ON PLANE BAND DESCRIPTIONS OF AN ST-MOVE

MAKI NAGURA

Abstract. Local moves have attached much attention in knot theory. On this speak, we focus on a local move that is called an ST-move and show that there is a plane band description of it. Moreover we have a necessary and sufficient condition for a local move to be an extended ST-move. Here an ST-move is a local move whose two tangle diagrams are not equal and have no crossings.

1. Definitions

Throughout this paper we work in PL category. We shall study tangles that were introduced by J. Conway in [1]. Please refer to [1] for details of tangles. First we begin with some definitions.

Definition 1.1. *Let* $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leqq 1\}$ *be a unit 3-ball. Let* $T = \cup_{i=1}^{n} t_i$ *be a union of* n *pairwise disjoint arcs* t_i *embedded properly in* B *and let* $\partial T = \partial(\cup_{i=1}^{n} t_i) = \cup_{i=1}^{n} \partial t_i = \big\{(\cos \frac{j}{n}\pi\, , \sin \frac{j}{n}\pi\, , 0) \mid j = 1, 2, \ldots, 2n\big\}$*. Then* (B, T) *is called an* n*-tangle. An* n*-tangle* (B, T) *is called to be oriented if each arc* t_i *is oriented* $(i = 1, 2, \ldots, n)$*.*

Definition 1.2. *Let* (B, T) *be an oriented* n*-tangle and let* (D, T) *be a regular projection of* B *onto the unit disk* $D = \{(x, y, 0) \mid x^2 + y^2 \leqq 1\}$*. Then* (D, T) *is called a tangle diagram of* (B, T) *where we draw one arc close to a double point so that it appears to have been cut to express that the arc passes under the other arc.*

Definition 1.3. *Let* (D_1, T_1) *and* (D_2, T_2) *be oriented* n*-tangle diagrams. A local move is a pair of tangles* (D_1, T_1) *and* (D_2, T_2) *with* $\partial T_1 = \partial T_2$ *and* $I(\partial T_1) = I(\partial T_2)$*. It is denoted by* $(D_1, T_1) \leftrightarrow (D_2, T_2)$*,* $\mathcal{T} : T_1 \leftrightarrow T_2$ *or simply denoted by* $T_1 \leftrightarrow T_2$*.*

To describe the equivalence of local moves we shall define an operation, which is called a braiding operation, as follows.

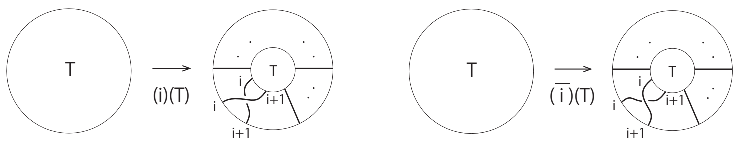

Fig. braiding operations (i) and $(\bar{i})$

Definition 1.4. *Let (D,T) be an n-tangle diagram and let $(\widetilde{D},\widetilde{T})$ be the n-tangle diagram shrunk (D,T) into $\widetilde{D}=\{(x,y,z)\,|\,x^2+y^2+z^2\leqq 1/2\}$ with T. Let $A(i)=\cup_{k=1}^{n}t_k$ ($A(\bar{i})=\cup_{k=1}^{n}t_k$, resp.) be a union of n pairwise disjoint arcs t_k embedded properly in $D\setminus Int(\widetilde{D})$ for any $k\neq i,i+1$ as follows ($i=1,2,\ldots,2n$):*

$\partial t_k=\big\{1/2(\cos\frac{k}{n}\pi,\,\sin\frac{k}{n}\pi,\,0),\,(\cos\frac{k}{n}\pi,\,\sin\frac{k}{n}\pi,\,0)\big\}$,

$\partial t_i=\big\{1/2(\cos\frac{i}{n}\pi,\,\sin\frac{i}{n}\pi,\,0),\,(\cos\frac{i+1}{n}\pi\,,\,\sin\frac{i+1}{n}\pi\,,0)\big\}$,

$\partial t_{i+1}=\big\{1/2(\cos\frac{i+1}{n}\pi,\,\sin\frac{i+1}{n}\pi,\,0),\,(\cos\frac{i}{n}\pi,\,\sin\frac{i}{n}\pi,\,0)\big\}$.

Here the arc t_i has only one crossing of upper (lower, resp.) with the arc t_{i+1} in the diagram $(D\setminus Int(\widetilde{D}),\cup_{k=1}^{n}t_k)$ and any arc t_k ($k\neq i,i+1$) has no crossings with the other arcs. Then an operation that transforms (D,T) into $(D,T')=(\widetilde{D},\widetilde{T})\cup_{\partial\widetilde{D}}(D\setminus Int(\widetilde{D}),A(i))$ ($(D,T')=(\widetilde{D},\widetilde{T})\cup_{\partial\widetilde{D}}(D\setminus Int(\widetilde{D}),A(\bar{i}))$, resp.) is called a braiding operation (i) ($(\bar{i})$, resp.) on (D,T) and we write $T'=(i)T$ ($T'=(\bar{i})T$, resp.). see Fig.

Definition 1.5. *Two local moves $\mathfrak{T}_1:T_1\to T_1'$ and $\mathfrak{T}_2:T_2\to T_2'$ are equivalent, denoted by $\mathfrak{T}_1=\mathfrak{T}_2$, if there are a finite number of braiding operations (i_1), (i_2), $\cdots$, (i_l) such that $T_2=(i_l)\cdots(i_2)(i_1)(T_1)$ and $T_2'=(i_l)\cdots(i_2)(i_1)(T_1')$ where $i_l=1,2,\ldots,2n$ or $\overline{1},\overline{2},\ldots,\overline{2n}$ ($l=1,2,\ldots,m$).*

In [3], a local move $(D_1,T_1)\leftrightarrow(D_2,T_2)$ is called an ST-move if the tangle diagrams (D_1,T_1) and (D_2,T_2) are both trivial and not equivalent ([3]). In this paper, we shall extend ST-moves as follows.

Definition 1.6. *A local move $\mathfrak{T}_1:T_1\leftrightarrow T_1'$ is called an extended $ST(n)$-move if there is a local move $\mathfrak{T}_2:T_2\leftrightarrow T_2'$ so that (D_2,T_2) and (D_2,T_2') are both trivial, $\mathfrak{T}_1=\mathfrak{T}_2$ and $(D_2,T_2)\neq(D_2,T_2')$. When we take no notice of the number of arcs in the tangle diagram, an extended $ST(n)$-move is called simply an extended ST-move.*

Next we shall discuss about band descriptions of a local move. Let m be a natural number.

Definition 1.7. *Let (B,T) be an oriented n-tangle and $T=\cup_{i=1}^{n}t_i$. Let I_k be a copy of the closed unit interval $I=[0,1]$ and let I_k^2 be oriented where $k=1,2,\ldots,m$. Let $b:\amalg_{k=1}^{m}I_k^2\to B$ be an embedding from the disjoint union $\amalg_{k=1}^{m}I_k^2$ of I_k^2 to the ball B as follows: For any $k\in\{1,2,\cdots,m\}$, there exist i and j in $\{1,2,\cdots,n\}$ such that $b(I_k^2)\cap t_i=b(I_k\times\{0\})$, $b(I_k^2)\cap t_j=b(I_k\times\{1\})$ and $b(I_k^2)\cap t_l=\emptyset$ for any $l\neq i,j\in\{1,2,\cdots,n\}$ where the orientation of $b(I_k^2)\cap t_i$ ($b(I_k^2)\cap t_j$, resp.) and the orientation of $b(I_k\times\{0\})$ ($b(I_k\times\{1\})$, resp.) are opposite each other. Then the embedding b and $b(I_k^2)$ are called a band map on (B,T) and a band on (B,T), respectively. The tangle replaced this image $b(\amalg_{k=1}^{m}I_k^2)$ with $b(\{0,1\}\times I_1\cup\cdots\cup\{0,1\}\times I_m)$ is called the band sum of T by $b(\amalg_{k=1}^{m}I_k^2)$ and we write T_b.*

Definition 1.8. *Let $\mathfrak{T}:T\leftrightarrow T'$ be a local move of tangle diagrams T and T'. If there exists a band map $b:\amalg_{k=1}^{m}I_k^2\to B$ on (B,T) such that $T'=T_b$, then $(B,T\cup b(\amalg_{k=1}^{m}I_k^2))$ is called a band description of $T\to T'$.*

If there is a band description of $T \to T'$, then there is also a band description of $T' \to T$. The pair of their band descriptions is called a band description of $\mathfrak{T}$.

2. Main Theorem

Theorem 1. *For any ST-move $\mathfrak{T}$, there is a band description $(B, T \cup b(\amalg_{k=1}^{m} I_k^2))$ of $\mathfrak{T}$ so that each band $b(I_k)$ is unknotted and plane where $k = 1, 2, \ldots, m$.*

A band description that is constructed in the proof of Theorem 1 is called a plane band description.

Theorem 2. *A local move $\mathfrak{T}$ is an extended ST-move if and only if there is a plane band description of $\mathfrak{T}$.*

References

[1] J. Conway, An Enumeration of Knots and Links, and Some of Their Algebraic Properties, Computational Problems in Abstract Algebra. Oxford, England. Pergamon Press (1970), 329–358.

[2] J. Hoste, Y. Nakanishi and K. Taniyama, Unknotting Operation Involving Trivial Tangles, Osaka J. Math. **27** (1990), 555–566.

[3] M. Nagura, Unknotting Operations by Using Oriented Trivial Tangle Diagrams, J. Knot Theory and Its Ramifications, Vol. 8, No. 7 (1999), 901–929.

[4] K. Reidemeister, Elementare Begrundung der Knotentheorie, Abh. Math. Sem. Univ. Hamburg 5 (1926), 24–32

E-mail address: maki@ynu.ac.jp