Twisted Torus Knots and Essential Surfaces

by

Kanji Morimoto

Konan University, Japan

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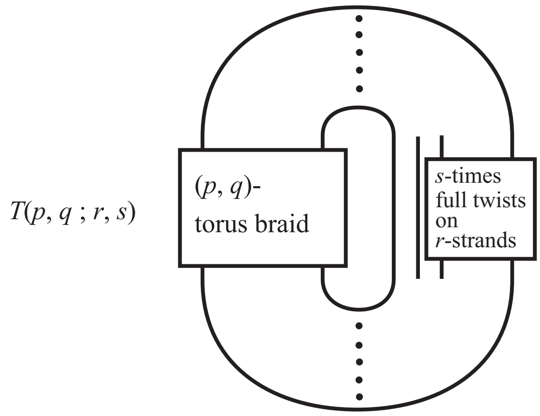
1. Twisted torus knots $T(p, q; r, s)$

Let $p,\ q$ be coprime integers with $p > q > 1$, $r,\ s$ integers with $p > r > 1$ and $s \neq 0$. Consider the torus knot $T(p, q)$, take r-strands in the parallel p-strings of $T(p, q)$, and perform s-times full twists on the r-strands.

Then we get the knot illustrated in the next figure, denoted by $T(p, q; r, s)$. It is called the twisted torus knot of type $(p, q; r, s)$ by J.C.Dean ([C-D-R]).



2. Essential surfaces

Let K be a knot in the 3-sphere S^3, $N(K)$ the regular neighborhood of K in S^3, and $E(K) = cl(S^3 - N(K))$ be the exterior. Let F be a surface (i.e. a connected 2-manifold) properly embedded in $E(K)$. Then we say that F is essential in $E(K)$ if F is closed, incompressible and not parallel to the torus $\partial E(K)$ or if F is bounded, incompressible and not parallel to an annulus $\subset E(K)$. Then we ask the following :

Problem 1 Which twisted torus knots have closed essential surfaces ?

In particular,

Problem 2 Which twisted torus knots have essential tori ?

Concerning these problems, the first result is :

Theorem 1 ([M1]) *If $r = 2$, then $T(p, q; r, s)$ has no closed essential surfaces.*

In addition, on essential tori in the exteriors, we have :

Theorem 2 ([M-Y]) *If $r = qm$ and $p = qn+1$ with $n \geq m > 1$, then $T(p, q; r, s)$ is a q-cable knot along $T(m, ms + 1)$. Hence $T(qn + 1, q; qm, s)$ has an essential torus if $(m, s) \neq (2, -1)$.*

Theorem 3 ([SL1]) *Suppose $r = qm$ with $m > 1$, then $T(p, q; r, s)$ is a q-cable knot along $T(m, ms + 1)$. Hence $T(p, q; qm, s)$ has an essential torus if $(m, s) \neq (2, -1)$.*

Example Put $n = m = 2$ and $q = 2$. Then $p = 5,\ r = 4$ and $T(5, 2; 4, s)$ has an essential torus $(s \neq 0, -1)$.

Hence we can ask :

Problem 3 Are there twisted torus knots with $r = 3$ which have closed essential surfaces ?

By Theorem 3, it is understandable to conjecture that if $T(p, q; r, s)$ has an essential torus then $r = qm$ for some m. But the situation is not so simple as we see in the next section.

3. Composite twisted torus knots

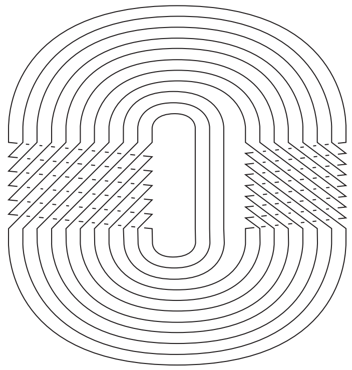
On the primeness of twisted torus knots, we have :

Theorem 4 ([M2]) *Let $e > 0, k_1 > 1, k_2 > 1$ be integers, and put $p = (e + 1)(k_1 + k_2) + 1$, $q = e(k_1 + k_2) + 1$, $r = p - k_1$ and $s = -1$. Then $T(p, q; r, s)$ is the connected sum of $T(k_1, ek_1 + 1)$ and $T(k_2, -(e + 1)k_2 - 1)$.*

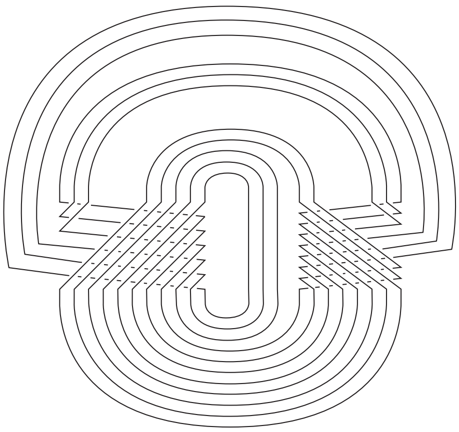
Example Put $e = 1, k_1 = 3, k_2 = 2$, then $p = 11, q = 6, r = 8$ and we have

$$T(11, 6; 8, -1) = T(3, 4)\#T(2, -5)$$

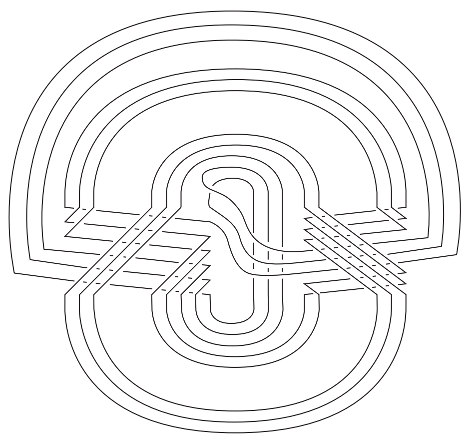
Theorem 4 shows that there are twisted torus knots with essential tori even if $r \neq qm$ because composite knots have essential tori.



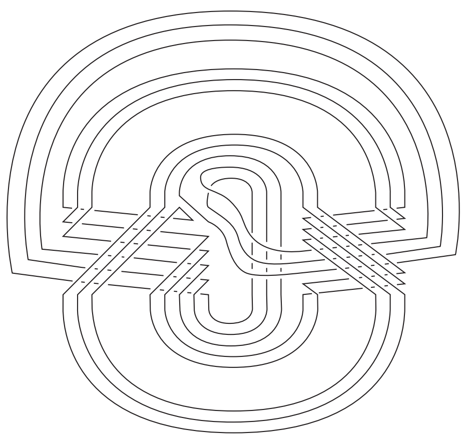
$T(11, 6; 8, -1) - (1)$



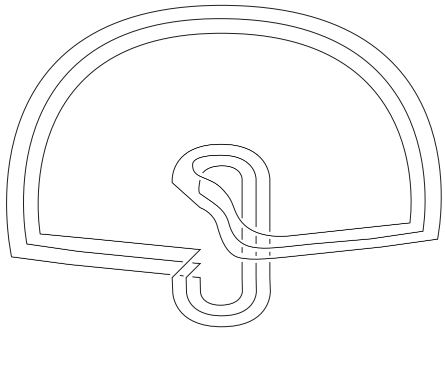
$T(11, 6; 8, -1) - (2)$



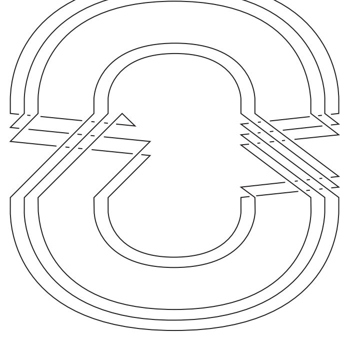
$T(11, 6; 8, -1) - (3)$



$T(11, 6; 8, -1) = T(3, 4)\#T(2, -5)$



$T(3, 4)$



$T(2, -5)$

4. Tangle decompositions of twisted torus knots

As a generalization of Theorem 4 we get the following :

Theorem 5 ([M3]) *Let* $e > 0$, $k_1 > 1$, $k_2 > 1$, $x_1 > 0$, $x_2 > 0$ *be integers with* $\gcd(x_1, x_2) = 1$. *Put*

$p = ((e+1)(k_1+k_2-1)+1)x_1 + (e+1)x_2,$

$q = (e(k_1+k_2-1)+1)x_1 + ex_2,$

$r = ((e+1)(k_1+k_2-1) - k_1 + 2)x_1 + ex_2$ *and* $s = -1$.

Then we have :

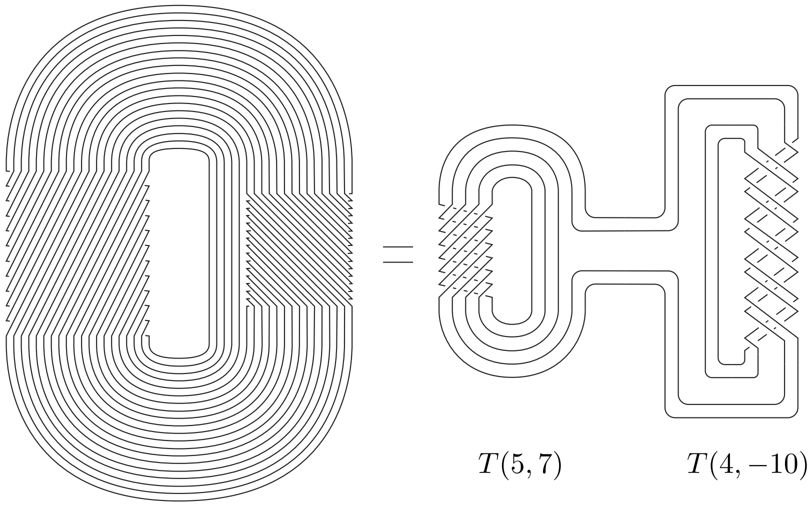
(1) $T(p, q; r, s)$ *has an* x_1*-string essential tangle decomposition.*

(2) *The decomposition is obtained by the* x_1*-string fusion of the torus knot* $T((k_1-1)x_1 + x_2, e((k_1-1)x_1 + x_2) + x_1)$ *and the torus link* $T(k_2x_1, -((e+1)k_2+1)x_1)$.

(3) $T(p, q; r, s)$ *has an essential torus in the exterior whose companion is the torus knot* $T(k_2, -(e+1)k_2 - 1)$.

By Theorem 5, for any integer $n > 0$, by putting $x_1 = n$ we get infinitely many twisted torus knots with n-string essential tangle decompositions.

Example Put $e = 1, k_1 = 2, k_2 = 2, x_1 = 2, x_2 = 3$, then $p = 20, q = 11, r = 15$ and we see that $T(20, 11; 15, -1)$ is the 2-string fusion of $T(5, 7)$ and $T(4, -10)$ as illustrated in the next figure. By tubing the decomposing 2-sphere along the torus link $T(4, -10)$, we have an essential torus whose companion is the torus knot $T(2, -5)$.

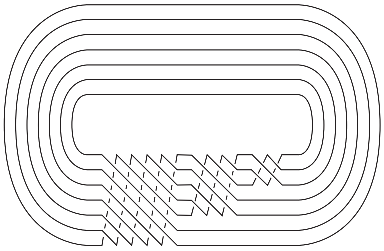


5. T-links and Lorenz links

Let $1 < r_1 < r_2 < \cdots < r_{k-1} < r_k$ be integers, and $s_i > 0$ $(i = 1, 2, \cdots, k)$ integers. Then by combining torus knots $T(r_1, s_1)$, $T(r_2, s_2)$, $\cdots$, $T(r_k, s_k)$, we get the link illustrated in the next figure. This link is called a T-link, and denoted by $T((r_1, s_1), (r_2, s_2), \cdots, (r_k, s_k))$. Then Birman-Kofman showed in [B-K] that "every T-link is a Lorenz link".

Example

$T((3, 2), (5, 3), (7, 5))$



By the definition of T-links, we see that $T(p, q; r, s)$ is the T-link $T((r, rs), (p, q))$ for $s > 0$. By combining the above result [B-K] and the result of R. Williams, we have :

Theorem 6 ([B-K], [W]) *Lorenz links are prime, and hence T-links are prime.*

This theorem says that if $T(p, q; r, s)$ is composite then $s < 0$. Moreover, Sy.Lee has shown the following :

Proposition 7 ([SL2]) *If $T(p, q; r, s)$ with $r \neq qm$ has an essential torus, then $|s| \leq 2$.*

This implies that if $T(p, q; r, s)$ is composite then $s = -1$ or -2.

6. Conjecture and more problems

By Theorem 3 and Theorem 5, we conjecture the following :

Conjecture *If a twisted torus knot $T(p, q; r, s)$ has an essential torus, then it is a knot in Theorem 3 or a knot in Theorem 5.*

i.e., (1) $r = qm$ $(m > 1)$ *or*

(2) $p = ((e + 1)(k_1 + k_2 - 1) + 1)x_1 + (e + 1)x_2$,

$q = (e(k_1 + k_2 - 1) + 1)x_1 + ex_2$,

$r = ((e + 1)(k_1 + k_2 - 1) - k_1 + 2)x_1 + ex_2$ *and* $s = -1$.

On the knot types of twisted torus knots, we can ask :

Problem 4 Characterize the knot types of twisted torus knots which are trivial knots ?

Problem 5 Characterize the knot types of twisted torus knots which are torus knots ?

8. Tunnel numbers of twisted torus knots

On the tunnel numbers of twisted torus knots we have :

Fact *If $r = 2$, then $T(p, q; r, s)$ has tunnel number one.*

Theorem 8 ([JL]) *If $r = 3$, then $T(p, q; r, s)$ has tunnel number one.*

Theorem 9 ([SL1], [M4]) *Suppose $r = qm$ with $m > 1$, then $T(p, q; r, s)$ has tunnel number one if and only if $p - qm = 1$.*

Hence we ask :

Problem 7 Characterize the knot types of tunnel number one twisted torus knots.

References

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