothesis I. Therefore, we have a unique 2A-framed VOA structure on

$$V(2) := \oplus_{\alpha\in S^\natural} V(2)^\alpha$$

with structure codes $(D(0), S^\natural)$ by Theorem 7.4.5. Following Theorem 7.4.9, define

$$V^\natural := \mathrm{Ind}_{D(0)}^{D^\natural} V(2) = \bigoplus_{\alpha\in S^\natural} \mathrm{Ind}_{U_{D(0)}}^{U_{D^\natural}} V(2)^\alpha.$$

Then $V^\natural$ has a unique structure of a 2A-framed VOA with structure codes $(D^\natural, S^\natural)$.

Theorem 7.5.3. *([M5]) The 2A-framed VOA $V^\natural$ with structure codes $(D^\natural, S^\natural)$ is isomorphic to the moonshine VOA constructed by Frenkel-Lepowsky-Muerman [FLM] and the full automorphism group* $\mathrm{Aut}(V^\natural)$ *is the Monster finite sporadic simple group* $\mathbb{M}$.

Proof: This is the main theorem of [M5] and the proof is not easy. Here we give a brief explanation. Let $\omega = e^1 + \cdots + e^{48}$ be the 2A-frame of $V^\natural$. Then we have mutually commutative 48 Miyamoto involutions $\{\tau_{e^i} \mid 1 \le i \le 48\}$ acting on $V^\natural$. One can easily check that $\theta = \tau_{e^1}\tau_{e^2}$ is of order 2. It is shown in [M5] that the fixed point subalgebra $(V^\natural)^{\langle\theta\rangle}$ is isomorphic to the $\mathbb{Z}_2$-orbifold V_Λ^+ of the Leech lattice VOA V_Λ. By our construction, it is not difficult to see that our $V^\natural$ has no weight one subspace. Therefore, by the classification of irreducible V_Λ^+-modules, $V^\natural \simeq V_\Lambda^+ \oplus (V_\Lambda^T)^+$. Since the structure of a $\mathbb{Z}_2$-graded extension $V_\Lambda^+ \oplus (V_\Lambda^T)^+$ is unique over $\mathbb{C}$, our moonshine VOA is isomorphic to FLM's moonshine VOA.

One can prove that $V^\natural$ is generated by its weight two subspace by considering codes $D^\natural$ and $S^\natural$. Then by the results of [C], [G] and [Ti], we can prove that $\mathrm{Aut}(V^\natural) \simeq \mathbb{M}$. For a different proof, see [M5]. ∎

Remark 7.5.4. It is shown in [C] [M1] that $\theta = \tau_{e^1}\tau_{e^2}$ belongs to the 2B-conjugacy class of the Monster.

Chapter 8

Applications to the Moonshine VOA

8.1 2A-involution and the baby-monster SVOA

In this subsection we consider an application of the theory of simple current extensions to the theory of Miyamoto involutions.

8.1.1 Commutant superalgebra associated to the Ising model

Let $(V, Y_V(\cdot, z), \mathbb{1}, \omega)$ be a simple VOA. Suppose that V contains a conformal vector e with central charge $1/2$. We assume that the Virasoro sub VOA generated by e is simple and we denote it by $\mathrm{Vir}(e)$.

Remark 8.1.1. Given a conformal vector e with central charge $1/2$, we can determine whether e generates a simple Virasoro VOA $L_{\mathrm{Vir}}(1/2, 0)$ by checking whether the singular vector

$$\big(64(e_{(-1)})^3 + 93(e_{(-2)})^2 - 264 e_{(-3)} e_{(-1)} - 108 e_{(-5)}\big)\,\mathbb{1}$$

vanishes in V or not (cf. [DMZ]).

Since $\mathrm{Vir}(e)$ is rational, we can decompose V into a direct sum of irreducible $\mathrm{Vir}(e)$-modules:

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16),$$

where $V_e(h)$ is the sum of all irreducible $\mathrm{Vir}(e)$-submodules isomorphic to $L(1/2, h)$, $h \in \{0, 1/2, 1/16\}$. Recall the Miyamoto involutions $\tau_e \in \mathrm{Aut}(V)$ and $\sigma_e \in \mathrm{Aut}(V^{\langle \sigma_e \rangle})$. By definition, τ_e acts on $V_e(0) \oplus V_e(1/2)$ as a scalar 1 and on $V_e(1/16)$ as a scalar -1, and σ_e acts on $V_e(0)$ as a scalar 1 and on $V_e(1/2)$ as a scalar -1.

Define the space of highest weight vectors by $T_e(h) := \{v \in V \mid e_{(1)} v = hv\}$ for $h \in \{0, 1/2, 1/16\}$. Then as vector spaces we have isomorphisms $V_e(h) \simeq L(1/2, h) \otimes T_e(h)$. By the Miyamoto involution σ_e, the subspace $V_e(0) \oplus V_e(1/2) = L(1/2, 0) \otimes T_e(0) \oplus L(1/2, 1/2) \otimes T_e(1/2)$ has a structure of a simple $\mathbb{Z}_2$-graded VOA. We also know that

$L(1/2,0) \oplus L(1/2,1/2)$ is a simple SVOA. So it is likely to hold that the commutant subalgebra $T_e(0)$ affords a $\mathbb{Z}_2$-graded extension $T_e(0) \oplus T_e(1/2)$. We prove that this is true. First, we show that a decomposition $\omega = e + (\omega - e)$ is orthogonal if V is of CFT-type.

Lemma 8.1.2. *If V is of CFT-type, then $\omega = e + (\omega - e)$ is an orthogonal decomposition.*

Proof: We compute $e_{(1)}\omega_{(2)}e$.

$$\begin{aligned} e_{(1)}\omega_{(2)}e &= \omega_{(2)}e_{(1)}e + [e_{(1)}, \omega_{(2)}]e \\ &= 2\omega_{(2)}e - [\omega_{(2)}, e_{(1)}]e \\ &= 2\omega_{(2)}e - \{(\omega_{(0)}e)_{(3)} + 2(\omega_{(1)}e)_{(2)} + (\omega_{(2)}e)_{(1)}\}e \\ &= 2\omega_{(2)}e - (\omega_{(2)}e)_{(1)}e. \end{aligned}$$

By the skew-symmetry, we have $(\omega_{(2)}e)_{(1)}e = e_{(1)}\omega_{(2)}e - \omega_{(0)}e_{(2)}\omega_{(2)}e$. Since $e_{(2)}\omega_{(2)}e \in V_0 = \mathbb{C}\mathbb{1}$, $\omega_{(0)}e_{(2)}\omega_{(2)}e = 0$ and so $(\omega_{(2)}e)_{(1)}e = e_{(1)}\omega_{(2)}e$. Substituting this into the equality above, we get $e_{(1)}\omega_{(2)}e = \omega_{(2)}e$. Namely, $\omega_{(2)}e$ is an eigenvector for $e_{(1)}$ with eigenvalue 1. Since V is a module for $\mathrm{Vir}(e)$, there is no eigenvector with $e_{(1)}$-weight 1. Hence $\omega_{(2)}e = 0$. Then the assertion follows from Lemma 3.8.4. ∎

By this lemma, we will assume that V is of CFT-type.

Proposition 8.1.3. *(1) $T_e(0) = \mathrm{Ker}_V e_{(0)} = \mathrm{Com}_V(\mathrm{Vir}(e))$ is a simple sub VOA with the Virasoro vector $\omega - e$.*
(2) $T_e(1/2)$ is an irreducible $T_e(0)$-module.
(3) $\mathrm{Vir}(e) = \mathrm{Ker}_V(\omega - e)_{(0)} = \mathrm{Com}_V(T_e(0))$.

Proof: (1): Let $v \in V$. Since $e_{(1)}v = 0$ implies $e_{(0)}v = 0$, $T_e(0) = \mathrm{Ker}_V e_{(1)} = \mathrm{Ker}_V e_{(0)}$. So we only need to show that $T_e(0)$ is simple. Since V is simple, the τ_e-orbifold $V^{\langle \tau_e \rangle} = V_e(0) \oplus V_e(1/2)$ is simple. Then the σ_e-orbifold $(V^{\langle \tau_e \rangle})^{\langle \sigma_e \rangle} = V_e(0)$ is also simple. Since $\mathrm{Vir}(e) \otimes T_e(0) \ni a \otimes b \mapsto a_{(-1)}b \in V_e(0)$ is an isomorphism of VOAs, $T_e(0)$ is also simple.
(2): Since both $V^{\langle \tau_e \rangle} = V_e(0) \oplus V_e(1/2)$ and $V_e(0)$ are simple VOAs, $V_e(1/2)$ is an irreducible $V_e(0)$-module. So $T_e(1/2)$ is also irreducible.
(3): As $\omega - e$ is a conformal vector, $\mathrm{Ker}_V(\omega - e)_{(0)}$ is generally contained in $\mathrm{Ker}_V(\omega - e)_{(1)}$. On the other hand, since V is of CFT-type, $\mathrm{Ker}_V(\omega - e)_{(1)} = \mathrm{Vir}(e)$. Then

$$\mathrm{Vir}(e) \subset \mathrm{Com}_V(\mathrm{Com}_V(\mathrm{Vir}(e))) = \mathrm{Ker}_V(\omega - e)_{(0)}$$

implies $\mathrm{Vir}(e) = \mathrm{Ker}_V(\omega - e)_{(0)}$. ∎

Recall the construction of code VOAs in Section 7.3.1. We note that our construction in Section 7.3.1 works in particular to define a tensor product of any two SVOAs.

Theorem 8.1.4. *Suppose that $T_e(1/2) \neq 0$. Then there exists a simple SVOA structure on $T_e(0) \oplus T_e(1/2)$ such that the even part of a tensor product of SVOAs*

$$\{L(1/2,0) \oplus L(1/2,1/2)\} \otimes \{T_e(0) \oplus T_e(1/2)\}$$

is isomorphic to $V_e(0) \oplus V_e(1/2)$ as a VOA.

Proof: We shall define vertex operators on an abstract space $T_e(0) \oplus T_e(1/2)$. First, we show an existence of a $T_e(0)$-intertwining operator of type $T_e(1/2) \times T_e(1/2) \to T_e(0)$. Write $Y_V(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ and $Y_V(e, z) = \sum_{n\in\mathbb{Z}} L^e(n)z^{-n-2}$. Since $L(0) - L^e(0)$ semisimply acts on both $T_e(0)$ and $T_e(1/2)$, we can take bases $\{a^\gamma \mid \gamma \in \Gamma\}$ and $\{u^\lambda \mid \lambda \in \Lambda\}$ of $T_e(0)$ and $T_e(1/2)$, respectively, consisting of eigen vectors for $L(0) - L^e(0)$. Let $\pi_\gamma : V_e(0) \to L(1/2, 0) \otimes a^\gamma$, $\gamma \in \Gamma$, be a projection map. For $\gamma \in \Gamma$ and $\lambda, \mu \in \Lambda$, we define a linear operator $I^\gamma_{\lambda\mu}(\cdot, z)$ of type $L(1/2, 1/2) \times L(1/2, 1/2) \to L(1/2, 0) \otimes a^\gamma$ by

$$\begin{aligned} I^\gamma_{\lambda\mu}(x, z)y \;&:= z^{-L(0)+L^e(0)}\pi_\gamma Y(z^{L(0)-L^e(0)}x \otimes u^\lambda, z)z^{L(0)-L^e(0)}y \otimes u^\mu \\ &= z^{-|\gamma|+|\lambda|+|\mu|}\pi_\gamma Y(x \otimes u^\lambda, z)y \otimes u^\mu, \end{aligned}$$

for $x, y \in L(1/2, 1/2)$, where $|\lambda|$, $|\mu|$ and $|\gamma|$ denote the $(L(0) - L^e(0))$-weight of u^λ, u^μ and a^γ, respectively. Then by Proposition 3.8.9 the operator $I^\gamma_{\lambda\mu}(\cdot, z)$ is an $L(1/2, 0)$-intertwining operator of type $L(1/2, 1/2) \times L(1/2, 1/2) \to L(1/2, 0)$. Since the space of intertwining operators of that type is one-dimensional, each $I^\gamma_{\lambda\mu}(\cdot, z)$ is proportional to the vertex operator map $Y_M(\cdot, z)$ on the SVOA $M = L(1/2, 0) \oplus L(1/2, 1/2)$ which we constructed explicitly in Section 7.1.2. Thus there exist scalars $c^\gamma_{\lambda\mu} \in \mathbb{C}$ such that $I^\gamma_{\lambda\mu}(\cdot, z) = c^\gamma_{\lambda\mu} Y_M(\cdot, z)$. Then the vertex operator of $x \otimes u^\lambda \in L(1/2, 1/2) \otimes T_e(1/2)$ on $V_e(1/2)$ can be written as follows:

$$Y_V(x \otimes u^\lambda, z)y \otimes u^\mu = Y_M(x, z)y \otimes \sum_{\gamma\in\Gamma} c^\gamma_{\lambda\mu} a^\gamma z^{|\gamma|-|\lambda|-|\mu|}.$$

Thus, by setting $J(u^\lambda, z)u^\mu := \displaystyle\sum_{\gamma\in\Gamma} c^\gamma_{\lambda\mu} a^\gamma z^{|\gamma|-|\lambda|-|\mu|}$, we obtain a decomposition

$$Y_V(x \otimes u^\lambda, z)y \otimes u^\mu = Y_M(x, z)y \otimes J(u^\lambda, z)u^\mu$$

for $x \otimes u^\lambda$, $y \otimes u^\mu \in L(1/2, 1/2) \otimes V_e(1/2)$. We claim that $J(\cdot, z)$ is a $T_e(0)$-intertwining operator of type $T_e(1/2) \times T_e(1/2) \to T_e(0)$. It is obvious that $J(u, z)v$ contains finitely many negative powers of z and the $(\omega - e)_{(0)}$-derivation property $J((\omega - e)_0 u, z)v = \frac{d}{dz} J(u, z)v$ hold for all $u, v \in T_e(1/2)$. So we should show that $J(\cdot, z)$ satisfies both the commutativity and the associativity. Let $a \in T_e(0)$ and $u, v \in T_e(1/2)$ be arbitrary elements. Then the commutativity of vertex operators on V leads to

$$\begin{aligned} &(z_1 - z_2)^N Y_V(\mathbb{1} \otimes a, z_1)Y_V(\psi_{-\frac{1}{2}}\mathbb{1} \otimes u, z_2)\psi_{-\frac{1}{2}}\mathbb{1} \otimes v \\ &= (z_1 - z_2)^N Y_V(\psi_{-\frac{1}{2}}\mathbb{1} \otimes u, z_2)Y_V(\mathbb{1} \otimes a, z_1)\psi_{-\frac{1}{2}}\mathbb{1} \otimes v. \end{aligned}$$

for sufficiently large N. Rewriting the equality above we get

$$\begin{aligned}&(z_1-z_2)^N Y_M(\psi_{-\frac{1}{2}}\mathbb{1}, z_2)\psi_{-\frac{1}{2}}\mathbb{1}\otimes Y_{T_e(0)}(a,z_1)J(u,z_2)v\\&=(z_1-z_2)^N Y_M(\psi_{-\frac{1}{2}}\mathbb{1}, z_2)\psi_{-\frac{1}{2}}\mathbb{1}\otimes J(u,z_2)Y_{T_e(1/2)}(a,z_1)v,\end{aligned}$$

where $Y_{T_e(0)}(a,z)$ and $Y_{T_e(1/2)}(\cdot,z)$ denote the vertex operator of $a\in T_e(0)$ on $T_e(0)$ and $T_e(1/2)$, respectively. By comparing the coefficients of $(\psi_{-\frac{1}{2}}\mathbb{1})_{(0)}\psi_{-\frac{1}{2}}\mathbb{1}=\mathbb{1}$, we get the commutativity:

$$(z_1-z_2)^N Y_{T_e(0)}(a,z_1)J(u,z_2)v=(z_1-z_2)^N J(u,z_2)Y_{T_e(1/2)}(a,z_1)v.$$

Similarly, by considering coefficients of $Y_V(Y_V(\mathbb{1}\otimes a,z_0)\psi_{-\frac{1}{2}}\mathbb{1}\otimes u,z_2)\psi_{-\frac{1}{2}}\mathbb{1}\otimes v$ in V, we obtain the associativity:

$$(z_0+z_2)^N Y_{T_e(0)}(a,z_0+z_2)J(u,z_2)v=(z_2+z_0)^N J(Y_{T_e(1/2)}(a,z_0)u,z_2)v.$$

Hence, $J(\cdot,z)$ is a $T_e(0)$-intertwining operator of the desired type.

Using $Y_V(\cdot,z)$ and $J(\cdot,z)$, we introduce a vertex operator map $\hat{Y}(\cdot,z)$ on $T_e(0)\oplus T_e(1/2)$. Let $a,b\in T_e(0)$ and $u,v\in T_e(1/2)$. We define

$$\begin{aligned}&\mathbb{1}\otimes\hat{Y}(a,z)b:=Y_V(\mathbb{1}\otimes a,z)\mathbb{1}\otimes b,\quad \psi_{-\frac{1}{2}}\mathbb{1}\otimes\hat{Y}(a,z)u:=Y_V(\mathbb{1}\otimes a,z)\psi_{-\frac{1}{2}}\mathbb{1}\otimes u,\\&\psi_{-\frac{1}{2}}\mathbb{1}\otimes\hat{Y}(u,z)a:=e^{z(L(-1)-L^e(-1))}Y_V(\mathbb{1}\otimes a,z)\psi_{-\frac{1}{2}}\mathbb{1}\otimes u,\quad \hat{Y}(u,z)v:=J(u,z)v.\end{aligned}$$

Then all $\hat{Y}(\cdot,z)$ are $T_e(0)$-intertwining operators. We note that $\hat{Y}(\cdot,z)$ satisfies the vacuum condition:

$$\hat{Y}(x,z)\mathbb{1}\in x+(T_e(0)\oplus T_e(1/2))\,[[z]]z$$

for any $x\in T_e(0)\oplus T_e(1/2)$. Hence, to prove that $T_e(0)\oplus T_e(1/2)$ is a simple SVOA, it is sufficient to show that the vertex operator map $\hat{Y}(\cdot,z)$ defined above satisfies the commutativity. By our definition, the vertex operator map $Y_V(a\otimes x,z)$ of $a\otimes x\in L(1/2,h)\otimes T_e(h)=V_e(h)$, $h=0,1/2$, can be written as $Y_M(a,z)\otimes\hat{Y}(x,z)$. Because of our manifest construction of $Y_M(\cdot,z)$ in Section 7.1.2, we can perform explicit computations of the vertex operator $Y_M(\cdot,z)$ on $L(1/2,0)\oplus L(1/2,1/2)$. Therefore, by comparing the coefficients of vertex operators on V, we can prove that $\hat{Y}(\cdot,z)$ satisfies the (super-)commutativity. Thus, by our definition, $(T_e(0)\oplus T_e(1/2),\hat{Y}(\cdot,z),\mathbb{1},\omega-e)$ carries a structure of a simple SVOA. The rest of the assertion is now clear. ∎

Remark 8.1.5. There is another proof of Theorem 8.1.4 in [Hö1]. In [Hö1], he assumed the existence of a positive definite invariant bilinear form on a real form of V. However, our argument does not need the assumption on the unitary form.

Since $\tau_e^2 = 1$ on V, the space $V_e(1/16)$ is an irreducible $V^{\langle\tau_e\rangle}$-module. As a $(V^{\langle\tau_e\rangle})^{\langle\sigma_e\rangle} = \mathrm{Vir}(e) \otimes T_e(0)$-module, $V_e(1/16)$ can be written as $L(1/2, 1/16) \otimes T_e(1/16)$. It is not clear that $T_e(1/16)$ is irreducible under $T_e(0)$. However, we can prove that it is irreducible under $T_e(0) \oplus T_e(1/2)$.

Theorem 8.1.6. *Suppose that $V_e(1/16) \neq 0$. Then $T_e(1/16)$ carries a structure of an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. Moreover, $V_e(1/16)$ is isomorphic to a tensor product of an irreducible $\mathbb{Z}_2$-twisted $L(1/2, 0) \oplus L(1/2, 1/2)$-module $L(1/2, 1/16)^+$ and an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module $T_e(1/16)$.*

Proof: The idea of the proof is the same as that of Theorem 8.1.4. Computing vertex operators on $L(1/2, 1/16)^+$ and then comparing the coefficients in V, we will reach the assertion. Denote by $Y_N(\cdot, z)$ the vertex operator map on the $\mathbb{Z}_2$-twisted $L(1/2, 0) \oplus L(1/2, 1/2)$-module $L(1/2, 1/16)^+$ as we constructed in Section 7.1.2. Let $a \otimes b \in L(1/2, h) \otimes T_e(h)$ with $h = 0$ or $1/2$ and $x \otimes y \in L(1/2, 1/16) \otimes T_e(1/16)$. As we did before, we can find $T_e(0)$-intertwining operators $Y_{T_e(h)\times T_e(1/16)}(\cdot, z)$ of types $T_e(h) \times T_e(1/16) \to T_e(1/16)$ such that

$$Y_V(a \otimes b, z)x \otimes y = Y_N(a, z)x \otimes Y_{T_e(h)\times T_e(\frac{1}{16})}(b, z)y. \tag{8.1.1}$$

Define $\hat{Y}(b, z)y := Y_{T_e(h)\times T_e(\frac{1}{16})}(b, z)y$ for $b \in T_e(h)$, $h = 0, 1/2$ and $y \in T_e(1/16)$. By direct computations, we can prove that the $\mathbb{Z}_2$-twisted Jacobi identity for $Y_N(\cdot, z)$ together with the Jacobi identity for $Y_V(\cdot, z)$ supplies the $\mathbb{Z}_2$-twisted Jacobi identity for $\hat{Y}(\cdot, z)$. Thus, $(T_e(1/16), \hat{Y}(\cdot, z))$ is a $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. Since $V_e(1/16) = L(1/2, 1/16) \otimes T_e(1/16)$ is irreducible under $V_e(0) \oplus V_e(1/2)$, the irreducibility of $T_e(1/16)$ is obvious. ∎

8.1.2 One-point stabilizer

We keep the setup of previous subsection. Assume that $V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16)$ with $V_e(h) \neq 0$ for $h = 0, 1/2, 1/16$. Define the one-point stabilizer by $C_{\mathrm{Aut}(V)}(e) := \{\rho \in \mathrm{Aut}(V) \mid \rho(e) = e\}$. Then by $\tau_{\rho(e)} = \rho\tau_e\rho^{-1}$ for any $\rho \in \mathrm{Aut}(V)$, we have $C_{\mathrm{Aut}(V)}(e) \subset C_{\mathrm{Aut}(V)}(\tau_e)$, where $C_{\mathrm{Aut}(V)}(\tau_e)$ denotes the centralizer of an involution $\tau_e \in \mathrm{Aut}(V)$.

Lemma 8.1.7. *There are group homomorphisms $\psi_1 : C_{\mathrm{Aut}(V)}(e) \to C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ and $\psi_2 : C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \to \mathrm{Aut}(T_e(0))$ such that $\mathrm{Ker}(\psi_1) = \langle\tau_e\rangle$ and $\mathrm{Ker}(\psi_2) = \langle\sigma_e\rangle$.*

Proof: Let $\rho \in C_{\mathrm{Aut}(V)}(e)$. Then ρ preserves the space of highest weight vectors $T_e(h)$ where $h \in \{0, 1/2, 1/16\}$. Then we can define the actions of ρ on the space of highest weight vectors $T_e(h)$ and the eigenspaces $V_e(h)$ for $h \in \{0, 1/2, 1/16\}$. In particular, we have group homomorphisms $\psi_1 : C_{\mathrm{Aut}(V)}(e) \to C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ and ψ_2 :

$C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \to \mathrm{Aut}(T_e(0))$ by a natural way. Assume that $\psi_1(\rho) = \mathrm{id}_{V^{\langle\tau_e\rangle}}$ for $\rho \in C_{\mathrm{Aut}(V)}(e)$. Since $\rho \in C_{\mathrm{Aut}(V)}(\tau_e)$, ρ acts on $V_e(1/16)$ commutes with the action of $V^{\langle\tau_e\rangle} = V_e(0) \oplus V_e(1/2)$ on its module $V_e(1/2)$. Therefore, ρ is a scalar by Schur's lemma and hence $\rho \in \langle\tau_e\rangle \subset C_{\mathrm{Aut}(V)}(\tau_e)$. Similarly, if $\psi_1(\rho') = \mathrm{id}_{T_e(0)}$ for $\rho' \in C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$, then $\rho' \in \langle\sigma_e\rangle \subset C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. ∎

Below we will work over the following setup.

Hypothesis II

V is a simple holomorphic C_2-cofinite VOA of CFT type such that

(1) V contains a conformal vector e which generates a simple Virasoro VOA $\mathrm{Vir}(e) \simeq L(1/2,0)$.

(2) V has a decomposition $V_e(0) \oplus V_e(1/2) \oplus V_e(1/16)$ such that $V_e(h) \neq 0$ for $h \in \{0, 1/2, 1/16\}$.

(3) $T_e(0)$ is a simple rational C_2-cofinite VOA of CFT-type.

(4) $V_e(1/16)$ is a simple current $V^{\langle\tau_e\rangle}$-module and $T_e(1/2)$ is a simple current $T_e(0)$-module.

Theorem 8.1.8. *Under Hypothesis II, $V^{\langle\tau_e\rangle}$ has exactly four inequivalent irreducible modules $V^{\langle\tau_e\rangle}$, $V_e(1/16)$, $W^0 := L(1/2,0)\otimes T_e(1/2) \oplus L(1/2,1/2)\otimes T_e(0)$ and $W^1 := V_e(1/16) \boxtimes_{V^{\langle\tau_e\rangle}} W^0$.*

Proof: Note that $V_e(0) = \mathrm{Vir}(e)\otimes T_e(0)$ and $V^{\langle\tau_e\rangle}$ are simple rational C_2-cofinite VOAs of CFT-type under Hypothesis II. Therefore, we can apply a theory of fusion products here. Since $V = V^{\langle\tau_e\rangle} \oplus V_e(1/16)$ is a $\mathbb{Z}_2$-graded simple current extension of $V^{\langle\tau_e\rangle}$, every irreducible $V^{\langle\tau_e\rangle}$-module is lifted to be either an irreducible V-module or an irreducible τ_e-twisted V-module. Moreover, the τ_e-twisted V-module is unique up to isomorphism by Theorem 10.3 of [DLM2]. Consider a $V_e(0)$-module $L(1/2,1/2)\otimes T_e(0)$. Since $T_e(1/2)$ is a simple current $T_e(0)$-module, the space

$$W^0 = L(1/2,1/2)\otimes T_e(0) \oplus L(1/2,0)\otimes T_e(1/2)$$

has a unique structure of an irreducible $V^{\langle\tau_e\rangle}$-module by Theorem 4.5.3. Then the induced module

$$W = W^0 \oplus W^1, \quad W^1 = V_e(1/16) \boxtimes_{V^{\langle\tau_e\rangle}} W^0,$$

becomes an irreducible τ_e-twisted V-module again by Theorem 4.5.3. Therefore, $V^{\langle\tau_e\rangle}$ has exactly four irreducible modules as in the assertion. Finally we remark that $V^{\langle\tau_e\rangle}$, $V_e(1/16)$ and W^1 have integral top weights and W^0 has a top weight in $1/2+\mathbb{N}$. ∎

By the fusion rules (7.2.1), we note that W^1 as a $\mathrm{Vir}(e)$ is a direct sum of copies of $L(1/2,1/16)$. Set the space of highest weight vectors of W^1 by $Q_e(1/16) := \{v \in W^1 \mid$

$L^e(0)v = (1/16) \cdot v\}$. Then as a $\mathrm{Vir}(e) \otimes T_e(0)$-module, $W^1 \simeq L(1/2, 1/16) \otimes Q_e(1/16)$. In this case, we can also verify that the space $Q_e(1/16)$ naturally carries an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module structure such that $W^1 \simeq L(1/2, 1/16)^+ \otimes Q_e(1/16)$ as a $(L(1/2, 0) \oplus L(1/2, 1/2)) \otimes (T_e(0) \oplus T_e(1/2))$-module.

Proposition 8.1.9. *If the $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module $T_e(1/16)$ is irreducible as a $T_e(0)$-module, then its $\mathbb{Z}_2$-conjugate is isomorphic to $Q_e(1/16)$ as a $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. In this case there are three irreducible $T_e(0)$-modules, $T_e(0)$, $T_e(1/2)$ and $T_e(1/16)$. Conversely, if $T_e(1/16)$ as a $T_e(0)$-module is not irreducible, then so is $Q_e(1/16)$ and in this case there are six inequivalent irreducible $T_e(0)$-modules.*

Proof: Assume that $T_e(1/16)$ is irreducible as a $T_e(0)$-module. Then its $\mathbb{Z}_2$-conjugate is not isomorphic to $T_e(1/16)$ as a $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. We denote the $\mathbb{Z}_2$-conjugate of $T_e(1/16)$ by $T_e(1/16)^-$. It is not difficult to generalize Theorem 4.5.2 to include the case of SVOA by a similar argument (see Appendix). Then we see that every irreducible $T_e(0)$-module is lifted to be either an irreducible $T_e(0) \oplus T_e(1/2)$-module or an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. Then by the classification of irreducible $V^{\langle \tau_e \rangle}$-modules in Theorem 8.1.8, we see that any irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module is isomorphic to one and only one of $T_e(1/16)$ and $T_e(1/16)^- = Q_e(1/16)$.

Conversely, if $T_e(1/16)$ is not irreducible, then it is a direct sum of inequivalent two irreducible $T_e(0)$-module as $T_e(1/2)$ is a simple current $T_e(0)$-module. Then $Q_e(1/16)$ is also a direct sum of two inequivalent irreducible $T_e(0)$-modules and $Q_e(1/16) \not\simeq T_e(1/16)$ as $T_e(0)$-modules because of the classification of irreducible $V^{\langle \tau_e \rangle}$-modules. ∎

Corollary 8.1.10. *If $T_e(1/16)$ is irreducible as a $T_e(0)$-module, then $V^{\langle \tau_e \rangle} \oplus W^1$ is a $\mathbb{Z}_2$-graded simple current extension of $V^{\langle \tau_e \rangle}$ which is equivalent to $V = V^{\langle \tau_e \rangle} \oplus V_e(1/16)$.*

Proof: If $T_e(1/16)$ is an irreducible $T_e(0)$-module, then by the previous proposition the $\mathbb{Z}_2$-conjugate $V_e(0) \oplus V_e(1/2)$-module of $V_e(1/16) = L(1/2, 1/16) \otimes T_e(1/16)$ is isomorphic to $W^1 = L(1/2, 1/16) \otimes Q_e(1/16)$. Then as the $\mathbb{Z}_2$-conjugate extension of $V^{\langle \tau_e \rangle} = V_e(0) \oplus V_e(1/2)$, $V^{\langle \tau_e \rangle} \oplus W^1$ has a structure of a $\mathbb{Z}_2$-graded extension. ∎

Remark 8.1.11. The above corollary implies that the $\mathbb{Z}_2$-twisted orbifold construction applied to V in the case of $\mathbb{Z}_2 = \langle \tau_e \rangle$ yields again V itself.

Theorem 8.1.12. *Under the Hypothesis II,*
(i) ψ_2 is surjective, that is, $C_{\mathrm{Aut}(V^{\langle \tau_e \rangle})}(e) \simeq \langle \sigma_e \rangle.\mathrm{Aut}(T_e(0))$.
(ii) $\mathrm{Aut}(T_e(0) \oplus T_e(1/2)) \simeq 2.(C_{\mathrm{Aut}(V^{\langle \tau_e \rangle})}(e)/\langle \sigma_e \rangle)$, *where 2 denotes the canonical $\mathbb{Z}_2$-symmetry on the SVOA $T_e(0) \oplus T_e(1/2)$.*
(iii) $|C_{\mathrm{Aut}(V^{\langle \tau_e \rangle})}(e) : C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle| \le 2$.
(iv) If $C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle$ is simple or has an odd order, then the extensions in (i) and (ii)

split. That is, $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \simeq \langle\sigma_e\rangle \times C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ *and* $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) \simeq 2\times \mathrm{Aut}(T_e(0))$.

Proof: Since we have an injection from $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)/\langle\sigma_e\rangle$ to $\mathrm{Aut}(T_e(0))$ by Lemma 8.1.7, we show that every element in $\mathrm{Aut}(T_e(0))$ lifts to be an element in $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. By Proposition 8.1.9, every irreducible $T_e(0)$-module appears in the one of $T_e(0)$, $T_e(1/2)$, $T_e(1/16)$ or $Q_e(1/16)$ as a submodule. In particular, we find that $T_e(0)$ is the only irreducible $T_e(0)$-module whose top weight is integral and $T_e(1/2)$ is the only irreducible $T_e(0)$-module whose top weight is in $1/2+\mathbb{N}$. Let $\rho\in\mathrm{Aut}(T_e(0))$. Then by considering top weights we can immediately see that $T_e(0)^\rho \simeq T_e(0)$ and $T_e(1/2)^\rho\simeq T_e(1/2)$. Then by Theorem 4.2.9 we have a lifting $\tilde\rho\in\mathrm{Aut}(T_e(0)\oplus T_e(1/2))$ such that $\tilde\rho T_e(0)=T_e(0)$, $\tilde\rho T_e(1/2)=T_e(1/2)$ and $\tilde\rho|_{T_e(0)}=\rho$. Since this lifting is unique up to a multiple of the canonical $\mathbb{Z}_2$-symmetry on $T_e(0)\oplus T_e(1/2)$, we have $\mathrm{Aut}(T_e(0)\oplus T_e(1/2))\simeq 2.\mathrm{Aut}(T_e(0))$. Now consider the canonical extension of $\tilde\rho$ to $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. We define $\tilde{\tilde\rho}\in C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ by

$$\tilde{\tilde\rho}|_{L(1/2,h)\otimes T_e(h)} := \mathrm{id}_{L(1/2,h)}\otimes\tilde\rho$$

for $h=0,1/2$. Then by this lifting $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ contains a subgroup which is isomorphic to $2.\mathrm{Aut}(T_e(0))$. Moreover, the canonical $\mathbb{Z}_2$-symmetry on $T_e(0)\oplus T_e(1/2)$ is naturally extended to $\sigma_e\in C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. Clearly $\psi_2(\tilde{\tilde\rho})=\rho$ and so ψ_2 is surjective. Hence we have the desired isomorphisms $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)\simeq\langle\sigma_e\rangle.\mathrm{Aut}(T_e(0))$ and $\mathrm{Aut}(T_e(0)\oplus T_e(1/2))\simeq 2.(C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)/\langle\sigma_e\rangle)$. This completes the proof of (i) and (ii).

Consider (iii). By Theorem 8.1.8, there are exactly three irreducible $V^{\langle\tau_e\rangle}$-modules whose top weights are integral, namely, $V^{\langle\tau_e\rangle}$, $V_e(1/16)$ and W^1. Since $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ acts on the 2-point set $\{V_e(1/16), W^1\}$ as a permutation, there is a subgroup H of $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ of index at most 2 such that $V_e(1/16)^\pi\simeq V_e(1/16)$ as a $V^{\langle\tau_e\rangle}$-module for all $\pi\in H$. Then there is a lifting $\tilde\pi\in C_{\mathrm{Aut}(V)}(e)$ of π such that $\psi_1(\tilde\pi)=\pi$ for each $\pi\in H$ by Theorem 4.2.9. Thus $|C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) : C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle|\le 2$ and (iii) holds.

Consider (iv). Suppose that $C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ is simple or has an odd order. We know that $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ contains a subgroup isomorphic to $C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ with index at most 2 by (iii). However, since $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ contains a normal subgroup $\langle\sigma_e\rangle$ of order 2, the index $|C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) : C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle|$ must be 2 and hence we obtain the desired isomorphism $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)\simeq\langle\sigma_e\rangle\times C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$. In this case it is easy to see that the extension $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) = 2.\mathrm{Aut}(T_e(0))$ splits. ∎

Corollary 8.1.13. *If* $C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ *is simple or has an odd order, then* $V_e(1/16)$ *is an irreducible* $V_e(0)$*-module and* $T_e(1/16)$ *is an irreducible* $T_e(0)$*-module. Therefore,* $V^{\langle\tau_e\rangle}\oplus W^1$ *forms the* σ_e*-conjugate extension of* $V=V^{\langle\tau_e\rangle}\oplus V_e(1/16)$ *and is isomorphic to* V.

Proof: Let H be the subgroup of $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ which fixes $V_e(1/16)$ in the action on the 2-point set $\{V_e(1/16), W^1\}$. It is shown in in the proof of (iii) of Theorem 8.1.12 that we have inclusions

$$H \subset C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle \subset C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) = \langle\sigma_e\rangle \times C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle.$$

Therefore, $\sigma_e \notin H$ and hence σ_e permutes $V_e(1/16)$ and W^1. Then $V_e(1/16)$ is an irreducible $V_e(0)$-module by Proposition 8.1.9 and hence $T_e(1/16)$ as a $T_e(0)$ as a $T_e(0)$-module is irreducible. The rest of the assertion is now clear. ∎

Remark 8.1.14. The assertion (iii) of Theorem 8.1.12 is already established in [M11]. Also, we should note that the idea of the above proof is already developed in [Sh].

8.1.3 Superalgebras associated to 2A-framed VOA

Let V be a 2A-framed VOA with structure codes (D,S). We assume that the pair (D,S) satisfies the condition (1-ii) of Hypothesis I and $D = S^\perp$. Then V is a holomorphic VOA by Theorem 7.4.2. Let $\omega = e^1+\cdots+e^n$ be the 2A-frame of V. We consider the commutant subalgebra for $\mathrm{Vir}(e^1)$. For simplicity, set $e = e^1$. Then we have a decomposition

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16).$$

Here we also assume that $\{1\}\cap\mathrm{Supp}(S) \neq \emptyset$, that is, $V_e(1/16) \neq 0$. Then by the condition (1-ii) of Hypothesis I we have $V_e(1/2) \neq 0$. Let $V = \oplus_{\alpha\in S} V^\alpha$ be the decomposition according to the structure codes (D,S). Then V^0 is isomorphic to the code VOA U_D associated to the even linear code D and V_D is an S-graded simple current extension of V^0 by Lemma 7.4.3. Set $S^0 = \{\alpha \in S \mid \{1\}\cap\mathrm{Supp}(\alpha) = \emptyset\}$ and $S^1 = \{\alpha \in S \mid \{1\}\cap\mathrm{Supp}(\alpha) = \{1\}\}$. Then $S = S^0 \sqcup S^1$ (disjoint union) and we have a $\mathbb{Z}_2$-grading

$$V = (\oplus_{\alpha\in S^0} V^\alpha) \oplus \left(\oplus_{\beta\in S^1} V^\beta\right)$$

with $V_e(0)\oplus V_e(1/2) = \oplus_{\alpha\in S^0} V^\alpha$ and $V_e(1/16) = \oplus_{\beta\in S^1} V^\beta$. Since V^α, $\alpha\in S$, are simple current V^0-modules, $V_e(1/16)$ is a simple current $V^{\langle\tau_e\rangle} = V_e(0)\oplus V_e(1/2)$-module.

Write $V_e(h) = L(1/2,h)\otimes T_e(h)$ for $h = 0, 1/2, 1/16$ as we did before. Then by Theorem 8.1.4, we know that $T_e(0)\oplus T_e(1/2)$ is a simple SVOA. The Virasoro vector of $T_e(0)$ is given by $\omega - e^1 = e^2+\cdots+e^n$ and so $T_e(0)$ is a 2A-framed VOA. We compute the structure codes of $T_e(0)$. Define $\phi_\epsilon : (\mathbb{Z}/2\mathbb{Z})^{n-1} \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^n$ by $(\mathbb{Z}/2\mathbb{Z})^{n-1} \ni \alpha \mapsto (\epsilon,\alpha) \in (\mathbb{Z}/2\mathbb{Z})^n$ for $\epsilon = 0,1$, and set

$$D^\epsilon := \{\alpha\in(\mathbb{Z}/2\mathbb{Z})^{n-1} \mid \phi_\epsilon(\alpha)\in D\},\ \epsilon = 0,1, \quad S^{0,0} := \{\beta\in(\mathbb{Z}/2\mathbb{Z})^{n-1} \mid \phi_0(\beta)\in S^0\}.$$

Proposition 8.1.15. *The structure codes of $T_e(0)$ with respect to the 2A-frame $e^2+\cdots+e^n$ are $(D^0, S^{0,0})$.*

Proof: For $\alpha \in S^0$, define $V^{\alpha,\epsilon}$ to be the sum of all irreducible $\otimes_{i=1}^n \mathrm{Vir}(e^i)$-modules whose $\mathrm{Vir}(e^1)$-components are isomorphic to $L(1/2,\epsilon/2)$ for $\epsilon = 0,1$. Then $V_e(1/2) \neq 0$ implies that $V^{\alpha,\epsilon} \neq 0$ for all $\alpha \in S^0$ and $\epsilon = 0,1$. Therefore, $V^\alpha = V^{\alpha,0} \oplus V^{\alpha,1}$ and we obtain the 1/16-word decompositions $V_e(0) = \oplus_{\alpha\in S^0} V^{\alpha,0}$ and $V_e(1/2) = \oplus_{\alpha\in S^0} V^{\alpha,1}$. Since $D = \phi_0(D^0) \sqcup \phi_1(D^1)$, $V^{0,0} \simeq L(1/2,0) \otimes U_{D^0}$. Thus $T_e(0)$ has the 1/16-word decomposition $T_e(0) = \oplus_{\alpha\in S^{0,0}} T_e(0)^\alpha$ such that $\tau(T_e(0)^\alpha) = \alpha$ and $T_e(0)^0 \simeq U_{D^0}$. Hence the structure codes of $T_e(0)$ are $(D^0, S^{0,0})$. ∎

Remark 8.1.16. By the proof above, we find that $T_e(1/2)$ also has the 1/16-word decomposition $T_e(1/2) = \oplus_{\alpha\in S^{0,0}} T_e(1/2)^\alpha$ such that $\tau(T_e(1/2)^\alpha) = \alpha$. In particular, $T_e(1/2)^0$ is isomorphic to a coset module U_{D^1}.

The following is easy to see:

Lemma 8.1.17. *The structure codes $(D^0, S^{0,0})$ satisfy the condition (1) of Hypothesis I.*

Thus $T_e(0)$ is an S^0-graded simple current extension of $T_e(0)^0 = U_{D^0}$ by Lemma 7.4.3. Thanks to Theorem 7.4.5, we can construct $T_e(0)$ as a 2A-framed VOA without reference to V.

Proposition 8.1.18. $T_e(1/2)$ *is a simple current* $T_e(0)$*-module.*

Proof: By Remark 8.1.16, $T_e(1/2)$ is isomorphic to the induced module $\mathrm{Ind}_{U_{D^0}}^{T_e(0)} U_{D^1}$ given by Theorem 4.5.2. Therefore, we have the fusion rule

$$T_e(1/2) \boxtimes_{T_e(0)} T_e(1/2) = \mathrm{Ind}_{U_{D^0}}^{T_e(0)} (U_{D^1} \boxtimes_{U_{D^0}} U_{D^1}) = \mathrm{Ind}_{U_{D^0}}^{T_e(0)} U_{D^0} = T_e(0)$$

by Theorem 4.4.9. Thus $T_e(0)$ is a simple current $T_e(0)$-module by Lemma 4.1.2. ∎

Summarizing, we obtain:

Proposition 8.1.19. *Let V be a 2A-framed VOA with structure codes (D,S) and a 2A-frame $\omega = e^1 + \cdots + e^n$. Suppose that the pair (D,S) satisfies the condition (1-ii) of Hypothesis I and $V_{e^1}(1/16) \neq 0$. Then V and e^1 satisfy Hypothesis II.*

8.1.4 The baby-monster SVOA

Now consider the moonshine VOA $V^\natural$ which we have constructed as a 2A-framed VOA in Section 7.5.2. It has a 2A-frame $\omega^\natural = e^1 + \cdots + e^{48}$ whose associated structure codes are $(D^\natural, S^\natural)$ as in Section 7.5.2. By Proposition 8.1.19, the pair $(V^\natural, e^1)$ satisfies Hypothesis II. In the following we consider the commutant superalgebra associated to $V^\natural$.

The full automorphism group of $V^\natural$ known to be the Monster $\mathbb{M}$. For a conformal vector $e \in V^\natural$ with central charge 1/2, it is shown in [C] [M1] that the Miyamoto involution

τ_e belongs to the 2A-conjugacy class of the Monster and that the correspondence $e \rightsquigarrow \tau_e$ is one-to-one. In particular, all conformal vectors of $V^\natural$ with central charge 1/2 are conjugate under the Monster. The centralizer $C_{\mathbb{M}}(\tau_e)$ of a 2A-involution τ_e is the 2-fold central extension $\langle\tau_e\rangle \cdot \mathbb{B}$ of the baby monster finite sporadic simple group $\mathbb{B}$ [ATLAS]. So $\mathbb{B}$ acts on the τ_e-invariants of $V^\natural$ as an automorphism group of a VOA. Motivated by this fact, G. Höhn studied the τ_e-invariants of $V^\natural$ and found the baby-monster SVOA $V\!B$ on which $\mathbb{B}$ acts as an automorphism group in [Hö1]. In this article we present a slight different approach. Consider the decomposition of $V^\natural$ as an $\mathrm{Vir}(e^1)$-module:

$$V^\natural = L(1/2,0)\otimes T^\natural_{e^1}(0) \oplus L(1/2,1/2)\otimes T^\natural_{e^1}(1/2) \oplus L(1/2,1/16)\otimes T^\natural_{e^1}(1/16).$$

Then by Theorem 8.1.4 we obtain a simple SVOA $T^\natural_{e^1}(0)\oplus T^\natural_{e^1}(1/2)$ and its irreducible $\mathbb{Z}_2$-twisted module $T^\natural_{e^1}(1/16)$. Following Höhn [Hö1], we set $V\!B^0 := T^\natural_{e^1}(0)$, $V\!B^1 := T^\natural_{e^1}(1/2)$ and $V\!B := V\!B^0 \oplus V\!B^1$, and we call it the *baby-monster SVOA**. We also set $V\!B_T := T^\natural_{e^1}(1/16)$ which is an irreducible $\mathbb{Z}_2$-twisted $V\!B$-module. Since all the conformal vectors of $V^\natural$ with central charge 1/2 are conjugate under $\mathbb{M}$, the algebraic structures of $V\!B$ and $V\!B_T$ are independent of the choice of a conformal vector e^1. Now by applying Theorem 8.1.12 to $V^\natural$ and e^1, we have the following theorem (cf. [Hö1] [Hö2]):

Theorem 8.1.20. *(1) The SVOA $V\!B$ obtained from $V^\natural$ by cutting off the Ising model is a simple SVOA.*
(2) The piece $V\!B_T$ obtained from $V^\natural$ is an irreducible $\mathbb{Z}_2$-twisted $V\!B$-module.
(3) $\mathrm{Aut}(V\!B^0) \simeq \mathbb{B}$ *and* $\mathrm{Aut}(V\!B) \simeq 2\times\mathbb{B}$.
(4) $V\!B_T$ as a $V\!B^0$-module is irreducible. Thus, there are exactly three irreducible $V\!B^0$-modules, $V\!B^0$, $V\!B^1$ and $V\!B_T$.
(5) The fusion rules for irreducible $V\!B^0$-modules are as follows:

$$V\!B^1\times V\!B^1 = V\!B^0,\quad V\!B^1\times V\!B_T = V\!B_T,\quad V\!B_T\times V\!B_T = V\!B^0+V\!B^1.$$

Proof: The assertion (1) follows from Theorem 8.1.4 and (2) follows from Theorem 8.1.6. So consider (3). For simplicity, below we write e for e^1. By (i) of Theorem 8.1.12, we have $C_{\mathrm{Aut}((V^\natural)^{\langle\tau_e\rangle})}(e) \simeq \langle\sigma_e\rangle\times\mathrm{Aut}(V\!B^0)$. On the other hand, we have $|C_{\mathrm{Aut}((V^\natural)^{\langle\tau_e\rangle})}(e) : C_{\mathrm{Aut}(V^\natural)}(e)/\langle\tau_e\rangle| \le 2$ by (iii) of Theorem 8.1.12. Since the correspondence $e\rightsquigarrow\tau_e$ is one-to-one by [C] and [M1], $C_{\mathrm{Aut}(V^\natural)}(e) = C_{\mathrm{Aut}(V^\natural)}(\tau_e) = C_{\mathbb{M}}(\tau_e)\simeq\langle\tau_e\rangle\cdot\mathbb{B}$. Therefore, $C_{\mathrm{Aut}((V^\natural)^{\langle\tau_e\rangle})}(e)$ contains $\mathbb{B}$ with index at most 2. If $C_{\mathrm{Aut}((V^\natural)^{\langle\tau_e\rangle})}(e)=\mathbb{B}$, then the simple group $\mathbb{B}$ contains a normal subgroup $\langle\sigma_e\rangle$ of order 2, which is a contradiction. It is easy to see that $\mathbb{B}$ and σ_e commute. Thus $C_{\mathrm{Aut}((V^\natural)^{\langle\tau_e\rangle})}(e)\simeq\langle\sigma_e\rangle\times\mathbb{B}$ and $\mathrm{Aut}(V\!B^0)\simeq\mathbb{B}$. This completes the proof of (3).

*He also call it the *shorter moonshine module* i n [Hö2].

Consider (4). Since $C_{\mathrm{Aut}((V^\natural)^{\langle\tau_e\rangle})}(e)/(C_{\mathrm{Aut}(V^\natural)}(e)/\langle\tau_e\rangle) = \langle\sigma_e\rangle$ is of order 2, the involution σ_e permutes $V_e^\natural(1/16)$ and W^1, where W^1 is as in Theorem 8.1.8, by the proof of (iii) of Theorem 8.1.12. Then $V_e^\natural(1/16)$ and W^1 as $V_e^\natural(0)$-modules are isomorphic. So $V\!B_T$ is irreducible as a $V\!B^0$-module by Proposition 8.1.9. Therefore, the irreducible $V\!B^0$-modules are given by $V\!B^0$, $V\!B^1$ and $V\!B_T$ again by Proposition 8.1.9. This completes the proof of (4).

Finally, consider (5). We only have to show the fusion rule $V\!B_T \times V\!B_T = V\!B^0 + V\!B^1$. By considering the 1/16-word of $V\!B_T$, the fusion product $V\!B_T \times V\!B_T$ is contained in $\mathbb{N}V\!B^0 \oplus \mathbb{N}V\!B^1$ in the fusion algebra for $V\!B^0$. Write $V\!B_T \times V\!B_T = n_0 V\!B^0 + n_1 V\!B^1$ with $n_0, n_1 \in \mathbb{N}$. Then the simplicity of $V^\natural$ implies $n_0 \neq 0$ and $n_1 \neq 0$. And by applying $V\!B^1$ to $V\!B_T \times V\!B_T$, we see that $n_0 = n_1$. At last, since the dual module of $V\!B_T$ is isomorphic to $V\!B_T$, the space of $V\!B^0$-intertwining operator of type $V\!B_T \times V\!B_T \to V\!B^0$ is one-dimensional. Thus $n_0 = n_1 = 1$ as desired. ∎

Remark 8.1.21. The assertion (3) of Theorem 8.1.20 is already established in [Hö2]. In [Hö2], Höhn used many results from the finite group theory. On the other hand, our proof is mainly based on the structure theory of the moonshine VOA and we only use that $C_{\mathbb{M}}(\tau_e)/\langle\tau_e\rangle$ is a simple group.

The classification of irreducible $V\!B^0$-modules has many corollaries.

Corollary 8.1.22. *The irreducible 2A-twisted $V^\natural$-module has a shape*

$$L(1/2,1/2) \otimes V\!B^0 \oplus L(1/2,0) \otimes V\!B^1 \oplus L(1/2,1/16) \otimes V\!B_T.$$

Proof: Follows from Theorem 8.1.20, Theorem 8.1.8 and Proposition 8.1.9. ∎

Remark 8.1.23. A straightforward construction of the 2A-twisted (and 2B-twisted) $V^\natural$-module is given by Lam [L2]. In his construction, it is given as $U_{D^\natural+\gamma} \boxtimes_{U_{D^\natural}} V^\natural$ with $\gamma = (10^{47}) \in (\mathbb{Z}/2\mathbb{Z})^{48}$.

Corollary 8.1.24. *For any conformal vector $e \in V^\natural$ with central charge 1/2, there is no automorphism ρ on $V^\natural$ such that $\rho(V_e^\natural(h)) = V_e^\natural(h)$ for $h = 0, 1/2$ and $\rho|_{(V^\natural)^{\langle\tau_e\rangle}} = \sigma_e$.*

Proof: Suppose such an automorphism ρ exists. We remark that ρ also preserves the space $V_e(1/16)$ as $\rho \in C_{\mathrm{Aut}(V)}(e)$. We view $V_e^\natural(1/16)$ as a $(V^\natural)^{\langle\tau_e\rangle}$-module by a restriction of the vertex operator map $Y_{V^\natural}(\cdot, z)$ on $V^\natural$. Consider the σ_e-conjugate $(V^\natural)^{\langle\tau_e\rangle}$-module $V_e^\natural(1/16)^{\sigma_e}$. By Theorem 8.1.20 and Proposition 8.1.9, $V_e^\natural(1/16)^{\sigma_e}$ is not isomorphic to $V_e^\natural(1/16)$ as a $(V^\natural)^{\langle\tau_e\rangle}$-module. On the other hand, we can take a canonical linear isomorphism $\varphi : V_e^\natural(1/16) \to V_e^\natural(1/16)^{\sigma_e}$ such that $Y_{V_e^\natural(1/16)^{\sigma_e}}(a,z)\varphi v = \varphi Y_{V^\natural}(\sigma_e a, z)v$ for all $a \in (V^\natural)^{\langle\tau_e\rangle}$ and $v \in V_e^\natural(1/16)$ by definition of the conjugate module. Then we have

$$Y_{V_e^\natural(1/16)^{\sigma_e}}(a,z)\varphi\rho v = \varphi Y_{V^\natural}(\sigma_e a, z)\rho v = \varphi Y_{V^\natural}(\rho a, z)\rho v = \varphi\rho Y_{V^\natural}(a,z)v$$

for any $a \in (V^\natural)^{\langle \tau_e \rangle}$ and $v \in V_e^\natural(1/16)$. Thus $\varphi\rho$ defines a $(V^\natural)^{\langle \tau_e \rangle}$-isomorphism between $V_e^\natural(1/16)$ and $V_e^\natural(1/16)^{\sigma_e}$, which is a contradiction. ∎

Corollary 8.1.25. *The 2A-twisted orbifold construction applied to $V^\natural$ yields $V^\natural$ itself again.*

Proof: Follows from Theorem 8.1.20 and Corollary 8.1.13. ∎

Remark 8.1.26. The statement in the corollary above was conjectured by Tuite [Tu]. In [Tu], Tuite has shown that any $\mathbb{Z}_p$-orbifold construction of $V^\natural$ yields the moonshine VOA $V^\natural$ or the Leech lattice VOA V_Λ under the uniqueness conjecture of the moonshine VOA which states that $V^\natural$ constructed by Frenkel et. al. [FLM] is the unique holomorphic VOA with central charge 24 whose weight one subspace is trivial.

Finally, we end this subsection by presenting the modular transformations of characters of $V\!B^0$-modules. Here the character means the conformal character, not the q-dimension, of modules. Recall the characters of $L(1/2,0)$-modules. By our explicit construction in Section 7.1.1, one can easily prove the following (cf. [FFR] [FRW]):

$$\mathrm{ch}_{L(1/2,0)}(\tau) = (1/2)\cdot q^{-1/48}\left\{\textstyle\prod_{n=0}^{\infty}(1+q^{n+1/2}) + \prod_{n=0}^{\infty}(1-q^{n+1/2})\right\},$$

$$\mathrm{ch}_{L(1/2,1/2)}(\tau) = (1/2)\cdot q^{-1/48}\left\{\textstyle\prod_{n=0}^{\infty}(1+q^{n+1/2}) - \prod_{n=0}^{\infty}(1-q^{n+1/2})\right\},$$

$$\mathrm{ch}_{L(1/2,1/16)}(\tau) = q^{-1/24}\textstyle\prod_{n=1}^{\infty}(1+q^{n}).$$

The following modular transformations are well-known:

$$\begin{aligned}
\mathrm{ch}_{L(1/2,0)}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{L(1/2,0)}(\tau) + \frac{1}{2}\mathrm{ch}_{L(1/2,1/2)}(\tau) + \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,1/16)}(\tau),\\
\mathrm{ch}_{L(1/2,1/2)}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{L(1/2,0)}(\tau) + \frac{1}{2}\mathrm{ch}_{L(1/2,1/2)}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,1/16)}(\tau),\\
\mathrm{ch}_{L(1/2,1/16)}(-1/\tau) &= \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,0)}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,1/2)}(\tau).
\end{aligned}$$

Set $J(\tau) := j(\tau) - 744 = q^{-1} + 0 + 196884q + \cdots$, where $j(\tau)$ is the famous $\mathrm{SL}_2(\mathbb{Z})$-invariant j-function. Since $\mathrm{ch}_{V^\natural}(\tau) = J(\tau)$ and

$$\mathrm{ch}_{V^\natural}(\tau) = \mathrm{ch}_{L(1/2,0)}(\tau)\mathrm{ch}_{V\!B^0}(\tau) + \mathrm{ch}_{L(1/2,1/2)}(\tau)\mathrm{ch}_{V\!B^1}(\tau) + \mathrm{ch}_{L(1/2,1/16)}(\tau)\mathrm{ch}_{V\!B_T}(\tau),$$

we can write down the characters of irreducible $V\!B^0$-modules by using those of $V^\natural$ and $L(1/2,0)$-modules. This computation is already done in [Mat] by using Norton's trace formula. The results are written as a rational expression involving the functions $J(\tau)$, $\mathrm{ch}_{L(1/2,h)}(\tau)$, $h = 0, 1/2, 1/16$, their first and second derivatives and the Eisenstein series $E_2(\tau)$ and $E_4(\tau)$, see [Mat].

By Zhu's theorem [Z], the linear space spanned by $\{\mathrm{ch}_{V\!B^0}(\tau), \mathrm{ch}_{V\!B^1}(\tau), \mathrm{ch}_{V\!B_T}(\tau)\}$ affords an $\mathrm{SL}_2(\mathbb{Z})$-action. By using the modular transformations for $J(\tau)$ and $\mathrm{ch}_{L(1/2,h)}(\tau)$, $h = 0, 1/2, 1/16$, we can show the following modular transformations:

$$\begin{aligned}
\mathrm{ch}_{V\!B^0}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{V\!B^0}(\tau) + \frac{1}{2}\mathrm{ch}_{V\!B^1}(\tau) + \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B_T}(\tau), \\
\mathrm{ch}_{V\!B^1}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{V\!B^0}(\tau) + \frac{1}{2}\mathrm{ch}_{V\!B^1}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B_T}(\tau), \\
\mathrm{ch}_{V\!B_T}(-1/\tau) &= \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B^0}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B^1}(\tau).
\end{aligned}$$

Namely, we have exactly the same modular transformation laws for the Ising model $L(1/2, 0)$. As in Theorem 8.1.20, we also note that the fusion algebra for $V\!B^0$ is also canonically isomorphic to that of $L(1/2, 0)$. Therefore, we may say that $L(1/2, 0)$ and $V\!B^0$ form a dual-pair in the moonshine VOA $V^\natural$.

8.2 3A-algebra for the Monster

In this section we study an example of VOA which has a close connection to the 2A-involution and the 3A-triality of the Monster. The study of this algebra was first begun by Miyamoto [M8] and in [SY] Sakuma and the author made a great development. Moreover, the research in this direction has been recently comes to be more significant in the study of 2A-involution of the Monster and the results in this section are greatly generalized in [LYY].

8.2.1 Construction

Let $A_1^5 = \mathbb{Z}\alpha^1 \oplus \mathbb{Z}\alpha^2 \oplus \cdots \oplus \mathbb{Z}\alpha^5$ with $\langle \alpha^i, \alpha^j \rangle = 2\delta_{i,j}$ and set $L := A_1^5 \cup (\gamma + A_1^5)$ with $\gamma := \frac{1}{2}\alpha^1 + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^3 + \frac{1}{2}\alpha^4$. Then L is an even lattice so that we can construct a VOA V_L associated to L. We have an isomorphism $V_L = V_{A_1^5} \oplus V_{\gamma + A_1^5} \simeq \{L_{\mathfrak{g}}(1,0)^{\otimes 4} \oplus L_{\mathfrak{g}}(1,1)^{\otimes 4}\} \otimes L_{\mathfrak{g}}(1,0)$, where $L_{\mathfrak{g}}(\ell, j)$ denotes the integrable module for the affine VOA $L_{\mathfrak{g}}(\ell, 0)$ associated to $\hat{sl}_2(\mathbb{C})$ (cf. Section 2.6.3). By (5.4.3) and the fusion rules (3.7.3) and (5.4.2), we can show the following.

Lemma 8.2.1. *We have the following inclusions*

$$\begin{aligned}
L_{\mathfrak{g}}(1,0)^{\otimes 3} &\supset L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L_{\mathfrak{g}}(3, 0), \\
L_{\mathfrak{g}}(1,1)^{\otimes 3} &\supset L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0) \otimes L_{\mathfrak{g}}(3, 3).
\end{aligned}$$

Therefore, V_L contains a sub VOA isomorphic to

$$L_{\mathfrak{g}}(3,0) \otimes L_{\mathfrak{g}}(1,0) \otimes L_{\mathfrak{g}}(1,0) \oplus L_{\mathfrak{g}}(3,3) \otimes L_{\mathfrak{g}}(1,1) \otimes L_{\mathfrak{g}}(1,0).$$

Lemma 8.2.2. *We have the following decompositions:*

$$\begin{aligned}
&L_{\mathfrak{g}}(3,0)\otimes L_{\mathfrak{g}}(1,0)\otimes L_{\mathfrak{g}}(1,0)\\
&\quad\simeq \{L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},0)\oplus L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},5)\oplus L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{4}{3})\}\otimes L_{\mathfrak{g}}(5,0)\\
&\quad\oplus\{L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},\tfrac{5}{7})\oplus L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},\tfrac{12}{7})\oplus L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{1}{21})\}\otimes L_{\mathfrak{g}}(5,2)\\
&\quad\oplus\{L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},\tfrac{22}{7})\oplus L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},\tfrac{1}{7})\oplus L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{10}{21})\}\otimes L_{\mathfrak{g}}(5,4),
\end{aligned}$$

$$\begin{aligned}
&L_{\mathfrak{g}}(3,3)\otimes L_{\mathfrak{g}}(1,1)\otimes L_{\mathfrak{g}}(1,0)\\
&\quad\simeq \{L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},5)\oplus L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},0)\oplus L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{4}{3})\}\otimes L_{\mathfrak{g}}(5,0)\\
&\quad\oplus\{L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},\tfrac{12}{7})\oplus L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},\tfrac{5}{7})\oplus L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{1}{21})\}\otimes L_{\mathfrak{g}}(5,2)\\
&\quad\oplus\{L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},\tfrac{1}{7})\oplus L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},\tfrac{22}{7})\oplus L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{10}{21})\}\otimes L_{\mathfrak{g}}(5,4).
\end{aligned}$$

Hence, $L_{\mathfrak{g}}(3,0)\otimes L_{\mathfrak{g}}(1,0)\otimes L_{\mathfrak{g}}(1,0)\oplus L_{\mathfrak{g}}(3,3)\otimes L_{\mathfrak{g}}(1,1)\otimes L_{\mathfrak{g}}(1,0)$ *(and* V_L*) contains a sub VOA* U *isomorphic to*

$$\left[\begin{array}{c} L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},0)\\ \oplus\\ L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},5)\\ \oplus\\ L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{4}{3})\end{array}\right]\bigoplus\left[\begin{array}{c} L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},5)\\ \oplus\\ L(\tfrac{4}{5},3)\otimes L(\tfrac{6}{7},0)\\ \oplus\\ L(\tfrac{4}{5},\tfrac{2}{3})\otimes L(\tfrac{6}{7},\tfrac{4}{3})\end{array}\right]. \tag{8.2.1}$$

We can also define U in the following way. For $i=1,2,\dots,5$, set

$$\begin{aligned}
H^j &:= \alpha^1_{(-1)}\mathbb{1}+\cdots+\alpha^j_{(-1)}\mathbb{1},\\
E^j &:= e^{\alpha^1}+\cdots+e^{\alpha^j},\\
F^j &:= e^{-\alpha^1}+\cdots+e^{-\alpha^j},\\
\Omega^j &:= \frac{1}{2(j+2)}\left(\frac{1}{2}H^j_{(-1)}H^j+E^j_{(-1)}F^j+F^j_{(-1)}E^j\right),\\
\omega^i &:= \Omega^i+\frac{1}{4}\left(\alpha^{i+1}_{(-1)}\right)^2\mathbb{1}-\Omega^{i+1}.
\end{aligned}$$

Then H^j, E^j and F^j generate a simple affine sub VOA $L_{\mathfrak{g}}(j,0)$ and ω^i, $1\le i\le 4$, generate simple Virasoro sub VOAs $L(c_i,0)$ in V_L. Furthermore, we have an orthogonal decomposition of the Virasoro vector ω_{V_L} of V_L into a sum of mutually commutative Virasoro vectors as

$$\omega_{V_L}=\omega^1+\omega^2+\omega^3+\omega^4+\Omega^5.$$

Then we may define U to be as follows:

$$U=\{v\in V_L \mid \omega^1_{(1)}v=\omega^2_{(1)}v=\Omega^5_{(1)}v=0\}.$$

Set

$$\begin{aligned}
e\quad &:= \frac{1}{16}\left((\alpha^4-\alpha^5)_{(-1)}\right)^2\mathbb{1} - \frac{1}{4}(e^{\alpha^4-\alpha^5}+e^{-\alpha^4+\alpha^5}),\\
v^0\quad &:= \frac{5}{18}\omega^3+\frac{7}{9}\omega^4-\frac{16}{9}e,\\
v^1\quad &:= (9F^4-8F^5)_{(-1)}(4F^3-3F^4)_{(0)}e^{\frac{1}{2}(\alpha^1+\alpha^2+\alpha^3+\alpha^4)}\\
&\qquad -\frac{1}{2}(9H^4-8H^5)_{(-1)}F^4_{(0)}(4F^3-3F^4)_{(0)}e^{\frac{1}{2}(\alpha^1+\alpha^2+\alpha^3+\alpha^4)}\\
&\qquad\quad -\frac{1}{2}(9E^4-8E^5)_{(-1)}\left(F^4_{(0)}\right)^2(4F^3-3F^4)_{(0)}e^{\frac{1}{2}(\alpha^1+\alpha^2+\alpha^3+\alpha^4)}.
\end{aligned}\tag{8.2.2}$$

Then we can show that both e and v^i are contained in U_2 and $e_{(1)}e=2e$, $e_{(3)}e=\frac{1}{4}\mathbb{1}$, $\omega^3_{(1)}v^i=\frac{2}{3}v^i$ and $\omega^4_{(1)}v^i=\frac{4}{3}v^i$ for $i=0,1$. Therefore, e generates a sub VOA isomorphic to $L(\frac{1}{2},0)$ in U and v^i, $i=0,1$, are highest weight vectors for $\mathrm{Vir}(\omega^3)\otimes\mathrm{Vir}(\omega^4)\simeq L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$ with highest weight $(\frac{2}{3},\frac{4}{3})$. Since the weight 2 subspace of U is 4-dimensional, we note that ω^3,ω^4,v^0 and v^1 span U_2. In the next subsection we will show that they generate U as a VOA.

8.2.2 Structures

By Lemma 8.2.2, we know that there exists a structure of a VOA in (8.2.1). Here we will prove that there exists a unique VOA structure on it. By (8.2.1), U contains a tensor product of two extensions of the unitary Virasoro VOAs $W(0)=L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ and $N(0)=L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)$ (see Section 5.4). Since both $W(0)$ and $N(0)$ are rational, U is completely reducible as a $W(0)\otimes N(0)$-module. Therefore, U as a $W(0)\otimes N(0)$-module is isomorphic to

$$U\simeq W(0)\otimes N(0)\oplus W\left(\frac{2}{3}\right)^{\epsilon_1}\otimes N\left(\frac{4}{3}\right)^{\xi_1}\oplus W\left(\frac{2}{3}\right)^{\epsilon_2}\otimes N\left(\frac{4}{3}\right)^{\xi_2},$$

where $\epsilon_i,\xi_j=\pm$. Recall that both $W(0)$ and $N(0)$ have the canonical involutions σ_1 and σ_2, respectively. Then they can be lifted to involutions of $W(0)\otimes N(0)$ and we still denote them by σ_1 and σ_2, respectively. By our construction, U has a $\mathbb{Z}_2$-grading $U=U^+\oplus U^-$ with

$$\begin{aligned}
U^+&\subset L_{\mathfrak{g}}(3,0)\otimes L_{\mathfrak{g}}(1,0)\otimes L_{\mathfrak{g}}(1,0)\subset V_{A_1^5}\quad\text{and}\\
U^-&\subset L_{\mathfrak{g}}(3,3)\otimes L_{\mathfrak{g}}(1,1)\otimes L_{\mathfrak{g}}(1,0)\subset V_{\gamma+A_1^5}.
\end{aligned}\tag{8.2.3}$$

We note that the decomposition above defines a natural extension of an involution $\sigma_1\sigma_2$ on $W(0)\otimes N(0)$ to that on U, which we will also denote by $\sigma_1\sigma_2$. Therefore, by Lemma 4.2.8, we have $(W(\frac{2}{3})^{\epsilon_1}\otimes N(\frac{4}{3})^{\xi_1})^{\sigma_1\sigma_2}=W(\frac{2}{3})^{\epsilon_2}\otimes N(\frac{4}{3})^{\xi_2}$ and hence $\epsilon_2=-\epsilon_1$ and $\xi_2=-\xi_1$. Since we may rename the signs of the irreducible $N(0)$-modules of $\pm$-type (cf. Remark 5.4.5), we may assume that $\epsilon_1=\xi_1$.

Theorem 8.2.3. *A VOA U contains a sub VOA $W(0)\otimes N(0)$. As a $W(0)\otimes N(0)$-module, U is isomorphic to*

$$W(0)\otimes N(0)\oplus W\left(\frac{2}{3}\right)^+\otimes N\left(\frac{4}{3}\right)^+\oplus W\left(\frac{2}{3}\right)^-\otimes N\left(\frac{4}{3}\right)^- \tag{8.2.4}$$

after fixing suitable choice of $\pm$-type of $N(\frac{4}{3})^\pm$. Therefore, U is a simple VOA and generated by its weight 2 subspace as a VOA.

Proof: The decomposition is already shown. Since U is a sub VOA of V_L, we have $Y(x,z)y\neq 0$ for all $x,y\in U$. Then by fusion rules for $W(0)\otimes N(0)$-modules, U is a $\mathbb{Z}_3$-simple current extension of $W(0)\otimes N(0)$. Therefore, U is a simple VOA. So we should show that U_2 generates U. Since U_2 contains the Virasoro vectors ω^3 and ω^4 and highest weight vectors of $W(\frac{2}{3})^\pm\otimes N(\frac{4}{3})^\pm$, U_2 generates whole of $W(\frac{2}{3})^\pm\otimes N(\frac{4}{3})^\pm$. Since V_L is simple, for any non-zero vectors $u\in W(\frac{2}{3})^+\otimes N(\frac{4}{3})^+$ and $v\in W(\frac{2}{3})^-\otimes N(\frac{4}{3})^-$ we have $Y(u,z)v\neq 0$ in U (cf. [DL]). Therefore, by the fusion rules in Theorem 5.4.6 and Theorem 5.4.8, $W(\frac{2}{3})^\pm\otimes N(\frac{4}{3})^\pm$ generate $W(0)\otimes N(0)$ in U. Hence, U_2 generates whole of U. ∎

By Lemma 4.2.7, we note that there exists the following $\mathbb{Z}_3$-simple current extension of $W(0)\otimes N(0)$.

$$U'=W(0)\otimes N(0)\oplus W\left(\frac{2}{3}\right)^+\otimes N\left(\frac{4}{3}\right)^-\oplus W\left(\frac{2}{3}\right)^-\otimes N\left(\frac{4}{3}\right)^+. \tag{8.2.5}$$

Since U and U' are σ_1-conjugate extensions of each others, they are equivalent $\mathbb{Z}_3$-simple current extensions of $W(0)\otimes N(0)$. Thus, we get the following.

Theorem 8.2.4. *The following $\mathbb{Z}_3$-simple current extensions of $W(0)\otimes N(0)$ are equivalent:*

$$W(0)\otimes N(0)\oplus W\left(\frac{2}{3}\right)^+\otimes N\left(\frac{4}{3}\right)^\pm\oplus W\left(\frac{2}{3}\right)^-\otimes N\left(\frac{4}{3}\right)^\mp.$$

Hence, there is a unique $\mathbb{Z}_3$-graded VOA structure in (8.2.1).

8.2.3 Modules

Let U be the $\mathbb{Z}_3$-graded VOA as in (8.2.1). In this subsection we will classify all irreducible U-modules. Set $U=U^0\oplus U^1\oplus U^2$ with $U^0=W(0)\otimes N(0)$, $U^1=W(\frac{2}{3})^+\otimes N(\frac{4}{3})^+$ and $U^2=W(\frac{2}{3})^-\otimes N(\frac{4}{3})^-$.

Lemma 8.2.5. *Every irreducible U-modules is $\mathbb{Z}_3$-stable.*

Proof: Let M be an irreducible U-module. Take an irreducible U^0-submodule P of M. By Lemma 4.4.1, both $U^1\cdot P$ and $U^2\cdot P$ are non-zero irreducible U^0-submodules of

M. It follows from the fusion rules for $U^0 = W(0) \otimes N(0)$-modules that $U^i \cdot P \not\simeq U^j \cdot P$ as U^0-modules if $i \not\equiv j \mod 3$. Therefore, $M = P \oplus (U^1 \cdot P) \oplus (U^2 \cdot P)$ and hence M has a $\mathbb{Z}_3$-grading under the action of U. This completes the proof. ∎

By this lemma, the structure of an irreducible U-module is completely determined by its U^0-module structure.

Theorem 8.2.6. *An irreducible U-module is isomorphic to one of the following:*

$$
\begin{aligned}
&W(0) \otimes N(0) \oplus W(\tfrac{2}{3})^+ \otimes N(\tfrac{4}{3})^+ \oplus W(\tfrac{2}{3})^- \otimes N(\tfrac{4}{3})^-,\\
&W(0) \otimes N(\tfrac{1}{7}) \oplus W(\tfrac{2}{3})^+ \otimes N(\tfrac{10}{21})^+ \oplus W(\tfrac{2}{3})^- \otimes N(\tfrac{10}{21})^-,\\
&W(0) \otimes N(\tfrac{5}{7}) \oplus W(\tfrac{2}{3})^+ \otimes N(\tfrac{1}{21})^+ \oplus W(\tfrac{2}{3})^- \otimes N(\tfrac{1}{21})^-,\\
&W(\tfrac{2}{5}) \otimes N(0) \oplus W(\tfrac{1}{15})^+ \otimes N(\tfrac{4}{3})^+ \oplus W(\tfrac{1}{15})^- \otimes N(\tfrac{4}{3})^-,\\
&W(\tfrac{2}{5}) \otimes N(\tfrac{1}{7}) \oplus W(\tfrac{1}{15})^+ \otimes N(\tfrac{10}{21})^+ \oplus W(\tfrac{1}{15})^- \otimes N(\tfrac{10}{21})^-,\\
&W(\tfrac{2}{5}) \otimes N(\tfrac{5}{7}) \oplus W(\tfrac{1}{15})^+ \otimes N(\tfrac{1}{21})^+ \oplus W(\tfrac{1}{15})^- \otimes N(\tfrac{1}{21})^-.
\end{aligned}
$$

Proof: By Theorem 4.5.2, every irreducible U^0-module is uniquely lifted to either an irreducible U-module or an irreducible $\mathbb{Z}_3$-twisted U-module according to its top weight. Thus the assertion follows. ∎

8.2.4 Conformal vectors

In this subsection we study the Griess algebra of U. Recall e, v_0, $v_1 \in U_2$ defined by (8.2.2). Set

$$
\begin{aligned}
\omega &:= \omega^3 + \omega^4, & a &:= \frac{105}{2^8}(\omega - e),\\
b &:= \frac{3^2}{2^8}(-5\omega^3 + 7\omega^4 - 4e), & c &:= kv^1,
\end{aligned}
$$

where the scalar $k \in \mathbb{R}$ is determined by the condition $\langle c, c\rangle = 3^5/2^{11}$. Then $\{e, a, b, c\}$ is a set of basis of U_2. By direct calculations one can show that the multiplications and inner products in the Griess algebra of U are given as follows:

$$
\begin{aligned}
e_{(1)}a &= 0, & e_{(1)}b &= \frac{1}{2}b, & e_{(1)}c &= \frac{1}{16}c,\\
a_{(1)}a &= \frac{105}{2^7}a, & a_{(1)}b &= \frac{3^2 \cdot 5 \cdot 7}{2^9}b, & a_{(1)}c &= \frac{31 \cdot 105}{2^{12}}c,\\
b_{(1)}b &= \frac{3^7}{2^{15}}e + \frac{3^3}{2^7}a, & b_{(1)}c &= \frac{3^2 \cdot 23}{2^{10}}c, & c_{(1)}c &= \frac{3^5}{2^{13}}e + \frac{31}{2^5}a + \frac{23}{2^5}b,\\
\langle a, a\rangle &= \frac{3^6 \cdot 5 \cdot 7}{2^{18}}, & \langle b, b\rangle &= \frac{3^7}{2^{16}}, & \langle c, c\rangle &= \frac{3^5}{2^{11}}.
\end{aligned}
$$

Hence, we note that the Griess algebra of our VOA U is isomorphic to that of $\mathrm{VA}(e,f)$ with $\langle e,f\rangle = 13/2^{10}$ in [M8]. Therefore, by tracing calculations in [M8] we can find the following conformal vectors with central charge $1/2$ in U_2.

$$f := \frac{13}{2^8}e + a + b + c, \quad \tilde{f} := \frac{13}{2^8}e + a + b - c.$$

And by a calculation we get

$$\begin{aligned} e_{(1)}f &= -\frac{105}{2^9}\omega + \frac{9}{2^5}e + \frac{9}{2^5}f + \frac{7}{2^5}\tilde{f}, & e_{(1)}\tilde{f} &= -\frac{105}{2^9}\omega + \frac{9}{2^5}e + \frac{7}{2^5}f + \frac{9}{2^5}\tilde{f}, \\ f_{(1)}\tilde{f} &= -\frac{105}{2^9}\omega + \frac{7}{2^5}e + \frac{9}{2^5}f + \frac{9}{2^5}\tilde{f}, & \langle e,f\rangle &= \langle e,\tilde{f}\rangle = \langle f,\tilde{f}\rangle = \frac{13}{2^{10}}. \end{aligned}$$

Using these equalities, we can show that the Griess algebra U_2 is generated by two conformal vectors e and f. Since U_2 generates U as a VOA by Theorem 8.2.3, U is generated by two conformal vectors e and f. Thus

Theorem 8.2.7. *U is generated by two conformal vectors e and f with central charge $1/2$ such that $\langle e,f\rangle = 13/2^{10}$.*

Now we can classify all conformal vectors in U. First, we seek all conformal vectors with central charge $1/2$. It is shown in [M1] that there exists a one-to-one correspondence between the set of conformal vectors with central charge c in U and the set of idempotents with squared length $c/8$ in U_2. So we should determine all idempotents with squared length $1/16$ in U_2. Let $X = \alpha\omega + \beta e + \gamma f + \delta\tilde{f}$ be a conformal vector with central charge $1/2$. Then we should solve the equations $(X/2)(1)(X/2) = (X/2)$ and $\langle X,X\rangle = 1/16$. By direct calculations, the solutions of $(\alpha,\beta,\gamma,\delta)$ are $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. Therefore,

Theorem 8.2.8. *There are exactly three conformal vectors with central charge $1/2$ in U_2, namely e, f and $\tilde{f}$.*

The rest of conformal vectors can be obtained in the following way. We should seek all idempotents and their squared lengths in U_2. Since we have a set of basis $\{\omega, e, f, \tilde{f}\}$ of U_2 and all multiplications and inner products are known, we can get them by direct calculations. After some computations, we reach that the possible central charges are $1/2$, $81/70$, $58/35$, $4/5$ and $6/7$. In the following, $(\alpha,\beta,\gamma,\delta)$ denotes $\alpha\omega + \beta e + \gamma f + \delta\tilde{f}$.

(1) Central charge $1/2$: $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$.

(2) Central charge $81/70$: $(1,-1,0,0)$, $(1,0,-1,0)$, $(1,0,0,-1)$.

(3) Central charge $58/35$: $(1,0,0,0)$.

(4) Central charge $4/5$: $(14/9,-32/27,-32/27,-32/27)$, $(-7/18,14/27,32/27,32/27)$,

$(-7/18, 32/27, 14/27, 32/27)$, $(-7/18, 32/27, 32/27, 14/27)$.

(5) Central charge 6/7: $(-5/9, 32/27, 32/27, 32/27)$, $(25/18, -14/27, -32/27, -32/27)$, $(25/18, -32/27, -14/27, -32/27)$, $(25/18, -32/27, -32/27, -14/27)$.

8.2.5 Automorphisms

By Theorem 8.2.8, U has three conformal vectors e, f and $\tilde{f}$. Since $e^{\tau_f} \neq e$ nor f and $f^{\tau_e} \neq f$ nor e, we must have $e^{\tau_f} = f^{\tau_e} = \tilde{f}$. Therefore, $\tau_e\tau_f\tau_e = \tau_{f^{\tau_e}} = \tau_{e^{\tau_f}} = \tau_f\tau_e\tau_f$ and so $(\tau_e\tau_f)^3 = 1$. It is clear that both τ_e and τ_f are non-trivial involutions acting on U and $\tau_e \neq \tau_f$. Hence τ_e and τ_f generate S_3 in $\mathrm{Aut}(U)$. We prove that $\langle \tau_e, \tau_f\rangle = \mathrm{Aut}(U)$.

Theorem 8.2.9. $\mathrm{Aut}(U) = \langle \tau_e, \tau_f\rangle$.

Proof: Let $\sigma \in \mathrm{Aut}(U)$. Since U is generated by e and f, the action of σ on U is completely determined by its actions on e and f. By Theorem 8.2.8, the set of conformal vectors with central charge 1/2 in U is $\{e, f, \tilde{f}\}$ so that we get an injection from $\mathrm{Aut}(U)$ to S_3. Since $\langle \tau_e, \tau_f\rangle$ acts on $\{e, f, \tilde{f}\}$ as S_3, we obtain $\mathrm{Aut}(U) = \langle \tau_e, \tau_f\rangle$. ∎

Remark 8.2.10. We note that both ω^3 and ω^4 are S_3-invariant so that the orthogonal decomposition $\omega = \omega^3 + \omega^4$ is the *characteristic* decomposition of ω in U.

Summarizing everything, we have already shown that U is generated by two conformal vectors e and f whose inner product is $\langle e, f\rangle = 13/2^{10}$ and its automorphism group is generated by two involutions τ_e and τ_f with $(\tau_e\tau_f)^3 = 1$. Hence, we conclude that our VOA U is the same as $\mathrm{VA}(e, f)$ in [M8] and gives a positive solution for Theorem 5.6 (4) of [M8].

Theorem 8.2.11. *The fixed-point subalgebra $U^{\langle \tau_e, \tau_f\rangle}$ is isomorphic to $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \oplus L(\frac{4}{5}, 3) \otimes L(\frac{6}{7}, 5)$ as a $\mathrm{Vir}(\omega^3) \otimes \mathrm{Vir}(\omega^4)$-module. It is a rational VOA.*

Proof: Since we may identify U as $\mathrm{VA}(e, f)$ in [M8], we can use the results obtained in [M8]. It is shown in [M8] that $\mathrm{Vir}(\omega^3) \otimes \mathrm{Vir}(\omega^4) = L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$ is a proper sub VOA of $U^{\langle \tau_e, \tau_f\rangle}$. Since U has both a $\mathbb{Z}_2$-grading (8.2.3) and a $\mathbb{Z}_3$-grading (8.2.4), all irreducible $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$-submodules but $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$ and $L(\frac{4}{5}, 3) \otimes L(\frac{6}{7}, 5)$ cannot be contained in $U^{\langle \tau_e, \tau_f\rangle}$. Hence, $U^{\langle \tau_e, \tau_f\rangle}$ must be as stated. Since $U^{\langle \tau_e, \tau_f\rangle}$ is a $\mathbb{Z}_2$-graded simple current extension of $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$, the rationality is clear. ∎

8.2.6 Fusion rules

Here we determine all fusion rules for irreducible U-modules. Set $U = U^0 \oplus U^1 \oplus U^2$ with $U^0 = W(0) \otimes N(0)$, $U^1 = W(\frac{2}{3})^+ \otimes N(\frac{4}{3})^+$ and $U^2 = W(\frac{2}{3})^- \otimes N(\frac{4}{3})^-$. Recall the list of all irreducible U-modules shown in Theorem 8.2.6. We note that all of them are

$\mathbb{Z}_3$-stable and each irreducible U-module is induced from one and only one of the following irreducible U^0-modules:

$$W(h)\otimes N(k),\ h=0,\frac{2}{5},\ k=0,\frac{1}{7},\frac{5}{7}.$$

Therefore, all irreducible U-modules are given by the induced modules:

$$\mathrm{Ind}_{U^0}^{U} W(h)\otimes N(k)=W(h)\otimes N(k)\oplus\{U^1\boxtimes_{U^0}(W(h)\otimes N(k))\}\oplus\{U^2\boxtimes_{U^0}(W(h)\otimes N(k))\}$$

where $h=0,\frac{2}{5}$ and $k=0,\frac{1}{7},\frac{5}{7}$. The fusion products for irreducible U-modules can be computed as follows:

Theorem 8.2.12. *All fusion rules for irreducible U-modules are given by the following formula:*

$$\begin{aligned}&\dim_{\mathbb{C}}\begin{pmatrix} & \mathrm{Ind}_{U^0}^{U}W(h_3)\otimes N(k_3) & \\ \mathrm{Ind}_{U^0}^{U}W(h_1)\otimes N(k_1) & & \mathrm{Ind}_{U^0}^{U}W(h_2)\otimes N(k_2)\end{pmatrix}_U \\ &= \quad \dim_{\mathbb{C}}\begin{pmatrix} & U\boxtimes_{U^0}(W(h_3)\otimes N(k_3)) & \\ W(h_1)\otimes N(k_1) & & W(h_2)\otimes N(k_2)\end{pmatrix}_{U^0},\end{aligned} \tag{8.2.6}$$

where $h_1,h_2,h_3\in\{0,\frac{2}{5}\}$ and $k_1,k_2,k_3\in\{0,\frac{1}{7},\frac{5}{7}\}$.

Proof: Since all irreducible U-modules are $\mathbb{Z}_3$-stable, the assertion immediately follows from Theorem 4.4.9. ∎

8.2.7 Relation to the 3A-triality of the Monster

In this section, we work over the real number field $\mathbb{R}$. We make it a rule to denote the complexification $\mathbb{C}\otimes_{\mathbb{R}}A$ of a vector space A over $\mathbb{R}$ by $\mathbb{C}A$. Recall the construction of our VOA U in Section 4.1. In it, we only used a formula (5.4.3), which was shown by Goddard et al. by using a character formula in [GKO]. Therefore, we can construct U even if we work over $\mathbb{R}$. To avoid confusions, we denote the real form of U by $U_{\mathbb{R}}$. We also note that the calculations on the Griess algebra of $U_{\mathbb{R}}$ in Section 4.4 is still correct even if we work over $\mathbb{R}$.

Definition 8.2.13. A VOA V over $\mathbb{R}$ is said to be *of moonshine type* if it admits a weight space decomposition $V=\oplus_{n=0}^{\infty}V_n$ with $V_0=\mathbb{R}\mathbb{1}$ and $V_1=0$ and it possesses a positive definite invariant bilinear form $\langle\cdot,\cdot\rangle$ such that $\langle\mathbb{1},\mathbb{1}\rangle=1$.

Assume that a VOA V of moonshine type contains two distinct conformal vectors e and f with central charge 1/2. In [M8], Miyamoto studied a vertex algebra $\mathrm{VA}(e,f)$ generated by e and f in the case where their Miyamoto involutions τ_e and τ_f generate S_3. In this subsection, we shall complete the classification of $\mathrm{VA}(e,f)$ in [M8] in the case where the inner product $\langle e,f\rangle$ is $13/2^{10}$.

Theorem 8.2.14. *([M8]) Under the assumption above, the inner product* $\langle e,f\rangle$ *is either* $1/2^8$ *or* $13/2^{10}$*. When the inner product is equal to* $13/2^{10}$*, a vertex algebra* $\mathrm{VA}(e,f)$ *generated by* e *and* f *forms a sub VOA in* V*. Denote by* $\mathrm{VA}(e,f)^{(\tau_e\pm)}$ *the eigen spaces for* τ_e *with eigenvalues* ±1*, respectively. The Griess algebra* $\mathrm{VA}(e,f)_2$ *is of dimension 4 and we can choose a basis* $\mathrm{VA}(e,f)_2^{(\tau_e+)}=\mathbb{R}\omega^3\perp\mathbb{R}\omega^4\perp\mathbb{R}v^0$ *and* $\mathrm{VA}(e,f)^{(\tau_e-)}=\mathbb{R}v^1$ *such that* $\omega^3+\omega^4$ *is the Virasoro vector of* $\mathrm{VA}(e,f)$ *and the multiplications and inner products in* $\mathrm{VA}(e,f)_2$ *are given as*

$$
\begin{aligned}
&\omega^3_{(1)}\omega^3=2\omega^3,\quad \omega^3_{(1)}\omega^4=0,\quad \omega^3_{(1)}v^0=\frac{2}{3}v^0,\quad \omega^3_{(1)}v^1=\frac{2}{3}v^1,\quad \omega^4_{(1)}\omega^4_{(1)}=2\omega^4,\\
&\omega^4_{(1)}v^0=\frac{4}{3}v^0,\quad \omega^4_{(1)}v^1=\frac{4}{3}v^1,\quad v^0_{(1)}v^0_{(1)}=\frac{5}{6}\omega^3+\frac{14}{9}\omega^4-\frac{10}{9}v^0,\quad v^0_{(1)}v^1=\frac{10}{9}v^1,\\
&\langle\omega^3,\omega^3\rangle=\frac{2}{5},\quad \langle\omega^4,\omega^4\rangle=\frac{3}{7},\quad \langle v^0,v^0\rangle=\frac{1}{2},\quad \langle v^1,v^1\rangle=1.
\end{aligned}
$$

The complexification $\mathbb{C}\mathrm{VA}(e,f)$ *has a* $\mathbb{Z}_3$*-grading* $\mathbb{C}\mathrm{VA}(e,f)=X^0\oplus X^1\oplus X^2$ *and as* $\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)\simeq L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$*-modules, they are isomorphic to one of the following:*

$$
\begin{aligned}
(i)\quad & X^0=\{L(\tfrac45,0)\oplus L(\tfrac45,3)\}\otimes L(\tfrac67,0),\quad X^1=L(\tfrac45,\tfrac23)^+\otimes L(\tfrac67,\tfrac43),\\
& X^2=L(\tfrac45,\tfrac23)^-\otimes L(\tfrac67,\tfrac43);\\
(ii)\quad & X^0=L(\tfrac45,0)\otimes\{L(\tfrac67,0)\oplus L(\tfrac67,5)\},\quad X^1=L(\tfrac45,\tfrac23)\otimes L(\tfrac67,\tfrac43)^+,\\
& X^2=L(\tfrac45,\tfrac23)\otimes L(\tfrac67,\tfrac43)^-;\\
(iii)\quad & X^0=L(\tfrac45,0)\otimes L(\tfrac67,0)\oplus L(\tfrac45,3)\otimes L(\tfrac67,5),\quad X^1=\{L(\tfrac45,\tfrac23)\otimes L(\tfrac67,\tfrac43)\}^+,\\
& X^2=\{L(\tfrac45,\tfrac23)\otimes L(\tfrac67,\tfrac43)\}^-;\\
(iv)\quad & X^0=\{L(\tfrac45,0)\oplus L(\tfrac45,3)\}\otimes\{L(\tfrac67,0)\oplus L(\tfrac67,5)\},\quad X^1=L(\tfrac45,\tfrac23)^+\otimes L(\tfrac67,\tfrac43)^\pm,\\
& X^2=L(\tfrac45,\tfrac23)^-\otimes L(\tfrac67,\tfrac43)^\mp.
\end{aligned}
$$

In the above, M^- *indicates a* $\mathbb{Z}_2$*-conjugate module of* M^+*.*

We will prove the following.

Theorem 8.2.15. *With reference to Theorem 8.2.14, only the case (iv) occurs. Therefore,* $\mathbb{C}\mathrm{VA}(e,f)$ *is isomorphic to* $U=\mathbb{C}U_{\mathbb{R}}$ *constructed in Section 8.2.1.*

Proof: The symmetric group $S_3=\langle\tau_e,\tau_f\rangle$ on three letters has three irreducible representations $W_0=\mathbb{C}w^0$, $W_1=\mathbb{C}w^1$ and $W_2=\mathbb{C}w^2\oplus\mathbb{C}w^3$, where W_0 is a trivial module, τ_e and τ_f act on w^1 as a scalar -1, and τ_e acts on w^2 and w^3 as scalars respectively 1 and -1. By the quantum Galois theory, we can decompose $\mathbb{C}\mathrm{VA}(e,f)$ as follows:

$$\mathbb{C}\mathrm{VA}(e,f)=\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle}\otimes W_0\bigoplus M_1\otimes W_1\bigoplus M_2\otimes W_2,$$

where M_1 and M_2 are inequivalent irreducible $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle}$-modules. In the proof of Theorem 8.2.14 in [M8], Miyamoto found that only the following two cases could be occur: $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle} = \mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$ or $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle} \supsetneq \mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$, and the former corresponds to the case (i)-(iii) and the latter does the case (iv). We assume that $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle} = \mathbb{C}\mathrm{VA}(\omega^3,\omega^4) \simeq L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$. In this case, seen as a $\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$-module, M^1 is isomorphic to: $L(\frac{4}{5},3)\otimes L(\frac{6}{7},0)$ in the case (i), $L(\frac{4}{5},0)\otimes L(\frac{6}{7},5)$ in the case (ii) and $L(\frac{4}{5},3)\otimes L(\frac{6}{7},5)$ in the case (iii), and M^2 as a $\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$-module is isomorphic to $L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})$ in each case. Therefore, $\mathbb{C}\mathrm{VA}(e,f)^{(\tau_e-)}$ has the following shapes:

$$\mathbb{C}\mathrm{VA}(e,f)^{(\tau_e-)} = \begin{cases} L(\frac{4}{5},3)\otimes L(\frac{6}{7},0)\otimes w^1 \bigoplus L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})\otimes w^3 & \text{in the case (i),} \\ L(\frac{4}{5},0)\otimes L(\frac{6}{7},5)\otimes w^1 \bigoplus L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})\otimes w^3 & \text{in the case (ii),} \\ L(\frac{4}{5},3)\otimes L(\frac{6}{7},5)\otimes w^1 \bigoplus L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})\otimes w^3 & \text{in the case (iii).} \end{cases}$$

We show that $\dim \mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)} = 3$. Since $\mathbb{C}\mathrm{VA}(e,f)_2^{(\tau_e-)} = \mathbb{C}v^1$ and v^1 is a highest weight vector with highest weight $(\frac{2}{3},\frac{4}{3})$, $\omega^3_{(0)}v^1$ and $\omega^4_{(0)}v^1$ are linearly independent vectors in $\mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)}$. We claim that $\{\omega^3_{(0)}v^1, \omega^4_{(0)}v^1, v^0_{(0)}v^1\}$ is a set of linearly independent vectors in $\mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)}$. Set $x^1 = \omega^3_{(0)}v^1$, $x^2 = \omega^4_{(0)}v^1$ and $x^3 = v^0_{(0)}v^1$. Using the commutator formula $[a_{(m)},b_{(n)}] = \sum_{i\ge 0}\binom{m}{i}(a_{(i)}b)_{(m+n-i)}$, an invariant property $\langle a_{(m)}b^1,b^2\rangle = \langle b^1, a_{(-n+2)}b^2\rangle$ for $a\in\mathbb{C}\mathrm{VA}(e,f)_2$ and an identity $(a_{(0)}b)_{(m)} = [a_{(1)},b_{(m-1)}] - (a_{(1)}b)_{(m-1)}$, we can calculate all $\langle x^i,x^j\rangle$, $1\le i,j\le 3$. For example, we compute $\langle x^3,x^3\rangle = \langle v^0_{(0)}v^1, v^0_{(0)}v^1\rangle$ as follows:

$$\begin{aligned} \langle v^0_{(0)}v^1, v^0_{(0)}v^1\rangle &= \langle v^1, v^0_{(2)}v^0_{(0)}v^1\rangle = \langle v^1, [v^0_{(2)}, v^0_{(0)}]v^1\rangle \\ &= \left\langle v^1, \left((v^0_{(0)}v^0)_{(2)} + 2(v^0_{(1)}v^0)_{(1)} + (v^0_{(2)}v^0)_{(0)}\right) v^1\right\rangle \\ &= \left\langle v^1, \left([v^0_{(1)}, v^0_{(1)}] + (v^0_{(1)}v^0)_{(1)}\right) v^1\right\rangle \\ &= \frac{5}{6}\langle v^1, \omega^3_{(1)}v^1\rangle + \frac{14}{9}\langle v^1, \omega^4_{(1)}v^1\rangle - \frac{10}{9}\langle v^1, v^0_{(1)}v^1\rangle \\ &= \frac{113}{81}. \end{aligned}$$

By a similar way, we can compute all $\langle x^i,x^j\rangle$, $1\le i,j\le 3$, and it is a routine work to check that $\det(\langle x^i,x^j\rangle)_{1\le i,j\le 3}\ne 0$. Since $\mathrm{VA}(e,f) = \mathrm{VA}(e,f)^{(\tau_e+)} \perp \mathrm{VA}(e,f)^{(\tau_e-)}$, the non-singularity of a matrix $(\langle x^i,x^j\rangle)_{1\le i,j\le 3}$ implies that x^1, x^2 and x^3 are linearly independent. Therefore, $\dim \mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)} = 3$. One can also see that

$$v^2 := v^0_{(0)}v^1 - \frac{5}{9}(\omega^3_{(0)} + \omega^4_{(0)})v^1$$

is a non-zero highest weight vector for $L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$ with highest weight $(3,0)$. Thus, the possibility of $\mathbb{C}\mathrm{VA}(e,f)$ is only the case (i). We next show that $\dim \mathbb{C}\mathrm{VA}(e,f)_5^{(\tau_e-)} =$

12. Set

$$y^1 = \omega^3_{(-2)} v^1, \quad y^2 = \omega^3_{(-1)}\omega^3_{(0)} v^1, \quad y^3 = \omega^3_{(-1)}\omega^4_{(0)} v^1, \quad y^4 = \omega^3_{(0)}\omega^3_{(0)}\omega^4_{(0)} v^1,$$
$$y^5 = \omega^3_{(0)}\omega^4_{(-1)} v^1, \quad y^6 = \omega^3_{(0)}\omega^4_{(0)}\omega^4_{(0)} v^1, \quad y^7 = \omega^4_{(-2)} v^1, \quad y^8 = \omega^4_{(-1)}\omega^4_{(0)} v^1,$$
$$y^9 = \omega^3_{(-1)} v^2, \quad y^{10} = \omega^3_{(0)}\omega^3_{(0)} v^2, \quad y^{11} = \omega^4_{(-1)} v^2, \quad y^{12} = v^0_{(-2)} v^1.$$

By a similar method used in computations of $\langle x^i, x^j\rangle$, we can calculate all $\langle y^i, y^j\rangle$, $1 \le i, j \le 12$, based on the informations of the Griess algebra of $\mathrm{VA}(e,f)$ and it is also a routine work to show that $\det(\langle y^i, y^j\rangle)_{1\le i,j\le 12} \neq 0$. Therefore, y^i, $1 \le i \le 12$, are linearly independent vectors in $\mathbb{C}\mathrm{VA}(e,f)_5^{(\tau_e -)}$. On the other hand, the dimension of the weight 5 subspace of the case (i) is 11, which is a contradiction. Therefore, we have $\mathbb{C}\mathrm{VA}(e,f)^{\langle \tau_e, \tau_f\rangle} \supsetneq \mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$ and hence only the case (iv) occurs. We can also write down the highest weight vector explicitly. Set

$$\begin{aligned} v^3 &= \frac{5^2}{3^4}\left(\frac{11}{3}\omega^3_{(-2)} - 2\omega^3_{(-1)}\omega^3_{(-0)}\right) v^1 + \frac{7}{3^4}\left(\frac{20}{3}\omega^4_{(-2)} - \omega^4_{(-1)}\omega^4_{(0)}\right) v^1 \\ &+ \frac{5^2}{2^3\cdot 3^2}\left(2\omega^3_{(-1)} - \omega^3_{(0)}\omega^3_{(0)}\right)\omega^4_{(0)} v^1 + \frac{7}{2^2\cdot 3^2\cdot 5}\left(8\omega^4_{(-1)} - \omega^4_{(0)}\omega^4_{(0)}\right)\omega^3_{(0)} v^1 \\ &- \frac{5}{2\cdot 13}\left(\frac{1}{3}\omega^3_{(-1)} - \frac{3}{5}\omega^3_{(0)}\omega^3_{(0)}\right) v^2 + \frac{28}{9}\omega^4_{(-1)} v^2 - v^0_{(-2)} v^1. \end{aligned}$$

Then one can verify that v^3 is a non-zero highest weight vector for $L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$ with highest weight $(0,5)$ by checking that

$$\langle \mathbb{C}\mathrm{VA}(e,f)_4^{(\tau_e -)}, \omega^s_{(2)} v^3\rangle = \langle \mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e -)}, \omega^s_{(3)} v^3\rangle = 0$$

for $s = 3,4$ and $\langle v^0_{(-2)} v^1, v^3\rangle = 1405/3^7$. Since $\mathbb{C}\mathrm{VA}(e,f)$ and $\mathbb{C}U_{\mathbb{R}}$ have unique VOA-structures, $\mathbb{C}\mathrm{VA}(e,f) \simeq \mathbb{C}U_{\mathbb{R}} = U$. ∎

Remark 8.2.16. In the proof above, we note that all $\langle x^i, x^j\rangle$, $1 \le i,j \le 3$ and all $\langle y^p, y^q\rangle$, $1 \le p, q \le 12$, are completely determined by the Griess algebra of $\mathrm{VA}(e,f)$. Therefore, the existence of the case (iv) immediately implies the uniqueness of $\mathbb{C}\mathrm{VA}(e,f)$.

By the theorem above, we can find an application of U to the moonshine VOA. Let $V^\natural_{\mathbb{R}}$ be the real form of the moonshine VOA over $\mathbb{R}$ constructed in [FLM] and [M5]. Since $V^\natural_{\mathbb{R}}$ is (of course) a VOA of moonshine type, its weight two subspace forms a commutative algebra, called the monstrous Griess algebra. As shown in [C] and in [M1], there is a one-to-one correspondence between the 2A-involutions of the Monster and conformal vectors with central charge 1/2 in $(V^\natural_{\mathbb{R}})_2$. Hence, there is a pair $\{e,f\}$ of conformal vectors with central charge 1/2 in $V^\natural_{\mathbb{R}}$ such that $\tau_e\tau_f$ defines a 3A-triality of $\mathbb{M}$. It is shown in [C] that the inner product $\langle e,f\rangle$ of such a pair is equal to $13/2^{10}$. Therefore, the complexification of the moonshine VOA $\mathbb{C}V^\natural_{\mathbb{R}}$ contains a sub VOA isomorphic to U by Theorem 8.2.15. As expected in [KMY], [Mat] and [M7], we can understand the 3A-triality of the Monster through the $\mathbb{Z}_3$-symmetry of the fusion algebra for the 3-state Potts model $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$.

Theorem 8.2.17. *There exists a sub VOA isomorphic to U in the complexificated moonshine VOA $\mathbb{C}V^\natural_{\mathbb{R}}$. Therefore, $\mathbb{C}V^\natural_{\mathbb{R}}$ contains both the 3-state Potts model $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ and the tricritical 3-state Potts model $L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)$ and we can define a 3A-triality of the Monster by the $\mathbb{Z}_3$-symmetries of the fusion algebras for these models.*

Chapter 9

Appendix

9.1 Simple Current Super-Extension

In this appendix we present a theory of simple current super-extensions.

Definition 9.1.1. Let V^0 be a simple VOA. A *simple current super-extension* of V^0 is a simple vertex operator superalgebra $V = V^0 \oplus V^1$ such that the even part is V^0 and the odd part V^1 is a simple current V^0-module.

Proposition 9.1.2. *Let $(V = V^0 \oplus V^1, Y_V(\cdot, z))$ be a simple current super-extension of V^0. Then the SVOA structure on V is unique over $\mathbb{C}$.*

Proof: Let $(V, Y_V^1(\cdot, z))$ be another simple SVOA structure on V. Since $Y_V(\mathbb{1}, z) = Y_V^1(\mathbb{1}, z) = \mathrm{id}_V$, we have $Y_V(a, z) = Y_V^1(a, z)$ for all $a \in V^0$. Then by the skew-symmetry we have $Y_V^1(u, z)a = e^{zL(-1)} Y_V^1(a, z)u = e^{zL(-1)} Y_V(a, z)u = Y_V(u, z)a$ for all $a \in V^0$ and $u \in V^1$. Since V^1 is a simple current, there is a scalar $\lambda \in \mathbb{C}^*$ such that $Y_V^1(u, z)v = \lambda Y_V(u, z)v$ for all $u, v \in V^1$. Then define a V^0-isomorphism $\varphi : (V^0 \oplus V^1, Y_V(\cdot, z)) \to (V^0 \oplus V^1, Y_V^1(\cdot, z))$ by $(a, u) \mapsto (a, u/\sqrt{\lambda})$ for $a \in V^0$ and $u \in V^1$. Then for $a \in V^0$ and $u, v \in V^1$, we have

$$\begin{aligned} &Y_V^1(\varphi u, z)\varphi(a, v) = (1/\sqrt{\lambda}) Y_V^1(u, z)(a, v/\sqrt{\lambda}) = (1/\sqrt{\lambda})(Y_V^1(u, z)v/\sqrt{\lambda}, Y_V^1(u, z)a) \\ &= (1/\sqrt{\lambda})(\sqrt{\lambda} Y_V(u, z)v, Y_V(u, z)a) = \varphi(Y_V(u, z)v, Y_V(u, z)a) = \varphi Y_V(u, z)(a, v). \end{aligned}$$

Thus φ defines an SVOA-isomorphism between $(V, Y_V(\cdot, z))$ and $(V, Y_V^1(\cdot, z))$. ∎

Let us recall the definition of $\mathbb{Z}_2$-twisted modules of an SVOA.

Definition 9.1.3. Let $V = V^0 \oplus V^1$ be an SVOA. A *graded $\mathbb{Z}_2$-twisted V-module* is a pair $(M, Y_M(\cdot, z))$ consisting of an $\mathbb{N}$-graded vector space $M = \oplus_{n \in \mathbb{N}} M(n)$ and a linear map

$$Y_M(\cdot\,, z) : V \ni a \mapsto Y_M(a, z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}(M)[[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]]$$

with the following conditions:

(i) For $a \in V^i$, the module vertex operator has the shape $Y_M(a,z) = \sum_{n \in \frac{i}{2}+\mathbb{Z}} a_{(n)} z^{-n-1}$;

(ii) For $a \in V$ and $v \in M$, $a_{(n)}v = 0$ for sufficiently large $n \in \frac{1}{2}\mathbb{Z}$;

(iii) $Y_M(\mathbb{1}, z) = \mathrm{id}_M$;

(iv) $a_{(n)}M(s) \subset M(s + \mathrm{wt}(a) - n - 1)$;

(v) For $\mathbb{Z}_2$-homogeneous $a, b \in V$, the following $\mathbb{Z}_2$-twisted Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a,z_1)Y_M(b,z_2) - (-1)^{\epsilon(a,b)} z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y_M(b,z_2)Y_M(a,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-\frac{\epsilon(a,a)}{2}} Y_M(Y_V(a,z_0)b, z_2). \tag{9.1.1}$$

In the rest of this section, we always assume that V^0 is a simple rational C_2-cofinite VOA of CFT-type and $V = V^0 \oplus V^1$ is a simple current super-extension of V^0. Then all V-modules and all $\mathbb{Z}_2$-twisted modules automatically have $\frac{1}{2}\mathbb{N}$-grading resp. $\mathbb{N}$-grading.

Theorem 9.1.4. *(1) Every V-module is completely reducible. (2) Every $\mathbb{Z}_2$-twisted V-module is completely reducible.*

Proof: (1): Let M be a V-module. Take an irreducible V^0-submodule W, which is possible as V^0 is rational. Then $V^1 \cdot W$ is not zero and is irreducible V^0-module by Lemma 4.4.1. It is clear that the $V^1 \cdot W$ is not isomorphic to W as a V^0-module as the difference between the top weight of W and that of $V^1 \cdot W$ is in $1/2 + \mathbb{Z}$. Then $W + (V^1 \cdot W)$ is a direct sum and is an irreducible V-submodule of M. Thus M is completely reducible.

(2): Let M be a $\mathbb{Z}_2$-twisted V-module. Take an irreducible V^0-submodule W of M. By Lemma 4.4.1, $V^1 \cdot W$ is a non-trivial irreducible V^0-submodule of M. If $V^1 \cdot W$ is not isomorphic to W as a V^0-module, then $W + (V^1 \cdot W)$ is a direct sum and so is an irreducible V-submodule of M. In this case we are done so we assume that $V^1 \cdot W \simeq W$ as a V^0-module. If $V^1 \cdot W = W$, then we are also done so that we consider the case $W + (V^1 \cdot W)$ is a direct sum of irreducible V^0-submodules. Take a V^0-isomorphism $\varphi : W \to V^1 \cdot W$. Then $\varphi^{-1} Y_M(\cdot|_{V^1}, z)|_W$ and $Y_M(\cdot|_{V^1}, z)\varphi|_W$ are V^0-intertwining operators of type $V^1 \times W \to V^1 \cdot W$ so that there is a scalar $\lambda \in \mathbb{C}^*$ such that $\varphi^{-1} Y_M(u,z)|_W = \lambda Y_M(u,z)\varphi|_W$ for all $u \in V^1$. Then by replacing φ by $\sqrt{\lambda}\varphi$, we may assume that $\lambda = 1$. In $W \oplus (V^1 \cdot W)$, consider V^0-submodules $W^{\pm} = \{(w, \pm\varphi w) \mid w \in W\}$. Then $W \oplus (V^1 \cdot W) = W^+ \oplus W^-$ and $Y_M(u,z)(w, \pm\varphi w) = (\pm Y_M(u,z)\varphi w, Y_M(u,z)w) = (\pm\varphi^{-1} Y_M(u,z)w, \varphi \cdot \varphi^{-1} Y_M(u,z)w) = \pm(\varphi^{-1} Y_M(u,z)w, \pm\varphi \cdot \varphi^{-1} Y_M(u,z)w)$ for all $u \in V^1$. Therefore, $W^{\pm}$ are irreducible V-submodules and so $W \oplus (V^1 \cdot W)$ is a completely reducible V-submodule of M. Thus M is completely reducible V-module. ∎

Proposition 9.1.5. *(1) Let $(M, Y_M(\cdot,z))$ be an irreducible V-module. Then M as a V^0-module is a direct sum of two inequivalent irreducible V^0-submodule and the V-module structure on M is unique over $\mathbb{C}$.*
(2) Let $(M, Y_M(\cdot,z))$ be an irreducible $\mathbb{Z}_2$-twisted V-module. If M as a V^0-module is not irreducible, then M is a direct sum of two inequivalent V^0-submodules and the V-module structure on M is unique over $\mathbb{C}$. On the other hand, if M is irreducible as a V^0-module, then there are exactly two inequivalent irreducible $\mathbb{Z}_2$-twisted V-module structure on M and anther structure is given by its $\mathbb{Z}_2$-conjugate.

Proof: (1): Let W be an irreducible V^0-submodule of M. Since the powers of z in the vertex operator map $Y_M(\cdot,z)$ is contained in $\mathbb{Z}$, the difference between the top weight of $V^1 \boxtimes_{V^0} W$ and that of W is in $1/2+\mathbb{Z}$. Thus M as a V^0-module is a direct sum $W \oplus (V^1 \boxtimes_{V^0} W)$ of two inequivalent V^0-submodules by Lemma 4.4.1. For convenience, we set $W^0 := W$ and $W^1 := V^1 \boxtimes_{V^0} W$. Let $(M, Y_M^1(\cdot,z))$ be another irreducible V-module structure. Then we may assume that $Y_M(a,z) = Y_M^1(a,z)$ for all $a \in V^0$ as $Y_M(\mathbb{1},z) = Y_M^1(\mathbb{1},z) = \mathrm{id}_M$. Since V^1 is a simple current V^0-module, there is a scalar $\lambda \in \mathbb{C}^*$ such that $Y_M^1(u,z)w^0 = \lambda Y_M(u,z)w^0$ for all $u \in V^1$ and $w^0 \in W^0$. Then by the associativity of vertex operators (2.3.2) we have $Y_M^1(u,z)w^1 = (1/\lambda)\cdot Y_M(u,z)w^1$ for all $u \in V^1$ and $w^1 \in W^1$. Now define a linear isomorphism $\varphi : (W^0 \oplus W^1, Y_M(\cdot,z)) \to (W^0 \oplus W^1, Y_M^1(\cdot,z))$ by $\varphi(w^0,w^1) = (w^0/\sqrt{\lambda}, \sqrt{\lambda}w^1)$ for $w^i \in W^i$, $i = 0,1$. Then

$$\begin{aligned}&Y_M^1(u,z)\varphi(w^0,w^1) = Y_M^1(u,z)(w^0/\sqrt{\lambda}, \sqrt{\lambda}w^1) = (\sqrt{\lambda}Y_M^1(u,z)w^1, Y_M^1(u,z)w^0/\sqrt{\lambda})\\ &= (Y_M(u,z)w^1/\sqrt{\lambda}, \sqrt{\lambda}Y_M(u,z)w^0) = \varphi(Y_M(u,z)w^1, Y_M(u,z)w^0) = \varphi Y_M(u,z)(w^0,w^1).\end{aligned}$$

Thus φ defines a V-isomorphism between $(M, Y_M(\cdot,z))$ and $(M, Y_M^1(\cdot,z))$. This completes the proof of (1).

(2): Let W be an irreducible V^0-submodule of M. In this case it may happen that $V^1 \boxtimes_{V^0} W \simeq W$ as a V^0-module because the powers of z in the vertex operator map $Y_M(\cdot|_{V^1},z)$ are in $1/2+\mathbb{Z}$. If $V^1 \boxtimes_{V^0} W \not\simeq W$ as a V^0-module, then $M = W \oplus (V^1 \boxtimes_{V^0} W)$ and we can prove the uniqueness of $\mathbb{Z}_2$-twisted V-module structure on M by a similar way to (1). So consider the case $V^1 \boxtimes_{V^0} W \simeq W$. Then we see that $M = W$ by Theorem 9.1.4. Let $(M, Y_M^1(\cdot,z))$ be another irreducible $\mathbb{Z}_2$-twisted V-module structure on M. Then $Y_M(a,z) = Y_M^1(a,z)$ for all $a \in V^0$ and there is a scalar $\mu \in \mathbb{C}^*$ such that $Y_M^1(u,z) = \mu Y_M(u,z)$ for all $u \in V^1$. Then by the associativity of the vertex operator map (2.3.2) we have $\mu^2 = 1$. Thus $\mu = \pm 1$ and so $(M, Y_M^1(\cdot,z))$ is isomorphic to either $(M, Y_M(\cdot,z))$ or its $\mathbb{Z}_2$-conjugate module structure. So it remains to show that the $\mathbb{Z}_2$-conjugate module is not isomorphic to $(M, Y_M(\cdot,z))$. Denote by $(M, Y_M^-(\cdot,z))$ be the $\mathbb{Z}_2$-conjugate module of $(M, Y_M(\cdot,z))$. Suppose that there is a V-isomorphism $\psi : (M, Y_M(\cdot,z)) \to (M, Y_M^-(\cdot,z))$. By definition, there is a V^0-isomorphism $\sigma : M \to M$

such that $Y_M^-(u,z)\sigma = \sigma Y_M(-u,z)$ for all $u \in V^1$. Then $\sigma^{-1}\psi$ is also a V^0-isomorphism on M and so there is a scalar $\alpha \in \mathbb{C}^*$ such that $\sigma^{-1}\psi = \alpha$ by Schur's lemma. Since V is simple SVOA and M is an irreducible $\mathbb{Z}_2$-twisted V-module, $Y_M(u,z)w \neq 0$ for all $0 \neq u \in V^1$ and $0 \neq w \in M$. However, $0 \neq \psi Y_M(u,z)w = \alpha\sigma Y_M(u,z)w = \alpha Y_M^-(-u,z)\sigma w = -Y_M^-(u,z)\psi w = -\psi Y_M(u,z)w$, a contradiction. Thus $(M, Y_M(\cdot,z))$ and its $\mathbb{Z}_2$-conjugate module structure are not isomorphic. ∎

By the preceeding theorem and the proposition above, we have determined the structures of V-modules. Next, we complete a classification of V-modules.

Proposition 9.1.6. *Let $(W^0, Y_{W^0}(\cdot,z))$ be an irreducible V^0-module. If $V^1 \boxtimes_{V^0} W^0 \not\simeq W^0$, then the difference between the top weight of $V^1 \boxtimes_{V^0} W^0$ and that of W^0 is contained either in $\mathbb{Z}$ or in $1/2 + \mathbb{Z}$.*

Proof: The proof is similar to that of Lemma 4.5.1. Suppose $W^1 := V^1 \boxtimes_{V^0} W^0$ is not isomorphic to W^0 as a V^0-module. Then $V^1 \boxtimes_{V^0} W^1 = W^0$ by the associativity for the fusion product. Let $h_0 \in \mathbb{Q}$ be the top weight of W^0 and $h_1 \in \mathbb{Q}$ that of W^1, and set $k = -h_0 + h_1$. Let $I^0(\cdot,z)$ be a non-trivial V^0-intertwining operator of type $V^1 \times W^0 \to W^1$ and $I^1(\cdot,z)$ a non-trivial V^0-intertwining operator of type $V^1 \times W^1 \to W^0$. Then the powers of z in $I^0(\cdot,z)$ are contained in $k + \mathbb{Z}$ and those of z in $I^1(\cdot,z)$ are in $-k + \mathbb{Z}$ by definition of intertwining operators. By Theorem 3.7.5, there is a scalars $\lambda \in \mathbb{C}^*$ such that

$$\langle \nu, I^1(u,z_1)I^0(v,z_2)w^0\rangle = \lambda\langle \nu, I^1(v,z_2)I^0(u,z_1)w^0\rangle,$$

where $\nu \in (W^0 \oplus W^1)^*$, $a \in V^0$, $u, v \in V^1$ and $w^i \in W^i$, $i = 0, 1$. Since both

$$z_1^k z_2^{-k}\langle \nu, I^1(u,z_1)I^0(v,z_2)w\rangle \text{ and } z_1^{-k}z_2^k\langle \nu, I^1(v,z_2)I^0(u,z_1)w\rangle$$

contain only integral powers of z_1 and z_2, we obtain the following equality of the meromorphic functions for $N \gg 0$:

$$\begin{aligned}&(z_1 - z_2)^N \iota_{12}^{-1} z_1^k z_2^{-k}\langle \nu, I^1(u,z_1)I^0(v,z_2)w\rangle\\ &= \lambda(z_1 - z_2)^N \iota_{21}^{-1} z_1^{2k} z_2^{-2k} z_1^{-k} z_2^k \langle \nu, I^1(v,z_2)I^0(u,z_1)w\rangle.\end{aligned}$$

Therefore, the monodromy-freeness implies $2k \in \mathbb{Z}$. ∎

Theorem 9.1.7. *Every irreducible V^0-module lifts to be either an irreducible V-module or an irreducible $\mathbb{Z}_2$-twisted V-module. More precisely, let W be an irreducible V^0-module, then*
(i) if $V^1 \boxtimes_{V^0} W \not\simeq W$ as a V^0-module, then $W \oplus (V^1 \boxtimes_{V^0} W)$ is the unique irreducible V-module or irreducible $\mathbb{Z}_2$-twisted V-module containing W as a V^0-submodule.
(ii) if $V^1 \boxtimes_{V^0} W \simeq W$ as a V^0-module, then there are exactly two inequivalent irreducible $\mathbb{Z}_2$-twisted V-module structure on W.

Proof: Let $(W^0, Y_{W^0}(\cdot, z))$ be an irreducible V^0-module. First, assume that $V^1 \boxtimes_{V^0} W^0$ is not isomorphic to W^0 as a V^0-module. Set $W^1 := V^1 \boxtimes_{V^0} W^0$ and take non-trivial V^0-intertwining operators $I^0(\cdot, z)$, $I^1(\cdot, z)$ of types $V^1 \times W^0 \to W^1$, $V^1 \times W^1 \to W^0$, respectively. Then by Theorem 3.7.5 there are scalars $\lambda_0, \lambda_1 \in \mathbb{C}^*$ such that

$$\begin{aligned}
\langle \nu, I^1(u^1, z_1) I^0(u^2, z_2) w^0 \rangle &= \lambda_0 \langle \nu, Y_{W^0}(Y_V(u^1, z_0)u^2, z_2)w^0\rangle|_{z_0=z_1-z_2}, \\
\langle \nu, I^0(u^1, z_1) I^1(u^2, z_2) w^1 \rangle &= \lambda_1 \langle \nu, Y_{W^1}(Y_V(u^1, z_0)u^2, z_2)w^1\rangle|_{z_0=z_1-z_2},
\end{aligned}$$

where $u^1, u^2 \in V^1$, $w^i \in W^i$, $i = 0, 1$ and $\nu \in (W^0 \oplus W^1)^*$. Moreover, since $Y_{W^i}(\mathbb{1}, z) = \mathrm{id}_{W^i}$, we have

$$\langle \nu, I^i(u^1, z_1) Y_{W^i}(a, z_2) w^i \rangle = \langle \nu, I^i(Y_V(u^1, z_0)a, z_2)w^i\rangle|_{z_0=z_1-z_2}$$

for all $a \in V^0$. Then by considering

$$\langle \nu, I^i(u^1, z_1) I^{i+1}(u^2, z_2) I^i(u^3, z_3) w^i \rangle$$

for $u^1, u^2, u^3 \in V^1$, $w^i \in W^i$ and $\nu \in (W^0 \oplus W^1)^*$, we have $\lambda_0 = \lambda_1$. Then by replacing $I^i(\cdot, z)$ by $(1/\sqrt{\lambda_i}) I^i(\cdot, z)$, we may assume that $\lambda_0 = \lambda_1 = 1$. Now define the vertex operator map $\hat{Y}(\cdot, z)$ on $W^0 \oplus W^1$ by

$$\hat{Y}(a, z)w^i := Y_{W^i}(a, z)w^i, \quad \hat{Y}(u, z)w^i := I^i(u, z)w^i$$

for $a \in V^0$, $u \in V^1$ and $w^i \in W^i$, $i = 0, 1$. Then we already have the associativity for $\hat{Y}(\cdot, z)$. Since

$$\begin{aligned}
&\langle \nu, I^i(u^1, z_1) I^{i+1}(u^2, z_2) I^i(u^3, z_3) w^i \rangle \\
&= \langle \nu, Y_{W^{i+1}}(Y_V(u^1, z_4)u^2, z_2) I^i(u^3, z_3) w^i \rangle|_{z_4=z_1-z_2} \\
&= \langle \nu, I^i(Y_V(Y_V(u^1, z_4)u^2, z_5)u^3, z_3) w^i \rangle|_{z_4=z_1-z_2, z_5=z_2-z_3} \\
&= \langle \nu, I^i(Y_V(u^1, z_6)Y_V(u^2, z_5)u^3, z_3) w^i \rangle|_{z_6=z_1-z_3, z_5=z_2-z_3} \\
&= \langle \nu, I^i(Y_V(u^2, z_5)Y_V(u^1, z_6)u^3, z_3) w^i \rangle|_{z_6=z_1-z_3, z_5=z_2-z_3} \\
&= -\langle \nu, I^i(Y_V(Y_V(u^2, z_7)u^1, z_6)u^3, z_3) w^i \rangle|_{z_7=z_2-z_1, z_6=z_1-z_3} \\
&= -\langle \nu, Y_{W^{i+1}}(Y_V(u^2, z_7)u^1, z_1) I^i(u^3, z_3) w^i \rangle|_{z_7=z_2-z_1} \\
&= -\langle \nu, I^i(u^2, z_2) I^{i+1}(u^1, z_1) I^i(u^3, z_3) w^i \rangle,
\end{aligned}$$

we also have the commutativity for $\hat{Y}(\cdot, z)$. Thus $(W^0 \oplus W^1, \hat{Y}(\cdot, z))$ is either an irreducible V-module or an irreducible $\mathbb{Z}_2$-twisted V-module by Proposition 9.1.6. This completes the proof of (i).

Next, consider the case $V^1 \boxtimes_{V^0} W^0 \simeq W^0$ as a V^0-module. For simplicity, we write $(W, Y_W(\cdot, z))$ for $(W^0, Y_{W^0}(\cdot, z))$. Take a non-trivial V^0-intertwining operator $J(\cdot, z)$ of type $V^1 \times W \to W$. Then powers of z in $J(\cdot, z)$ are half-integral by definition of intertwining operators. By Theorem 3.7.5, we have

$$\langle \nu, J(u, z_1) Y_W(a, z_2) w \rangle = \langle J(Y_V(u, z_0)a, z_2) w \rangle|_{z_0 = z_1 - z_2}$$

for all $a \in V^0$, $u \in V^1$, $w \in W$ and $\nu \in W^*$. Moreover, there is a scalar $\lambda \in \mathbb{C}^*$ such that

$$\langle \nu, J(u^1, z_1) J(u^2, z_2) w \rangle = \lambda \langle \nu, Y_W(Y_V(u^1, z_0) u^2, z_2) w \rangle|_{z_0 = z_1 - z_2}$$

for $u^1, u^2 \in V^1$ again by Theorem 3.7.5. Then by replacing $J(\cdot, z)$ by $(1/\sqrt{\lambda}) J(\cdot, z)$, we may assume that $\lambda = 1$. Then by an argument similar to that in the proof of (i) we can prove the commutativity

$$\langle \nu, J(u^1, z_1) J(u^2, z_2) w \rangle = -\langle \nu, J(u^2, z_2) J(u^1, z_1) w \rangle.$$

Thus by defining the vertex operator map $\hat{Y}(\cdot, z)$ on W by

$$\hat{Y}(a, z) := Y_W(a, z), \quad \hat{Y}(u, z) := J(u, z)$$

for $a \in V^0$ and $u \in V^1$, we have an irreducible $\mathbb{Z}_2$-twisted V-module structure on $(W, \hat{Y}(\cdot, z))$. This completes the proof of (ii). ∎

We recall the following extension property in Theorem 4.6.1:

Theorem 9.1.8. *Let $V^{(0,0)}$ be a simple rational C_2-cofinite VOA of CFT-type, and D an abelian group. Assume that we have a set of inequivalent irreducible simple current $V^{(0,0)}$-modules $\{V^{(\alpha,\beta)} \mid \alpha \in D, \beta \in \mathbb{Z}_2 = \{0,1\}\}$ with $D \oplus \mathbb{Z}_2$-graded fusion rules $V^{(\alpha_1,\beta_1)} \boxtimes_{V^{(0,0)}} V^{(\alpha_2,\beta_2)} = V^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}$ for any $(\alpha_1,\beta_1), (\alpha_2,\beta_2) \in D \oplus \mathbb{Z}_2$. Moreover, assume that $V_D = \oplus_{\alpha \in D} V^{(\alpha,0)}$ forms a D-graded simple current extension of $V^{(0,0)}$ and $V^{(0,0)} \oplus V^{(0,1)}$ forms a simple current super-extension of $V^{(0,0)}$. Then $V_{D \oplus \mathbb{Z}_2} = \oplus_{(\alpha,\beta) \in D \oplus \mathbb{Z}_2} V^{(\alpha,\beta)}$ has a unique structure of a simple vertex operator superalgebra with even part V_D and odd part $\oplus_{\alpha \in D} V^{(\alpha,1)}$. Namely, $V_{D \oplus \mathbb{Z}_2}$ forms a simple current super-extension of V_D.*

Remark 9.1.9. By this theorem, we can introduce a simple SVOA structure on $V\!B = V\!B^0 \oplus V\!B^1$ without reference to $V^\natural$ as a simple current super-extension of $V\!B^0$.

Finally, we present a lifting property of automorphisms.

Theorem 9.1.10. *Suppose that $\sigma \in \mathrm{Aut}(V^0)$ satisfies $(V^1)^\sigma \simeq V^1$ as a V^0-module. Then there is a lifting $\tilde{\sigma} \in \mathrm{Aut}(V)$ such that $\sigma(V^0) = V^0$, $\sigma(V^1) = V^1$ and $\sigma|_{V^0} = \sigma$.*

Proof: By assumption, we can define a σ-conjugate super-extension V^σ of V. Then by the uniqueness of the SVOA structure in Proposition 9.1.2, we have a desired lifting $\tilde{\sigma} \in \mathrm{Aut}(V)$. Note that this lifting is unique up to multiple of the canonical $\mathbb{Z}_2$-symmetry on V. ∎

9.2 Proof of Theorem 5.1.12

In this appendix, we present the detailed proof of Theorem 5.1.12. We use the same notation as in Section 4.2. First, we list the defining relations and useful identities. Let $a \in V$, $u^i \in W^i$, $i = 1, 2, 3$. Then the defining relations are as follows.

$$E^{\pm}(h,z)\phi_i = \phi_i E^{\pm}(h,z), \quad E^{\pm}(h,z)\phi_i' = \phi_i' E^{\pm}(h,z),$$

$$h_{(0)}\phi_i = \phi_i\left(h_{(0)} + \gamma\right), \quad h_{(0)}\phi_i' = \phi_i'\left(h_{(0)} + \gamma\right),$$

$$\Delta(h,z)\phi_i = z^{\gamma}\phi_i\Delta(h,z), \quad \Delta(h,z)\phi_i' = z^{\gamma}\phi_i'\Delta(h,z),$$

$$Y(\phi_0 a, z)u^i = E^-(h,z)\phi_i Y(a,z)\Delta(h,\zeta z)u^i,$$

$$Y(\phi_0 a, z)\phi_i u^i = E^-(h,z)\pi_i^{-1}\phi_i'\phi_i Y(z^{\gamma}\Delta(h,z)a, z)\Delta(h,\zeta z)u^i,$$

$$I^{01}(u^1,z)\phi_2 u^2 = \phi_3 I^{00}(\Delta(h,z)u^1, z)u^2,$$

$$I^{10}(\phi_1 u^1, z)u^2 = E^-(h,z)\phi_3 I^{00}(u^1,z)\Delta(h,\zeta z)u^2,$$

$$I^{11}(\phi_1 u^1, z)\phi_2 u^2 = E^-(h,z)\pi_3^{-1}\phi_3'\phi_3 I^{00}(\phi_1^{-1}\Delta(h,z)\phi_1 u^1, z)\Delta(h,\zeta z)u^2.$$

And the followings are taken from Lemma 5.1.2.

$$\Delta(h,z_2)I^{ij}(x,z_0) = I^{ij}(\Delta(h,z_2+z_0)x, z_0)\Delta(h,z_2),$$

$$I^{ij}(x,z_2)E^-(h,z_1) = E^-(h,z_1)I^{ij}(\Delta(h,z_2-z_1)\Delta(-h,z_2)x, z_2),$$

$$I^{ij}(E^-(h,z_1)x, z_2) = E^-(h,z_1+z_2)E^-(-h,z_2)I^{ij}(x,z_2)\Delta(-h,\zeta z_2)\Delta(h,\zeta(z_2+z_1)).$$

The followings are given by choosing suitable π_i's.

$$I^{00}(\pi_1^{-1}\phi_1'\phi_1 u^1, z) = E^-(2h,z)\pi_3^{-1}\phi_3'\phi_3 I^{00}(u^1,z)\Delta(2h,\zeta z),$$

$$I^{00}(u^1,z)\pi_2^{-1}\phi_2'\phi_2 = \pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(2h,z)u^1, z).$$

In the following argument, we will freely use the relations and identities above. We shall show the following Jacobi identity.

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\frac{z_1-z_2}{z_0}\right)^{-\alpha_1} Y(\phi_0 a, z_1)\bar{I}(x,z_2)y \\
&-(-1)^{\epsilon(\gamma,x)}\zeta^{\alpha_1} z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\left(\frac{z_2-z_1}{z_0}\right)^{-\alpha_1}\bar{I}(x,z_2)Y(\phi_0 a, z_1)y \\
&= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\left(\frac{z_2+z_0}{z_1}\right)^{-\alpha_2}\bar{I}(Y(\phi_0 a, z_0)x, z_2)y,
\end{aligned} \tag{9.2.1}$$

where $a \in V$, $x = u^1$ or $\phi_1 u^1$, and $y = u^2$ or $\phi_2 u^2$. We divide the proof into four cases according to the parities of x and y.

(i): $(x, y) = (u^1, u^2)$. In this case, we have

$$\begin{aligned}
&Y(\phi_0 a, z_1) I^{00}(u^1, z_2) u^2 \\
&= E^-(h, z_1)\phi_3 Y(a, z_1)\Delta(h, \zeta z_1) I^{00}(u^1, z_2) u^2 \\
&= E^-(h, z_1)\phi_3 Y(a, z_1) I^{00}(\Delta(h, \zeta z_1 + z_2) u^1, z_2)\Delta(h, \zeta z_1) u^2, \\
&I^{01}(u^1, z_2) Y(\phi_0 a, z_1) u^2 \\
&= I^{01}(u^1, z_2) E^-(h, z_1)\phi_2 Y(a, z_1)\Delta(h, \zeta z_1) u^2 \\
&= E^-(h, z_1) I^{01}(\Delta(h, z_2 - z_1)\Delta(-h, z_2) u^1, z_2)\phi_2 Y(a, z_1)\Delta(h, \zeta z_1) u^2 \\
&= E^-(h, z_1)\phi_3 I^{00}(\Delta(h, z_2)\Delta(h, z_2 - z_1)\Delta(-h, z_2) u^1, z_2) Y(a, z_1)\Delta(h, \zeta z_1) u^2 \\
&= E^-(h, z_1)\phi_3 I^{00}(\Delta(h, z_2 - z_1) u^1, z_2) Y(a, z_1)\Delta(h, \zeta z_1) u^2.
\end{aligned}$$

By

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\Delta(h, \zeta z_1 + z_2) u^1 = z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\frac{z_1 - z_2}{z_0}\right)^{\alpha_1}\Delta(h, \zeta z_0) u^1$$

and

$$z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\Delta(h, z_2 - z_1) u^1 = z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\zeta^{-\alpha_1}\left(\frac{z_2 - z_1}{z_0}\right)^{\alpha_1}\Delta(h, \zeta z_0) u^1,$$

the left hand side of (9.2.1) is equal to

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) E^-(h, z_1)\phi_3 Y(a, z_1) I^{00}(\Delta(h, \zeta z_0) u^1, z_2)\Delta(h, \zeta z_1) u^2 \\
&\quad - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) E^-(h, z_1)\phi_3 I^{00}(\Delta(h, \zeta z_0) u^1, z_2) Y(a, z_1)\Delta(h, \zeta z_1) u^2 \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) E^-(h, z_1)\phi_3 I^{00}(Y(a, z_0)\Delta(h, \zeta z_0) u^1, z_2)\Delta(h, \zeta z_1) u^2.
\end{aligned}$$

On the other hand, the right hand side of (9.2.1) is equal to

$$\begin{aligned}
&z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} I^{10}(Y(\phi_0 a, z_0) u^1, z_2) u^2 \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} I^{10}(E^-(h, z_0)\phi_1 Y(a, z_0)\Delta(h, \zeta z_0) u^1, z_2) u^2 \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} E^-(h, z_0 + z_2) E^-(-h, z_2) \\
&\quad \times I^{10}\left(\phi_1 Y(a, z_0)\Delta(h, \zeta z_0) u^1, z_2\right)\Delta(-h, \zeta z_2)\Delta(h, \zeta(z_2 + z_0)) u^2
\end{aligned}$$

$$= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)E^-(h,z_1)E^-(-h,z_2)E^-(h,z_2)\phi_3$$
$$\times I^{00}\left(Y(a,z_0)\Delta(h,\zeta z_0)u^1,z_2\right)\Delta(h,\zeta z_2)\Delta(-h,\zeta z_2)\Delta(h,\zeta z_1)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)E^-(h,z_1)\phi_3 I^{00}(Y(a,z_0)\Delta(h,\zeta z_0)u^1,z_2)\Delta(h,\zeta z_1)u^2.$$

In the transformation above, we also used that

$$z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\left(\frac{z_2+z_0}{z_1}\right)^{-\alpha_2}\Delta(h,\zeta(z_2+z_0))u^2 = z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\Delta(h,\zeta z_1)u^2.$$

Hence, (9.2.1) holds in this case.

(ii): $(x,y)=(\phi_1u^1,u^2)$. We have

$$Y(\phi_0a,z_1)I^{10}(\phi_1u^1,z_2)u^2$$
$$= Y(\phi_0a,z_1)E^-(h,z_2)\phi_3I^{00}(u^1,z_2)\Delta(h,\zeta z_2)u^2$$
$$= E^-(h,z_1)\pi_3^{-1}\phi_3'\phi_3Y(z_1^{\gamma}\Delta(h,z_1)a,z_1)\Delta(h,\zeta z_1)E^-(h,z_2)I^{00}(u^1,z_2)\Delta(h,\zeta z_2)u^2$$
$$= z_1^{\gamma}E^-(h,z_1)\pi_3^{-1}\phi_3'\phi_3Y(\Delta(h,z_1)a,z_1)E^-(h,z_2)\left(1-\frac{z_2}{z_1}\right)^{\gamma}$$
$$\times\Delta(h,\zeta z_1)I^{00}(u^1,z_2)\Delta(h,\zeta z_2)u^2$$
$$= (z_1-z_2)^{\gamma}E^-(h,z_1)\pi_3^{-1}\phi_3'\phi_3E^-(h,z_2)Y(\Delta(h,z_1-z_2)\Delta(-h,z_1)\Delta(h,z_1)a,z_1)$$
$$\times I^{00}(\Delta(h,\zeta z_1+z_2)u^1,z_2)\Delta(h,\zeta z_1)\Delta(h,\zeta z_2)u^2$$
$$= (z_1-z_2)^{\gamma}E^-(h,z_1)E^-(h,z_2)\pi_3^{-1}\phi_3'\phi_3Y(\Delta(h,z_1-z_2)a,z_1)$$
$$\times I^{00}(\Delta(h,\zeta(z_1-z_2))u^1,z_2)\Delta(h,\zeta z_1)\Delta(h,\zeta z_2)u^2,$$

$$I^{11}(\phi_1u^1,z_2)Y(\phi_0a,z_1)u^2$$
$$= I^{11}(\phi_1u^1,z_2)E^-(h,z_1)\phi_2Y(a,z_1)\Delta(h,\zeta z_1)u^2$$
$$= E^-(h,z_2)\pi_3^{-1}\phi_3'\phi_3I^{00}(z_2^{\gamma}\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_2)E^-(h,z_1)Y(a,z_1)\Delta(h,\zeta z_1)u^2$$
$$= z_2^{\gamma}E^-(h,z_2)\pi_3^{-1}\phi_3'\phi_3I^{00}(\Delta(h,z_2)u^1,z_2)E^-(h,z_1)\left(1-\frac{z_1}{z_2}\right)^{\gamma}$$
$$\times\Delta(h,\zeta z_2)Y(a,z_1)\Delta(h,\zeta z_1)u^2$$
$$= (z_2-z_1)^{\gamma}E^-(h,z_2)\pi_3^{-1}\phi_3'\phi_3E^-(h,z_1)I^{00}(\Delta(h,z_2-z_1)\Delta(-h,z_2)\Delta(h,z_2)u^1,z_2)$$
$$\times Y(\Delta(h,\zeta z_2+z_1)a,z_1)\Delta(h,\zeta z_1)\Delta(h,\zeta z_2)u^2$$

$$= (z_2 - z_1)^{\gamma} E^-(h, z_1)E^-(h, z_2)\pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(h, z_2 - z_1)u^1, z_2)$$
$$\times Y(\Delta(h, -z_2 + z_1))a, z_1)\Delta(h, \zeta z_1)\Delta(h, \zeta z_2)u^2.$$

Therefore, the left hand side of (9.2.1) is equal to

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) z_0^{\gamma} E^-(h, z_1)E^-(h, z_2)\pi_3^{-1}\phi_3'\phi_3 Y(\Delta(h, z_0)a, z_1)I^{00}(\Delta(h, \zeta z_0)u^1, z_2)$$
$$\times\Delta(h, \zeta z_1)\Delta(h, \zeta z_2)u^2$$
$$-z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) z_0^{\gamma} E^-(h, z_1)E^-(h, z_2)\pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(h, \zeta z_0)u^1, z_2)$$
$$\times Y(\Delta(h, z_0)a, z_1)\Delta(h, \zeta z_1)\Delta(h, \zeta z_2)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} E^-(h, z_1)E^-(h, z_2)\pi_3^{-1}\phi_3'\phi_3 I^{00}(Y(\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)$$
$$\times\Delta(h, \zeta z_1)\Delta(h, \zeta z_2)u^2.$$

On the other hand, the right hand side of (9.2.1) is equal to

$$z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} I^{00}(Y(\phi_0 a, z_0)\phi_1 u^1, z_2)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2}$$
$$\times I^{00}(E^-(h, z_0)\pi_1^{-1}\phi_1'\phi_1 Y(z_0^{\gamma}\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} z_0^{\gamma} E^-(h, z_0 + z_2)E^-(-h, z_2)$$
$$\times I^{00}(\pi_1^{-1}\phi_1'\phi_1 Y(\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\Delta(-h, \zeta z_2)\Delta(h, \zeta(z_2 + z_0))u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} E^-(h, z_1)E^-(-h, z_2)$$
$$\times I^{00}(\pi_1^{-1}\phi_1'\phi_1 Y(\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\Delta(-h, \zeta z_2)\Delta(h, \zeta z_1)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} E^-(h, z_1)E^-(-h, z_2)E^-(2h, z_2)\pi_3^{-1}\phi_3'\phi_3$$
$$\times I^{00}(Y(\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\Delta(2h, \zeta z_2)\Delta(-h, \zeta z_2)\Delta(h, \zeta z_1)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} E^-(h, z_1)E^-(h, z_2)\pi_3^{-1}\phi_3'\phi_3$$
$$\times I^{00}(Y(\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\Delta(h, \zeta z_1)\Delta(h, \zeta z_2)u^2.$$

Hence, we obtain (9.2.1) in this case.

(iii): $(x, y) = (u^1, \phi_2 u^2)$. We have

$$
\begin{aligned}
&Y(\phi_0 a, z_1) I^{01}(u^1, z_2)\phi_2 u^2\\
&= Y(\phi_0 a, z_1)\phi_3 I^{00}(\Delta(h, z_2)u^1, z_2)u^2\\
&= E^-(h, z_1)\pi_3^{-1}\phi_3'\phi_3 Y(z_1^\gamma \Delta(h, z_1)a, z_1)\Delta(h, \zeta z_1) I^{00}(\Delta(h, z_2)u^1, z_2)u^2\\
&= z_1^\gamma E^-(h, z_1)\pi_3^{-1}\phi_3'\phi_3 Y(\Delta(h, z_1)a, z_1) I^{00}(\Delta(h, \zeta z_1 + z_2)\Delta(h, z_2)u^1, z_2)\Delta(h, \zeta z_1)u^2,\\
&I^{00}(u^1, z_2)Y(\phi_0 a, z_1)\phi_2 u^2\\
&= I^{00}(u^1, z_2)\pi_2^{-1}\phi_2'\phi_2 E^-(h, z_1)Y(z_1^\gamma \Delta(h, z_1)a, z_1)\Delta(h, \zeta z_1)u^2\\
&= z_1^\gamma \pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(2h, z_2)u^1, z_2)E^-(h, z_1)Y(\Delta(h, z_1)a, z_1)\Delta(h, \zeta z_1)u^2\\
&= z_1^\gamma \pi_3^{-1}\phi_3'\phi_3 E^-(h, z_1) I^{00}(\Delta(h, z_2 - z_1)\Delta(-h, z_2)\Delta(2h, z_2)u^1, z_2)\\
&\qquad\qquad \times Y(\Delta(h, z_1)a, z_1)\Delta(h, \zeta z_1)u^2\\
&= z_1^\gamma E^-(h, z_1)\pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(h, z_2 - z_1)\Delta(h, z_2)u^1, z_2)Y(\Delta(h, z_1)a, z_1)\Delta(h, \zeta z_1)u^2.
\end{aligned}
$$

Therefore, the left hand side of (9.2.1) is equal to

$$
\begin{aligned}
&z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_1^\gamma E^-(h, z_1)\pi_3^{-1}\phi_3'\phi_3\\
&\qquad \times I^{00}(Y(\Delta(h, z_1)a, z_0)\Delta(h, \zeta z_0)\Delta(h, z_2)u^1, z_2)\Delta(h, \zeta z_1)u^2.
\end{aligned}
$$

On the other hand, the right hand side of (9.2.1) is

$$
\begin{aligned}
&z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} I^{11}(Y(\phi_0 a, z_0)u^1, z_2)\phi_2 u^2\\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} I^{11}(E^-(h, z_0)\phi_1 Y(a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\phi_2 u^2\\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} E^-(h, z_0 + z_2)E^-(-h, z_2)\\
&\qquad \times I^{11}(\phi_1 Y(a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\Delta(-h, \zeta z_2)\Delta(h, \zeta(z_2 + z_0))\phi_2 u^2\\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) E^-(h, z_1)E^-(-h, z_2) I^{11}(\phi_1 Y(a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\\
&\qquad \times \phi_2(\zeta z_2)^{-\gamma}\Delta(-h, \zeta z_2)(\zeta z_1)^\gamma \Delta(h, \zeta z_1)u^2
\end{aligned}
$$

$$
\begin{aligned}
&= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) z_1^{\gamma} z_2^{-\gamma} E^-(h,z_1)E^-(-h,z_2)E^-(h,z_2)\pi_3^{-1}\phi_3'\phi_3 \\
&\qquad \times I^{00}(z_2^{\gamma}\Delta(h,z_2)Y(a,z_0)\Delta(h,\zeta z_0)u^1,z_2)\Delta(h,\zeta z_2)\Delta(-h,\zeta z_2)\Delta(h,\zeta z_1)u^2 \\
&= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) z_1^{\gamma} E^-(h,z_1)\pi_3^{-1}\phi_3'\phi_3 \\
&\qquad \times I^{00}(Y(\Delta(h,z_2+z_0)a,z_0)\Delta(h,z_2)\Delta(h,\zeta z_0)u^1,z_2)\Delta(h,\zeta z_1)u^2 \\
&= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) z_1^{\gamma} E^-(h,z_1)\pi_3^{-1}\phi_3'\phi_3 \\
&\qquad \times I^{00}(Y(\Delta(h,z_1)a,z_0)\Delta(h,\zeta z_0)\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_1)u^2.
\end{aligned}
$$

Hence, we get (9.2.1) in this case.

(iv): $(x,y)=(\phi_1 u^1,\phi_2 u^2)$. We have

$$
\begin{aligned}
&Y(\phi_0 a,z_1)I^{11}(\phi_1 u^1,z_2)\phi_2 u^2 \\
&= Y(\phi_0 a,z_1)E^-(h,z_2)\pi_3^{-1}\phi_3'\phi_3 I^{00}(z_2^{\gamma}\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_2)u^2 \\
&= z_2^{\gamma}E^-(h,z_2)Y(\Delta(h,z_1-z_2)\Delta(-h,z_1)\phi_0 a,z_1)\pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_2)u^2 \\
&= z_2^{\gamma}E^-(h,z_2)Y((z_1-z_2)^{\gamma}z_1^{-\gamma}\phi_0\Delta(h,z_1-z_2)\Delta(-h,z_1)a,z_1)\pi_3^{-1}\phi_3'\phi_3 \\
&\qquad \times I^{00}(\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_2)u^2 \\
&= z_1^{-\gamma}z_2^{\gamma}(z_1-z_2)^{\gamma}E^-(h,z_2)E^-(h,z_1)\phi_3 Y(\Delta(h,z_1-z_2)\Delta(-h,z_1)a,z_1)\Delta(h,\zeta z_1) \\
&\qquad \times \pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_2)u^2 \\
&= z_1^{-\gamma}z_2^{\gamma}(z_1-z_2)^{\gamma}E^-(h,z_1)E^-(h,z_2)\phi_3 Y(\Delta(h,z_1-z_2)\Delta(-h,z_1)a,z_1)\pi_3^{-1}\phi_3'\phi_3 \\
&\qquad \times (\zeta z_1)^{2\gamma}\Delta(h,\zeta z_1)I^{00}(\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_2)u^2 \\
&= z_1^{\gamma}z_2^{\gamma}(z_1-z_2)^{\gamma}E^-(h,z_1)E^-(h,z_2)\phi_3\pi_3^{-1}\phi_3'\phi_3 Y(\Delta(h,z_1-z_2)\Delta(h,z_1)a,z_1) \\
&\qquad \times I^{00}(\Delta(h,\zeta z_1+z_2)\Delta(h,z_2)u^1,z_2)\Delta(h,\zeta z_1)\Delta(h,\zeta z_2)u^2, \\
&I^{10}(\phi_1 u^1,z_2)Y(\phi_0 a,z_1)\phi_2 u^2 \\
&= I^{10}(\phi_1 u^1,z_2)E^-(h,z_1)\pi_2^{-1}\phi_2'\phi_2 Y(z_1^{\gamma}\Delta(h,z_1)a,z_1)\Delta(h,\zeta z_1)u^2 \\
&= z_1^{\gamma}E^-(h,z_1)I^{10}(\Delta(h,z_2-z_1)\Delta(-h,z_2)\phi_1 u^1,z_2)\pi_2^{-1}\phi_2'\phi_2 Y(\Delta(h,z_1)a,z_1)\Delta(h,\zeta z_1)u^2 \\
&= z_1^{\gamma}E^-(h,z_1)I^{10}((z_2-z_1)^{\gamma}z_2^{-\gamma}\phi_1\Delta(h,z_2-z_1)\Delta(-h,z_2)u^1,z_2)\pi_2^{-1}\phi_2'\phi_2 \\
&\qquad \times Y(\Delta(h,z_1)a,z_1)\Delta(h,\zeta z_1)u^2
\end{aligned}
$$

$$= z_1^{\gamma} z_2^{-\gamma}(z_2 - z_1)^{\gamma} E^-(h, z_1)E^-(h, z_2)\phi_3 I^{00}(\Delta(h, z_2 - z_1)\Delta(-h, z_2)u^1, z_2)\Delta(h, \zeta z_2)$$
$$\times \pi_2^{-1}\phi_2'\phi_2 Y(\Delta(h, z_1)a, z_1)\Delta(h, \zeta z_1)u^2$$
$$= z_1^{\gamma} z_2^{-\gamma}(z_2 - z_1)^{\gamma} E^-(h, z_1)E^-(h, z_2)\phi_3 I^{00}(\Delta(h, z_2 - z_1)\Delta(-h, z_2)u^1, z_2)\pi_2^{-1}\phi_2'\phi_2$$
$$\times(\zeta z_2)^{2\gamma} Y(\Delta(h, \zeta z_2 + z_1)\Delta(h, z_1)a, z_1)\Delta(h, \zeta z_2)\Delta(h, \zeta z_1)u^2$$
$$= z_1^{\gamma} z_2^{\gamma}(z_2 - z_1)^{\gamma} E^-(h, z_1)E^-(h, z_2)\phi_3\pi_3^{-1}\phi_3'\phi_3 I^{00}(\Delta(h, z_2 - z_1)\Delta(h, z_2)u^1, z_2)$$
$$\times Y(\Delta(h, \zeta z_2 + z_1)\Delta(h, z_1)a, z_1)\Delta(h, \zeta z_1)\Delta(h, \zeta z_2)u^2.$$

Thus, the left hand side of (9.2.1) is equal to

$$z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} z_1^{\gamma} z_2^{\gamma} E^-(h, z_1)E^-(h, z_2)\phi_3\pi_3^{-1}\phi_3'\phi_3$$
$$\times I^{00}(Y(\Delta(h, z_0)\Delta(h, z_1)a, z_0)\Delta(h, \zeta z_0)\Delta(h, z_2)u^1, z_2)\Delta(h, \zeta z_1)\Delta(h, \zeta z_2)u^2.$$

On the other hand, the right hand side of (9.2.1) is equal to

$$z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} I^{01}(Y(\phi_0 a, z_0)\phi_1 u^1, z_2)\phi_2 u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2}$$
$$\times I^{01}(E^-(h, z_0)\pi_1^{-1}\phi_1'\phi_1 Y(z_0^{\gamma}\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\phi_2 u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha_2} z_0^{\gamma} E^-(h, z_0 + z_2)E^-(-h, z_2)$$
$$\times I^{01}(\pi_1^{-1}\phi_1'\phi_1 Y(\Delta(h, z_0)a, z_0)\Delta(h, \zeta z_0)u^1, z_2)\Delta(-h, \zeta z_2)\Delta(h, \zeta(z_2 + z_0))\phi_2 u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} E^-(h, z_1)E^-(-h, z_2) I^{01}(\pi_1^{-1}\phi_1'\phi_1 Y(\Delta(h, z_0)a, z_0)$$
$$\times\Delta(h, \zeta z_0)u^1, z_2)\phi_2(\zeta z_2)^{-\gamma}(\zeta z_1)^{\gamma}\Delta(-h, \zeta z_2)\Delta(h, \zeta z_1)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} z_1^{\gamma} z_2^{-\gamma} E^-(h, z_1)E^-(-h, z_2)\phi_3 I^{00}(\Delta(h, z_2)\pi_1^{-1}\phi_1'\phi_1 Y(\Delta(h, z_0)a, z_0)$$
$$\times\Delta(h, \zeta z_0)u^1, z_2)\Delta(-h, \zeta z_2)\Delta(h, \zeta z_1)u^2$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) z_0^{\gamma} z_1^{\gamma} z_2^{-\gamma} E^-(h, z_1)E^-(-h, z_2)\phi_3$$
$$\times I^{00}(z_2^{2\gamma}\pi_1^{-1}\phi_1'\phi_1 Y(\Delta(h, z_2 + z_0)\Delta(h, z_0)a, z_0)\Delta(h, z_2)\Delta(h, \zeta z_0)u^1, z_2)$$
$$\times\Delta(-h, \zeta z_2)\Delta(h, \zeta z_1)u^2$$

$$
\begin{aligned}
&= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) z_0^{\gamma} z_1^{\gamma} z_2^{\gamma} E^-(h,z_1)E^-(-h,z_2)\phi_3 E^-(2h,z_2)\pi_3^{-1}\phi_3'\phi_3 \\
&\qquad \times I^{00}(Y(\Delta(h,z_1)\Delta(h,z_0)a,z_0)\Delta(h,z_2)\Delta(h,\zeta z_0)u^1,z_2)\Delta(2h,\zeta z_2) \\
&\qquad\qquad \times\Delta(-h,\zeta z_2)\Delta(h,\zeta z_1)u^2 \\
&= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) z_0^{\gamma} z_1^{\gamma} z_2^{\gamma} E^-(h,z_1)E^-(h,z_2)\phi_3\pi_3^{-1}\phi_3'\phi_3 \\
&\qquad \times I^{00}(Y(\Delta(h,z_1)\Delta(h,z_0)a,z_0)\Delta(h,z_2)\Delta(h,\zeta z_0)u^1,z_2)\Delta(h,\zeta z_1)\Delta(h,\zeta z_2)u^2.
\end{aligned}
$$

Hence, we have the desired identity (9.2.1). ∎

Bibliography

[ATLAS] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, ATLAS of finite groups, Clarendon Press, Oxford, 1985.

[Ab] T. Abe, Rationality of the vertex operator algebra V_L^+ for a positive definite even lattice L, math.QA/0311210.

[ABD] T. Abe, G. Buhl and C. Dong, Rationality, regularity, and C_2-cofiniteness, to appear in *Trans. Amer. Math. Soc.*, math.QA/0204021.

[AD] T. Abe and C. Dong, Classification of irreducible modules for the vertex operator algebra V_L^+: General case, to appear in *J. Algebra*, math.QA/0210274.

[ADL] T. Abe, C. Dong and H. Li, Fusion rules for the vertex operator algebras $M(1)^+$ and V_L^+, to appear in *Comm. Math. Phys.*, math.QA/0310425.

[AM] G. Anderson, and G. Moore, Rationality in conformal field theory, *Comm. Math. Phys.* **117** (1988), 441–450.

[Ast] A. Astashkevich, On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras, *Comm. Math. Phys.* **186** (1997), 531–562.

[BPZ] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nucl. Phys.* **B 241** (1984), 333–380.

[Bo1] R.E. Borcherds, Vertex algebras, Kac-Moody algebras and the Monster, *Proc. Nat. Acad. Sci. USA* **83** (1986), 3068–3071.

[Bo2] R.E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992), 405–444.

[Bu] G. Buhl, A spanning set for VOA modules, *J. Algebra* **254** (2002), 125–151.

[C] J. H. Conway, A simple construction for the Fischer-Griess monster group, *Invent. Math.* **79** (1985), 513–540.

[DVVV] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, The operator algebra of orbifold models, *Comm. Math. Phys.* **123** (1989), 485–526.

[D1] C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **160** (1993), 245–265.

[D2] C. Dong, Twisted modules for vertex algebras associated with even lattices, *J. Algebra* **165** (1994), 90–112.

[D3] C. Dong, Representations of the moonshine module vertex operator algebra, in: *Mathematical aspects of conformal and topological field theories and quantum groups*, Proc. Joint Summer Research Conference, Mount Holyoke, 1992, ed. P. Sally, M. Flato, J. Lepowsky, N. Reshetikhin and G. Zuckerman, Contemporary Math. **175**, Amer. Math. Soc., Providence, 1994, 27–36.

[DGH] C. Dong, R.L. Griess and G. Höhn, Framed vertex operator algebras, codes and the moonshine module, *Comm. Math. Phys.* **193** (1998), 407–448.

[DL] C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Progress in Math. **112**, Birkhäuser, Boston, 1993.

[DLM1] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571–600.

[DLM2] C. Dong, H. Li and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1–56.

[DLM3] C. Dong, H. Li and G. Mason, Compact automorphism groups of vertex operator algebras, *Internat. Math. Res. Notices* **18** (1996), 913–921.

[DLM4] C. Dong, H. Li and G. Mason, Hom functor and the associativity of tensor products of modules for Vertex operator algebras, *J. Algebra* **188** (1997), 443–475.

[DLM5] C. Dong, H. Li and G. Mason, Simple currents and extensions of vertex operator algebras, *Comm. Math. Phys.* **180**, 671–707.

[DLM6] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Algebra* **206** (1998), 67–96.

[DLM7] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras and associative algebras, *Internat. Math. Res. Notices* **8** (1998), 389–397.

[DLMN] C. Dong, H. Li, G. Mason and S.P. Norton, Associative subalgebras of Griess algebra and related topics, *Proc. of the Conference on the Monster and Lie algebra at the Ohio State University*, May 1996, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin - New York, 1998.

[DM1] C. Dong and G. Mason, On quantum Galois theory, *Duke Math. J.* **86** (1997), 305–321.

[DM2] C. Dong and G. Mason, Quantum Galois theory for compact Lie groups, *J. Algebra* **214** (1999), 92–102.

[DM3] C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, math.QA/0201318.

[DM4] C. Dong and G. Mason, Vertex operator algebras and Moonshine; A Survey, *Advanced Studies in Pure Mathematics* **24**, Progress in Algebraic Combinatorics, Mathematical Society of Japan.

[DMZ] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, Proc. Symp. Pure. Math., American Math. Soc. **56** II (1994), 295–316.

[DY] C. Dong and G. Yamskulna, Vertex operator algebras, generalized doubles and dual pairs, *Math. Z.* **241** (2002), 397–423.

[FF] B.L. Feigin and D.B. Fuchs, Verma modules over the Virasoro algebra, *Topology*, Lecture Notes in Math. **1060**, Springer-Verlag, Berlin, 1984, 230–245.

[FFR] A. J. Feingold, I. B. Frenkel and J. F.X. Ries, Spinor construction of vertex operator algebras, triality, and $E_8^{(1)}$, *Contemp. Math.* **121** (1991).

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs Amer. Math. Soc.* **104**, 1993.

[FLM] I.B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, New York, 1988.

[FQS] D. Frieden, Z. Qiu and S. Shenker, Conformal invariance, unitarity and two-dimensional critical exponents, *MSRI publ.* **3**, Springer-Verlag (1984), 419–449.

[FRW] A. J. Feingold, J. F.X. Ries and M. Weiner, Spinor construction of the $c = \frac{1}{2}$ minimal model. Moonshine, the Monster, and related topics (South Hadley, MA, 1994), 45–92, Contemp. Math., 193, Amer. Math. Soc., Providence, RI, 1996.

[FZ] I.B. Frenkel and Y. Zhu, Vertex operator algebras associated to representation of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.

[G] R.L. Griess, Jr., The friendly giant, *Invent. Math.* **69** (1982), 1–102.

[GH] R.L. Griess and G. Höhn, Virasoro frames and their stabilizers for the E_8 lattice type vertex operator algebra, *Journal für die reine und angewandte Mathematik (Crelle)* **561** (2003), 1–37.

[GN] M. Gaberdiel and A. Neitzke, Rationality, quasirationality, and finite W-algebras, DAMTP-200-111.

[GKO] P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro and super-Virasoro algebras, *Comm. Math. Phys.* **103** (1986), 105–119.

[Hö1] G. Höhn, Selbstduale Vertexoperatorsuperalgebren und das Babymonster, Ph.D. thesis, Bonn 1995.

[Hö2] G. Höhn, The group of symmetries of the shorter moonshine module, preprint, math.QA/0210076.

[H1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Algebra* **100** (1995), 173–216.

[H2] Y.-Z. Huang, Virasoro vertex operator algebras, (non-meromorphic) operator product expansion and the tensor product theory, *J. Algebra* **182** (1996), 201–234.

[H3] Y.-Z. Huang, A nonmeromorphic extension of the moonshine module vertex operator algebra, *in* "Moonshine, the Monster and Related Topics, Proc. Joint Summer Research Conference, Mount Holyoke, 1994" (C. Dong and G. Mason, Eds.), Contemporary Math., pp. 123–148, Amer. Math. Soc., Providence, RI, 1996.

[H4] Y.-Z. Huang, Differential equations and intertwining operators, math.QA/0206206.

[HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor product for representations of a vertex operator algebra, in Proc. 20th Intl. Conference on Diff. Geom. Methods in Theoretical Physics, New York, 1991, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344-354.

[HL2] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebra and vertex tensor categories, in: Lie Theory and Geometry, ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhauser, Boston, 1994, 349–383.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Sel. Math.* **1** (1995), 699-756.

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995), 141-171.

[HMT] A. Hanaki, M. Miyamoto and D. Tambara, Quantum Galois theory for finite groups, *Duke. Math. J.* **97** (1999), 541–544.

[K] V. G. Kac, Vertex algebras for beginners, second edition, Cambridge University Press, Cambridge, 1990.

[Kar] G. Karpilovsky, Group representations Vol. 2, Mathematical studies Vol. **177**, North-Holland, 1993.

[KMY] M. Kitazume, M. Miyamoto and H. Yamada, Ternary codes and vertex operator algebras, *J. Algebra* **223** (2000), 379–395.

[KR] V. G. Kac and A. K. Raina, "Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie algebras," World Scientific, Singapore, 1987.

[KW] V. G. Kac and W. Wang, Vertex operator superalgebras and their representations, *Contemp. Math.* **175** (1994).

[L1] C. H. Lam, Induced modules for orbifold vertex operator algebras, *J. Math. Soc. Japan* **53** (2001), 541–557.

[L2] C. H. Lam, Some twisted modules for framed vertex operator algebras, *J. Algebra* **231** (2000), 331–341.

[LLY] C. H. Lam, N. Lam and H. Yamauchi, Extension of Virasoro vertex operator algebra by a simple module, *Internat. Math. Res. Notices* **11** (2003), 577–611.

[LY] C. H. Lam and H. Yamada, Tricritical 3-state Potts model and vertex operator algebras constructed from ternary codes, preprint.

[LYY] C. H. Lam, H. Yamada and H. Yamauchi, Vertex operator algebras, extended E_8-diagram, and McKay's observation on the Monster simple group, preprint.

[Li0] H. Li, The theory of physical superselection sectors in terms of vertex operator algebra language, q-alg/9504026.

[Li1] H. Li, Local systems of vertex operators, vertex superalgebras and modules, *J. Pure Appl. Algebra* **109** (1996), 143–195.

[Li2] H. Li, Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, *in* "Mooshine, the Monster, and related topics" (C. Dong and G. Mason, Eds.), *Contemp. Math.* **193**, 203–236, Amer. Math. Soc., Providence, 1996.

[Li3] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, *J. Pure Appl. Algebra* **96** (1994), 279–297.

[Li4] H. Li, The physics superselection principal in vertex operator algebra theory, *J. Algebra* **196** (1997), 436–457.

[Li5] H. Li, Extension of vertex operator algebras by a self-dual simple module, *J. Algebra* **187** (1997), 236–267.

[Li6] H. Li, Determining fusion rules by $A(V)$-modules and bimodules, *J. Algebra* **212** (1999), 515–556.

[Li7] H. Li, An analogue of the Hom functor and a generalized nuclear democracy theorem, *Duke Math. J.* **93** (1998), 73–114.

[Li8] H. Li, Certain extensions of vertex operator algebras of affine type, *Comm. Math. Phys.* **217** (2001), 653–696.

[Li9] H. Li, On abelian coset generalized vertex algebras, *Comm. Contemp. Math.* **3** (2001), 287–340.

[Mas] G. Mason, The quantum double of a finite group and its role in conformal field theory, Proc. LMS Lecture Notes **212**, CUP.

[Mat] A. Matsuo, Norton's trace formulae for the Griess algebra of a vertex operator algebra with larger symmetry, *Comm. Math. Phys.* **224** (2001), 565–591.

[Mc] J. McKay, Graphs, sigularities, and finite groups, The Santa Cruz Conference on Finite Groups (Santa Cruz, 1979) (B. Cooperstein and G. Mason, eds.), Proc. Symp. Pure Math., vol 37, Amer. Math. Soc., Providence, RI, 1980, pp. 183–186.

[M1] M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, *J. Algebra* **179** (1996), 528–548.

[M2] M. Miyamoto, Binary codes and vertex operator (super)algebras, *J. Algebra* **181** (1996), 207–222.

[M3] M. Miyamoto, Representation theory of code vertex operator algebras, *J. Algebra* **201** (1998), 115-150.

[M4] M. Miyamoto, A Hamming code vertex operator algebra and construction of vertex operator algebras, *J. Algebra* **215** (1999), 509–530.

[M5] M. Miyamoto, A new construction of the moonshine vertex operator algebra over the real number field, to appear in *Ann. of Math.*

[M6] M. Miyamoto, A modular invariance on the theta functions defined on vertex operator algebras, *Duke Math. J.* **101** (2000), 221–236.

[M7] M. Miyamoto, 3-state Potts model and automorphisms of vertex operator algebras of order 3, *J. Algebra* **239** (2001), 56-76.

[M8] M. Miyamoto, VOAs generated by two conformal vectors whose τ-involutions generate S_3, *J. Algebra* **268** (2003), 653–671.

[M9] M. Miyamoto, Modular invariance of vertex operator algebras satysfying C_2-cofiniteness, to appear in *Duke Math. J.*

[M10] M. Miyamoto, A theory of tensor products for vertex operator algebra satsifying C_2-cofiniteness, math.QA/0309350.

[M11] M. Miyamoto, Automorphism groups of $\mathbb{Z}_2$-orbifold VOAs, preprint.

[MT] M. Miyamoto and K. Tanabe, Uniform product of $A_{g,n}(V)$ for an orbifold model V and G-twisted Zhu algebra, to appear in *J. Algebra*, math.QA/0112054.

[R1] M. Roitman, On free conformal and vertex algebras, *J. Algebra* **217** (1999), 496–527.

[R2] M. Roitman, Combinatorics of free vertex algebras, *J. Algebra* **255** (2002), 297–323.

[Sh] H. Shimakura, Automorphism group of the vertex operator algebra V_L^+ for an even lattice L without roots, math.QA/0311141.

[SY] S. Sakuma and H. Yamauchi, Vertex operator algebra with two Miyamoto involutions generating S_3, *J. Algebra* **267** (2003), 272–297.

[Ti] J. Tits, On R. Griess' "Friendly Giant," *Invent. Math.* **78** (1984), 491–499.

[Tu] M. Tuite, On the relationship between Monstrous Moonshine and the uniqueness of the Moonshine module, *Comm. Math. Phys.* **166** (1995), 495–532.

[Wak] M. Wakimoto, Lectures on infinite-dimensional Lie algebra, World Scientific, 2001.

[Wan] W. Wang, Rationality of Virasoro vertex operator algebras, *Internat. Math. Res. Notices* **71** (1993), 197–211.

[Yams] G. Yamskulna, C_2-cofiniteness of the vertex operator algebra V_L^+ when L is a rank one lattice, to appear in *Comm. Algebra*, math.QA/0202056.

[Y1] H. Yamauchi, Orbifold Zhu theory associated to intertwining operators, *J. Algebra* **265** (2003), 513–538.

[Y2] H. Yamauchi, Modularity on vertex operator algebras arising from semisimple primary vectors, to appear in *Internat. J. Math.*

[Y3] H. Yamauchi, Module category of simple current extensions of vertex operator algebras, to appear in *J. Pure Appl. Algebra.*

[Z] Y.Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–302.