特別な L 空間と絡み目の順序について

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Abstract. L 空間とは、Heegaard-Floer ホモロジーが階数 N をもつ自由加群となるようなホモロジー３球面のことである。ここで N は一次ホモロジー群の位数。この講演では、特別な場合として境界写像が消えているような L 空間について考える。その具体例を構成し、それらを交代的な絡み目で分岐する二重分岐被覆の観点から特徴付けする。その際に絡み目に対して定まるある種の順序を用いる。

1. Heegaard Floer homology and strong L-spaces

Let Y be a closed oriented three manifold. Ozsváth and Szabó introduced the Heegaard Floer homology $\widehat{HF}(Y)$ in [7]. This is a topological invariant of closed oriented three manifolds.

Now, we recall a *Heegaard decomposition* (Σ, α, β) of a closed oriented three manifold Y shortly. It consists of three data as follows.

U_α, U_β : genus g handlebodies

Σ : a closed oriented genus g-surface

$\alpha = \{\alpha_1, \cdots, \alpha_g\}$, $\beta = \{\beta_1, \cdots, \beta_g\}$: attaching circles

Then, Y can be represented uniquely by attaching these handlebodies (see Figure 1).

$$Y = U_\alpha \cup_\Sigma U_\beta$$

(Note that attaching circles are characterized as pairwise disjoint, homologically linearly independent, simple closed curves on Σ.)

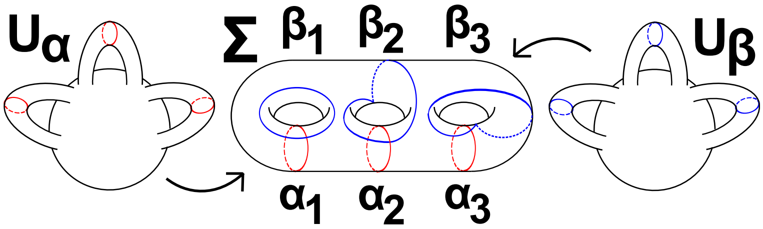

Figure 1. Heegaard decomposition

We sketch the definition of Heegaard Floer homology of (Σ, α, β). Define

$$T_\alpha \cap T_\beta := \{x = (x_{1\sigma(1)}, x_{2\sigma(2)}, \cdots, x_{g\sigma(g)}) | x_{i\sigma(i)} \in \alpha_i \cap \beta_{\sigma(i)}, \sigma \in S_g\},$$

where S_g is the symmetric group on g letters.

Then, the Heegaard Floer chain complex of (Σ, α, β) is defined by

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$\widehat{CF}(Y) = \{$the $\mathbb{Z}_2$-*vector space generated by the elements of* $T_\alpha \cap T_\beta\}$.

We use the following fact (see [7]).

Fact 1. $\exists \widehat{\partial} : \widehat{CF}(Y) \to \widehat{CF}(Y)$: homomorphism s.t. $\widehat{\partial} \cdot \widehat{\partial} = 0$.

Roughly speaking, the boundary map $\widehat{\partial}$ *counts* the pseudo holomorphic Whitney disk. Thus, we can define the Heegaard Floer homology of (Σ, α, β), i.e.,

$\rightsquigarrow \widehat{HF}(\Sigma, \alpha, \beta) = \mathrm{Ker}(\widehat{\partial})/\mathrm{Im}(\widehat{\partial})$

Actually, this homology is independent of the choice of the Heegaard diagram representing Y. That is, this becomes a topological invariant for closed three manifolds. We denote it by $\widehat{HF}(Y)$.

Definition 1.1 (L-space, strong L-space)**.** Let Y be a rational homology three-sphere.

Y is an <u>*L-space*</u> $\overset{\text{def}}{\Leftrightarrow} \widehat{HF}(Y) \cong \mathbb{Z}_2^{|H_1(Y;\mathbb{Z})|}$.

Y is a <u>*strong L-space*</u> $\overset{\text{def}}{\Leftrightarrow} \exists(\Sigma, \alpha, \beta)$ representing Y s.t.

$$|T_\alpha \cap T_\beta| = |H_1(Y;\mathbb{Z})|.$$

Remark. We can define a strong L-space in another way. Y is a <u>*strong L-space*</u>

$\Leftrightarrow \exists(\Sigma, \alpha, \beta)$ representing Y s.t. the boundary map $\widehat{\partial}$ is the *zero* map.

We call such a diagram (Σ, α, β) a <u>*strong diagram*</u>.

The above definitions first appeared in [1].

2. Smoothing order and Main theorem

Let D_L be a link diagram of a link L. A smoothing is the following operation.

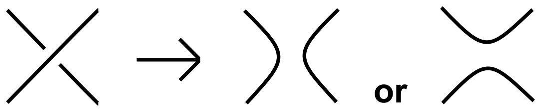


Figure 2. smoothing

Definition 2.1 ([2] and [9])**.** Let L_1 and L_2 be alternating links in S^3.

D_{L_1} and D_{L_2}: alternating link diagrams of L_1 and L_2

$\underline{D_{L_1} \subseteq D_{L_2}} \overset{\text{def}}{\Leftrightarrow} D_{L_2}$ contains D_{L_1} as a component after smoothing some crossing points of D_{L_2}.

$\underline{L_1 \leq L_2} \overset{\text{def}}{\Leftrightarrow} \forall\ D_{L_2}$: a minimal crossing alternating link diagram of L_2,

$\exists\ D_{L_1}$: a minimal crossing alternating link diagram of L_1 s.t.

$D_{L_1} \subseteq D_{L_2}$.

Note that we can define $\leq$ for any two links by ignoring alternating conditions (see [2]).

<u>Example</u>: It is easy to see that (the Hopf link) $<$ (the trefoil knot).

Let us define a class of alternating links by using this smoothing order.

Definition 2.2. $\mathcal{L}_{\overline{\mathrm{Brm}}} = \{$ an alternating link L in S^3 such that $\mathrm{Brm} \not\leq \mathrm{L}\}$,

where Brm is the Borromean rings. (We denote the following link diagram of Borromean rings by Brm too).

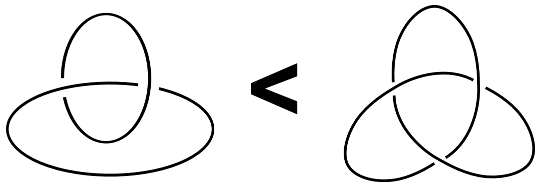


FIGURE 3

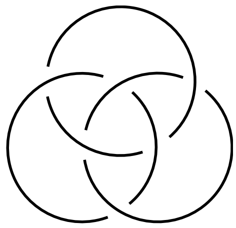


FIGURE 4. The Borromean

Now, let us denote $\Sigma(L)$ the *double branched covering of S^3 branched along a link L*.

Theorem 2.1 (Usui)**.** *Let L be an alternating link in S^3. If L satisfies the following conditions:*

- $L \in \mathcal{L}_{\overline{\mathrm{Brm}}}$ *, i.e.* Brm $\not\leq$ L,
- $\Sigma(L)$ *is a rational homology three sphere,*

then $\Sigma(L)$ is a strong L-space and a graph manifold (or a connected sum of graph-manifolds).

Note that the following theorem seems stronger than Theorem 2.1.

Theorem 2.2. [3] *L :an alternating link in S^3*
$\Sigma(L)$ *is a rational homology three sphere* $\Rightarrow$ $\Sigma(L)$ *is a strong L-space,*

However, we can prove Theorem 2.1 independently and systematically. In this article, we describe some ideas to prove Theorem 2.1.

3. IDEAS OF PROOF

There are three ideas to prove Theorem 1.1.

1. (1) Generalization : We start from the diagram of lens space.
2. (2) Transformation : We use the Montesinos trick.
3. (3) Induction : We replace the definition of $\mathcal{L}_{\overline{\mathrm{Brm}}}$ with an inductive difinition to prove Theorem 2.1 inductively.

(1) First, we review the Heegaard diagrams of lens spaces which are basic examples of (strong) L-spaces.
Example:
$\rightsquigarrow \widehat{CF}(L(p,q)) = \mathbb{Z}_2^p$ and $|T_\alpha \cap T_\beta| = p = |H_1(L(p,q);\mathbb{Z})|$.
$\rightsquigarrow \widehat{HF}(L(p,q)) = \mathbb{Z}_2^p$ and $L(p,q)$ are strong L-spaces.

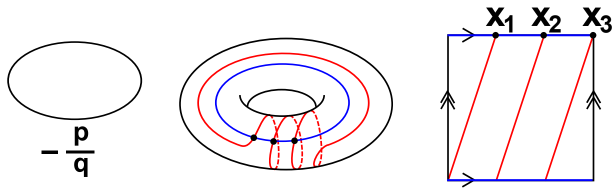
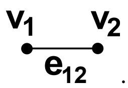


FIGURE 5. Lens space $L(p,q)$ and its Heegaard diagram ($p = 3, q = 1$)

<u>Question.</u> Can we find other strong diagrams in this way? (Are there any other strong diagrams which are naturally induced from some surgery representations of three manifolds?)

Definition 3.1. (T, σ, w) : an *alternatingly weighted tree* $\overset{\text{def}}{\Leftrightarrow}$

- T is a tree. Let $V(T)$ be the set of vertices of T.
- $\sigma : V(T) \to \{\pm 1\}$ is a map s.t $\sigma(v_1) = -\sigma(v_2)$ for  .
- $w : V(T) \to \{0, 1, \infty\}$ is a map.

Given an alternatingly weighted tree (T, σ, w), we can define a three manifold $Y_{(T,\sigma,w)}$ naturally by performing surgeries along the unknots in S^3 (see Figure 3.)

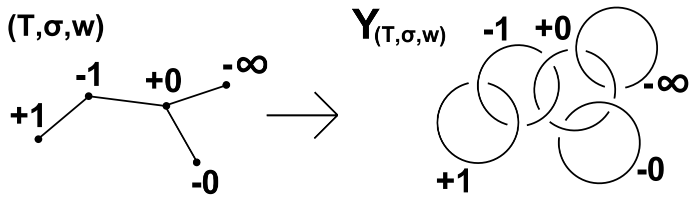


FIGURE 6

We can take a Heegaard diagram of $Y_{(T,\sigma,w)}$ naturally (see Figure 3). Moreover, this diagram becomes a strong diagram. (That is, we can prove $|T_\alpha \cap T_\beta| = |H_1(Y_{(T,\sigma,w)}; \mathbb{Z})|$.)

Proposition 3.1. (T, σ, w) *: an alternatingly-weighted tree*
If $Y_{(T,\sigma,w)}$ *is* $\mathbb{Q}HS$*, then* $Y_{(T,\sigma,w)}$ *is a <u>strong L-space</u> and a graph manifold (or a connected sum of graphmanifolds).*

<u>(2)</u> Next, we describe the Montesinos trick shortly by using an example (see 3, 3 and 3).
<u>Example</u>: $\Sigma(\text{trefoil}) = \mathrm{L}(3, 1)$.
<u>We</u> use the Montesinos trick to obtain the correspondence between double branched coverings of S^3 and surgeries along a link. In particular, we study the correspondence between the class of links $\mathcal{L}_{\overline{\text{Brm}}}$ and the class of manifolds $\{Y_{(T,\sigma,w)} : (T, \sigma, w)$ is an aternatingly weighted tree$\}$.
<u>(3)</u> We give another inductive definition of $\mathcal{L}_{\overline{\text{Brm}}}$.

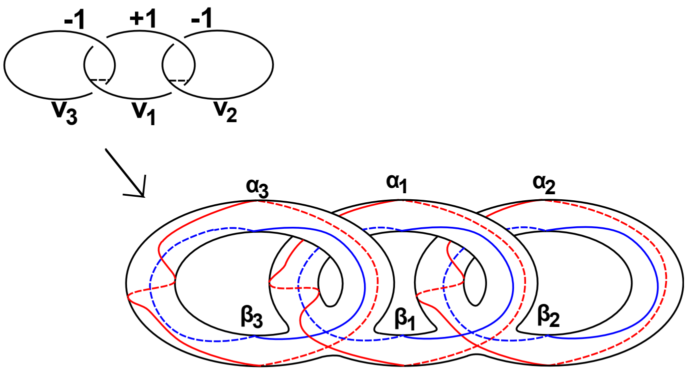


FIGURE 7

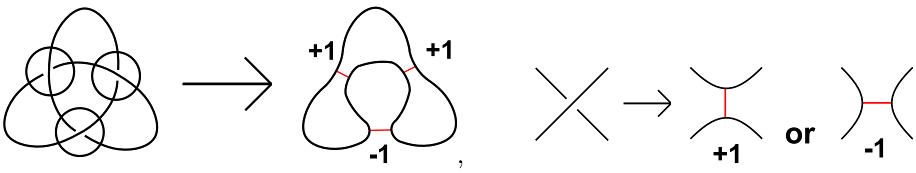


FIGURE 8

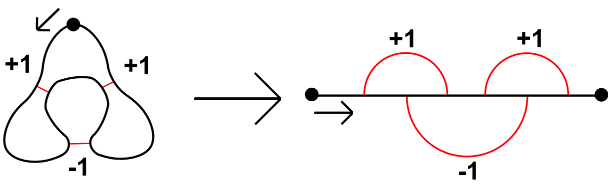


FIGURE 9

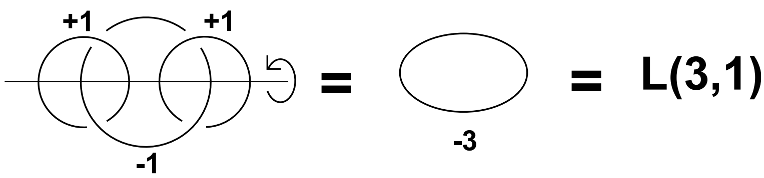


FIGURE 10

Definition 3.2. Let D_L be an alternating link diagram. Let B be an embedded disk in $\mathbb{R}^2$.

B is 1-reducible for D_L $\overset{\text{def}}{\Leftrightarrow}$ ∂B intersects with D_L at just one crossing point c and c looks as in Figure 3.2.

B is 2-reducible for D_L $\overset{\text{def}}{\Leftrightarrow}$ ∂B intersects with D_L at just two crossing points c_1 and c_2 and they look as in Figure 3.2.

B is reducible for D_L $\overset{\text{def}}{\Leftrightarrow}$ B is 1- or 2-reducible for D_L.

Definition 3.3. A class D_{red} is defined as follows:

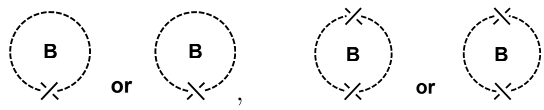


FIGURE 11. 1-reducible and 2-reducible

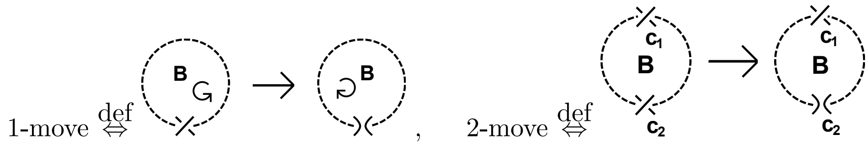


FIGURE 12. 1-move and 2-move

D_L : alternating link diagram
$D_L \in D_{red}$ if D_L satisfies the one of the following two properties:

- D_L is a disjoint union of finite number of the unknot diagrams.
- $\exists$ a sequence of embedded disks $B_1, \cdots, B_n$ and $\exists$ 1- or 2-moves s.t.
  - B_1 is reducible for D_L,
  - B_2 is reducible for $D_L(B_1)$,
  - B_3 is reducible for $D_L(B_1, B_2) = D_L(B_1)(B_2)$,

    $\vdots$
  - B_n is reducible for $D_L(B_1, \cdots, B_{n-1})$,
  - $D_L(B_1, \cdots, B_n)$ is a disjoint union of finite number of the unknot diagrams.

$\mathcal{L}_{red} = \{L; \exists D_L \in D_{red}\}$.
We say such diagrams and links B-*reducible.*

Example : The trefoil knot is B-reducible; i.e. (trefoil) $\in \mathcal{L}_{red}$

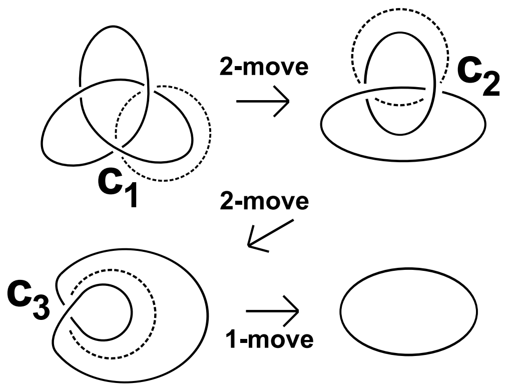


FIGURE 13

Sketch of proof:

Step1 $\quad \mathcal{L}_{\overline{\text{Brm}}} = \mathcal{L}_{red} = \{L; \exists D_L \in D_{red}\}$.
Step2 $\quad \forall L \in \mathcal{L}_{red}, \exists (T, \sigma, w)$ s.t $\Sigma(L) = Y_{(T,\sigma,w)}$.
Conversely, $\forall (T, \sigma, w)$, $\exists L \in \mathcal{L}_{red}$ s.t $Y_{(T,\sigma,w)} = \Sigma(L)$.

(We can prove this statements inductively by using the Montesinos trick.)

<u>Step3</u> $\forall(T,\sigma,w)$, $Y_{(T,\sigma,w)}$ is a strong L-space and a graph manifold. (Proposition 3.1)

$$\begin{array}{ccc}
\{\Sigma(L)\text{: the double branched covering of } S^3 \text{ branched along link } L\} & \longleftrightarrow & \{S^3(L')\text{: a surgery along a link } L' \} \\
\bigcup & & \bigcup \\
\{\Sigma(L) | L \in \mathcal{L}_{\overline{\mathrm{Brm}}}, \text{ i.e. alternating and } \mathrm{Brm} \not\leq \mathrm{L}\} & \overset{\text{step2}}{\longleftrightarrow} & \{Y_{(T,\sigma,w)} | (T,\sigma,w) \text{is an alternatingly-weighted tree}\} \\
\shortparallel \text{ step1} & & \text{step3} \Downarrow \text{ Proposition 1} \\
\{\Sigma(L) | L \in \mathcal{L}_{red}, \text{ i.e. } \exists D_L \in D_{red}\}. & & Y_{(T,\sigma,w)} \text{ are strong L-spaces.}
\end{array}$$

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References

[1] S.Boyer, C.McA.Gordon and L.Watson, *On L-spaces and left-ordarable fundamental groups*, preprint (2011), arXiv:1107.5016.

[2] T.Endo, T.Itoh and K.Taniyama, *A graph-theoretic approach to a partial order of knots and links*, Topology Appl. **157**(2010) 1002–1010

[3] J.Greene, *A spanning tree model for the Heegaard Floer homology of a branched double-cover*, preprint (2008), arXiv:0805.1381.

[4] A.S.Levine and S.Lewallen, *Strong L-spaces and left-orderability*, preprint (2011), arXiv:1110.0563.

[5] J.M.Montesinos, *Surgery on links and double branched covers of S^3*, Knots, Groups and 3-Manifolds, Ann. of Math. Studies 84, Princeton Univ. Press,. Princeton, 1975, pp. 227–259.

[6] P.S.Ozsváth and Z.Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. **159**(2004) 1159–1245

[7] P.S.Ozsváth and Z.Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. **159**(2004) 1027–1158

[8] P.S.Ozsváth and Z.Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194**(2005) 1–33

[9] K.Taniyama, *Knotted projections of planar graphs*, Proc. Amer. Math. Soc. **123**(1995) 3575–3579.