On lassos
and the Jones polynomial of satellite knots*

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Abstract

The purpose of the talk is to present in the first place a new family of knots in the solid torus called lassos and some of their properties. We will recall the idea of satellite knot and the formula for their Alexander polynomial. I will then investigate the Kauffman bracket and the Jones polynomial of knots in the solid torus, and straightaway give explicit formulae for their calculus for satellite knots. In this respect, by using lassos as patterns of satellite knots I will construct infinitely many knots having the same Alexander polynomial as the one of a chosen knot. In the last part I will prove for certain subfamilies of these satellite knots that they are actually different from each other by using their Jones polynomial.

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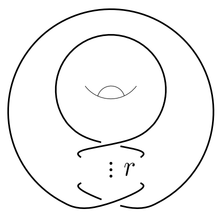
1 Introduction

We define the terms that will be used along the presentation:

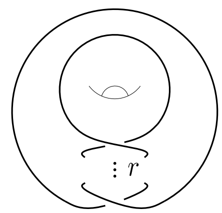
1.1 Lassos

The following definitions will be thought inside the solid torus $ST \simeq \mathbb{S}^1 \times \mathcal{D}^2$, and the unknots (trivial knots) that we will mention will be isotopic to $\mathbb{S}^1 \times \{0\}$ in ST (unless otherwise specified).

Definition 1. We call a *simple lasso* $L(r)$ to the r-twisted knot sum of two nested unknots:



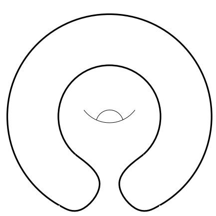
$r > 0$



$r < 0$

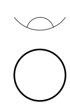
(the  symbol should be understood as the center of ST)[1]

r can be both positive or negative depending on the rotation direction of the twists, as shown in the pictures. The case $r = 0$ will hence represent the standard (untwisted) knot sum:



$L(0)$

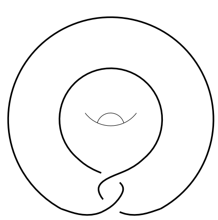
$\rightsquigarrow$



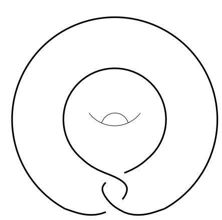
$L(0)$

and, therefore, we may use the terminology $L(0)$ to refer to a trivial knot inside ST.

Remark. Using this notation, a *Whitehead double* can be written as $L(\pm 2)$.



$L(2)$

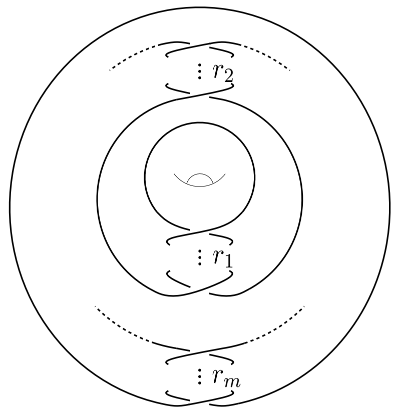


$L(-2)$

[1]We will not depict the outer bound of the solid torus, only its center. The diagrams are interpreted inside ST.

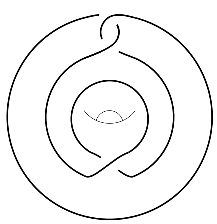
Now, we can generalize this definition to the following.

Definition 2. We call a (*general*) *lasso* $L(r_1, r_2, ..., r_m)$ to the consecutive r_i-twisted knot sum of $m+1$ nested unknots:

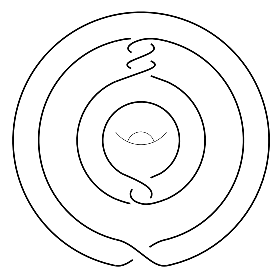


$L(r_1, r_2, ..., r_m)$

These depictions of lassos we will be refered to as *normal diagrams*, and will be used throughout this paper. Let us draw some concrete examples of lassos with normal diagrams:

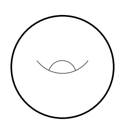


$L(1, 2)$



$L(-2, 3, -1)$

Remark. It is convenient to note that, in concordance with the definition of general lasso, $L(\emptyset)$ would correspond to only one “nested” unknot, and therefore:



$L(\emptyset)$

However, we will <u>not</u> consider any r_i to be 0, since this would lead to simplifiable cases where no 0 appears:

1. $L(0, r_2, r_3, ..., r_m) \simeq L(r_3, ..., r_m)$.

2. $L(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \simeq L(r_1, ..., r_{i-1} + r_{i+1}, ..., r_m)$.

3. $L(r_1, ..., r_{m-2}, r_{m-1}, 0) \simeq L(r_1, ..., r_{m-2})$. This last equivalence is an immediate consequence of the following result.

Proposition 1. *Let $L(r_1, r_2, ..., r_{m-1}, r_m)$ be a lasso. Then,*

$$L(r_m, r_{m-1}, ..., r_2, r_1) \simeq L(r_1, r_2, ..., r_{m-1}, r_m).$$

Definition 3. Let $L(r_1, r_2, ..., r_m)$ be a lasso. Let us consider the $(m+1)$-tuple $(\bar{r}_0, \bar{r}_1, ..., \bar{r}_m)$ where:

$$\bar{r}_0 = 1 \qquad , \qquad \bar{r}_i = \begin{cases} (-1)^{1+r_i} & \text{if } \bar{r}_{i-1} = 1 \\ 1 & \text{if } \bar{r}_{i-1} = -1 \end{cases} \qquad i \in \{1, ..., m\}.$$

Then, we define the *degree* of $L(r_1, r_2, ..., r_m)$ as:

$$\deg\big(L(r_1, r_2, ..., r_m)\big) = \sum_{i=0}^{m} \bar{r}_i.$$

Definition 4. We define the following sets:

$$\mathscr{L} = \{L \in Emb(\mathbb{S}^1, ST) \mid L \;\; lasso\},$$

$$\mathscr{L}_n = \{L \in \mathscr{L} \mid \deg(L) = n\}.$$

Having set the basic definitions, we now focus on the main results.

1.2 Satellite knots and the Alexander polynomial

Definition 5 (Satellite knot)**.** Let P be a knot (*pattern*) in ST. Let $e : ST \hookrightarrow \mathbb{S}^3$ be an embedding of ST onto a regular neighbourhood of a knot C (*companion*) in $\mathbb{S}^3$, that maps a longitude of ST onto a longitude of C (i.e., it is *faithful*). Then eP is the *satellite knot* of P and C, hereinafter $Sat(P, C)$.

Theorem 1 (Lickorish and others)**.** *Let P be a knot in ST, C be a knot in $\mathbb{S}^3$ and $Sat(P, C)$ be their satellite knot. Then,*

$$\Delta_{Sat(P,C)}(t) = \Delta_P(t)\Delta_C(t^n),$$

where P represents n times a generator of $H_1(ST)$.

This result is the basis to the following.

Proposition 2. *Let $L \in \mathscr{L}_d$, C be a knot in $\mathbb{S}^3$ and $Sat(L, C)$ be their satellite knot. Then,*

$$\Delta_{Sat(L,C)}(t) = \Delta_C(t^d).$$

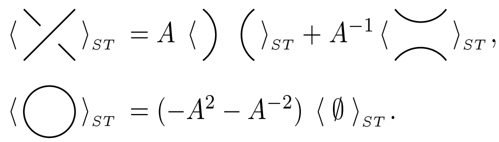
2 Kauffman & Jones

Definition 6 (Framed link)**.** A *framed link* in an oriented 3-manifold Y is a disjoint union of embedded circles, equipped with a non-zero normal vector field. They are considered up to isotopy.

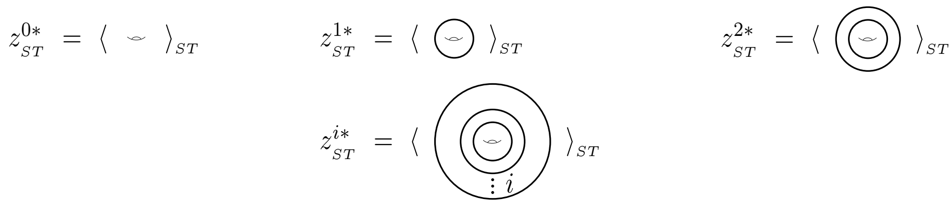
We will consider $Y = ST$, and let $\mathcal{B}$ be the set of isotopy classes of framed links in ST.

2.1 Kauffman bracket

Definition 7. We define the *Kauffman bracket skein module* (and write $S(ST)$) as the free $\mathbb{C}[A^{\pm 1}]$-module with basis $\mathcal{B}$ over the smallest submodule containing:



In our case, if we write $\mathcal{B}^* = \{z_{ST}^{i*} \mid i \geq 0\}$ being z_{ST} the core of a solid torus, $S(ST)$ has an algebra structure that is the polynomial algebra $\mathbb{C}[A^{\pm 1}; z_{ST}]$. Thus, the basis for $S(ST)$ would be:



However, in practice we will consider a *normalized* Kauffman bracket skein module (for practical reasons), which would be the previous where $z_{ST}^{0} = (-A^2 - A^{-2}) z_{ST}^{0*}$, so that in the subsequent part of the paper it matches with the Jones polynomial. Thus consider:

$$z_{ST}^{0} = (-A^2 - A^{-2}) z_{ST}^{0*} \qquad\qquad z_{ST}^{i} = z_{ST}^{i*}$$

$$\mathcal{B} = \{z_{ST}^{i} \mid i \geq 0\}.$$

Let us consider now $\mathscr{L}$ (the set of lassos) the object of study.

Proposition 3. *Let $L(r_1, r_2, ..., r_m) \in \mathscr{L}$ with normal diagram. Then:*

1. $\langle L(\emptyset) \rangle_{ST} = z_{ST}^{1}$;
2. $\langle L(0) \rangle_{ST} = z_{ST}^{0}$;
3. $\langle L(0, r_2, ..., r_m) \rangle_{ST} = T(r_2) \langle L(r_3, r_4, ..., r_m) \rangle_{ST}$;
4. $\langle L(r_1, r_2, ..., r_m) \rangle_{ST} = \begin{cases} A \langle L(r_1 - 1, r_2, ..., r_m) \rangle_{ST} + A^{-1} z_{ST}^{1} T(r_1 - 1) \langle L(r_2, r_3, ..., r_m) \rangle_{ST} & \text{if } r_1 > 0 \\ A^{-1} \langle L(r_1 + 1, r_2, ..., r_m) \rangle_{ST} + A z_{ST}^{1} T(r_1 + 1) \langle L(r_2, r_3, ..., r_m) \rangle_{ST} & \text{if } r_1 < 0 \end{cases}$

where $T(n) = (-A^{-3})^n$.

Corollary 1. *For the particular case of the simple lasso $L(r)$ with $r > 0$ we obtain the explicit formula:*

$$\langle L(r) \rangle_{ST} = A^r z_{ST}^{0} + z_{ST}^{2} T(r) \sum_{i=1}^{r} (-1)^i A^{4i-2}.$$

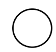
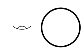
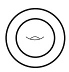
2.2 Jones polynomial

We use the *Kauffman bracket skein module* to redefine the Jones polynomial in ST.

Definition 8. Let L be an oriented framed link in ST with a diagram D. We define the *Jones polynomial* of L in ST (as resemblance of the Jones polynomial in $\mathbb{S}^3$) as:

$$J_{ST}(L) = T(wr(D))\langle D\rangle_{ST}\Big|_{t^{1/2}=A^{-2}},$$

where $T(n) = (-A^{-3})^n$ and $wr(D)$ is the writhe of the link diagram.[2]

In the case of $\mathbb{S}^3$ the only basis element of the bracket is $z^0_{\mathbb{S}^3} = \langle$  $\rangle_{\mathbb{S}^3}$ and in ST the basis considered is $\mathcal{B} = \{z^i_{ST} \mid i \geq 0\}$, that is, $z^0_{ST} = \langle$  $\rangle_{ST}$, $z^1_{ST} = \langle$  $\rangle_{ST}$, $z^2_{ST} = \langle$  $\rangle_{ST}$, etcetera.

Remark. z^0_{ST} was specially chosen for this purpose, so that when a link L in ST is <u>not</u> knotted around its center then:

$$J_{ST}(L) = J(L).$$

2.3 Working on satellites

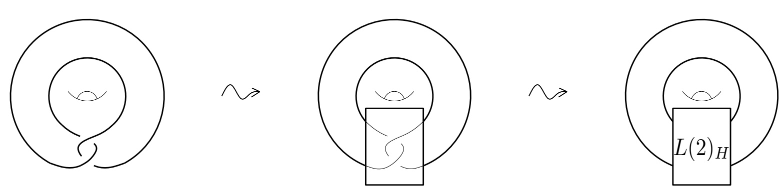
In this subsection we will consistently and constantly utilize the following notation:

- L : (oriented) framed link in $\mathbb{S}^3$.
- L^k : k-parallel of L.
- $L^k(n)$: k-parallel of L with $n \in \mathbb{Z}$ right-handed half-twists.
- U : trivial knot (unknot) in $\mathbb{S}^3$.
- M_P : *geometric degree* of a knot P in ST; it can be also regarded as the maximum power of z_{ST} in $\langle P\rangle_{ST} = \sum_{i=0}^{M_P} a_i z^i_{ST}$).

Remark. Using this notation, $U^k(n) = T(k,n)$ $\big((k,n)$-torus knot$\big)$.

Definition 9 (Hot zone)**.** We call a *hot zone* of a diagram to a region that contains all the crossings of the diagram (hence it also keeps all the information of the writhe). When abbreviated, we will represent it as a box with the label of the knot whose crossings contains, with H as subindex.

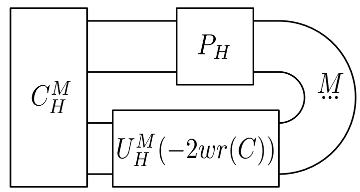
$L(2)$:



[2]The subindex ST will be only specified for the solid torus case. For the $\mathbb{S}^3$ case (the usual case), no subindex will be written for neither the Jones polynomial nor the Kauffman bracket.

Hereinafter we will use the same naming for knots and their diagrams when calculi on diagrams occur.

Definition 10 (Composite diagram)**.** Let P be a knot in ST and C be a knot in $\mathbb{S}^3$. Let M be the geometric degree of P. We call a *composite diagram* of $Sat(P, C)$ to:



Theorem 2. *Let P be a knot in ST, C be a knot in $\mathbb{S}^3$ and $Sat(P, C)$ be their satellite knot with composite diagram. Then,*

$$\langle Sat(P, C)\rangle = \langle P\rangle_{ST}\Big|_{z_{ST}^k = T(-wr(C))^{M-k}\langle C^k(-2wr(C))\rangle},$$

where $T(n) = (-A^{-3})^n$, $wr(C)$ is the writhe of C and $0 \leq k \leq M$.

Lemma 1. *Let P be a knot in ST, C be a knot in $\mathbb{S}^3$ and $Sat(P, C)$ be their satellite knot with composite diagram. The framing is blackboard. Then,*

$$wr(Sat(P, C)) = wr(P) + Mwr(c).$$

Theorem 3. *Let P be a knot in ST, C be a knot in $\mathbb{S}^3$ and $Sat(P, C)$ be their satellite knot. Then,*

$$J(Sat(P, C)) = J_{ST}(P)\Big|_{z_{ST}^k = J(C;k)},$$

where $0 \leq k \leq M$, and $J(C; k)$ represents the k-colored Jones polynomial.

Using the provided information, one can prove that the families of lassos $L(r)$ and $L(1, r)$ where $r \in \mathbb{Z}$ used as patterns of satellite knots generate knots with the same Alexander polyomial (those with lassos of the same degree), but they can be told apart with the Jones polynomial thanks to the last theorem.

3 Sum up

What have we got in the end? A definition of a new kind of knots (lassos) inside the solid torus, a powerful result regarding the Alexander polynomial of satellite knots, and an explicit formula for calculating the Jones polynomial of any satellite knot. Plus, the constructions of satellite knots using lassos are essentially distinct for certain families of lassos thanks to the Jones polynomial. This is, we can now create "by request" knots that have the same Alexander polynomial as any other given knot with t taken at any power, being sure that the knot we create is essentially different from the originally given. We will serve ourselves from this other well-known result:

$$\Delta_{K_1\#K_2}(t) = \Delta_{K_1}(t)\Delta_{K_2}(t).$$

By using it, now we can be told:

"I want a knot K whose Alexander polynomial is $\Delta_K(t) = \Delta_{5_1}(t)^2 \Delta_{8_{19}}(t^3)$".

And we proceed to build it. For us, in this case it suffices to use a lasso $L \in \mathscr{L}_3$ (since there is a t^3) and then claim that such knot could be:

$$K = 5_1 \# 5_1 \# Sat(L(1,1), 8_{19}).$$

But we could also give an infinite amount of examples, where we would only change the lasso to any other of the same degree so that the result would not change. For example:

$$K = 5_1 \# 5_1 \# Sat(L(-3,7), 8_{19}).$$

Or even use the same results to give other knots combinations such as:

$$K = 5_1 \# 5_1 \# Sat(L(1,1), 8_{19}) : \# Sat(L(2), 10_1 61).$$

Let me now finish this summary by recapitulating with a beautiful and simple example.
Let us consider the lasso $L(1,2)$. It happens to be the simplest lasso of degree 1. Consequently, by using it repeatedly we are able to build knots with the exact same Alexander polynomial as a given knot, without further constructions than a simple satellite composition. So given a knot K, we have:

$$\Delta_{Sat(L(1,2),K)}(t) = \Delta_K(t).$$

And using this fact, higher compositions can also be thought:

$$\Delta_K(t) = \Delta_{Sat(L(1,2),K)}(t) = \Delta_{Sat(L(1,2),Sat(L(1,2),K))}(t) = \dots .$$

4 Etymology

The word *lasso* is defined (by the Oxford dictionary) as "a rope with a noose at one end, used especially in North America for catching cattle". Its origin dates back to the beginning of the nineteenth century, coming from the Spanish word *lazo*, from Latin *laqueum*, also related to *lace*.
In a sense, the lassos here defined "catch" the center of the torus and "immobilize" it. But *torus* is a word also coming from Latin that degenerated into *toro* in Spanish, in which, in turn, happens to mean *bull* as well. Therefore the simile is round, and there was no better name for lassos than lassos.