Link-universal subsets of $\mathbb{R}^3$

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ABSTRACT

We show that every link has a polygonal representative all of whose vertices are contained in $\{0,1\} \times \{0,1\} \times \mathbb{N}$.

1 Link-universal subsets

First, we define the concepts “link-universal subsets”and “complete bipartite graph supporting subsets” in order to state our results.

Definition 1.

A subset $A \subseteq \mathbb{R}^3$ is said to be *link-universal* if for any link L , there exists a polygonal link L' such that L' is ambient isotopic to L and $V(L') \subseteq A$. Here $V(L')$ is the set of the vertices of L'. Clearly the property link-universal is invariant under affine transformations of $\mathbb{R}^3$.

Definition 2.

A subset $A \subseteq \mathbb{R}^3$ is said to be *complete bipartite graph supporting(CBGS)* if for any $m, n \in \mathbb{N}$, there exists a linear embedding $f : K_{m,n} \to \mathbb{R}^3$ such that $f(V(K_{m,n})) \subseteq A$. Here $K_{m,n}$ is the complete bipertite graph with a bipartition $V(K_{m,n}) = X \amalg Y$ of the vertices with $|X| = m, |Y| = n$.

Proposition 1.

If A is CBGS, A is link-universal.

This proposition follows from a theorem of Miki Shimabara [4].

Main theorem.

The set $\{0,1\} \times \{0,1\} \times \mathbb{N}$ *is link-universal.*

A proof using a universal template in [5] will appear in [2].

Remark.
Let B be a set containing exactly three points in $\mathbb{R}^2$. Then, $B \times \mathbb{N}$ is not link-universal. Because $B \times \mathbb{N}$ is contained in an unknotted open annulus in $\mathbb{R}^3$.

2 Height of links

Definition 3.

Let L be a link. Then, $h(L)$ is the smallest n such that there exists a polygonal link L' that is ambient isotopic to L and $V(L') \subseteq \{0,1\} \times \{0,1\} \times \{0,\cdots,n\}$. Then, $h(L)$ is a link invariant. Links of $h(L) = 0,1$ have been completely determined,but links of $h(L) = 2,3,\cdots$ have not yet.

Links of $h(L) = 0,1,2$ are shown in the figures below.

The link of $h(L) = 0$

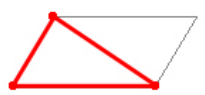

Figure 1: A trivial knot

The link of $h(L) = 1$

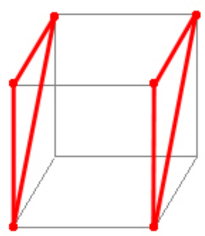

Figure 2: A trivial 2-component link

The links of $h(L) = 2$

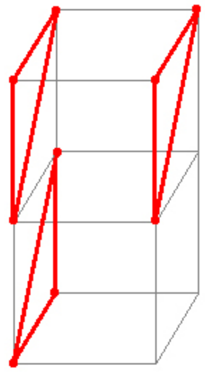

Figure 3: A trivial 3-component link

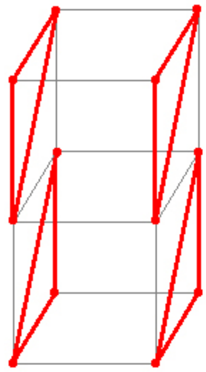

Figure 4: A trivial 4-component link

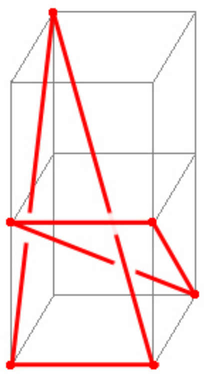

Figure 5: A Hopf link

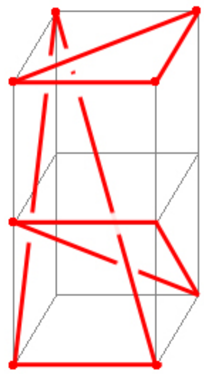

Figure 6: A split union of a Hopf link and a trivial knot

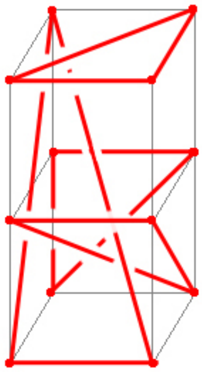

Figure 7: A split union of a Hopf link and two trivial knots

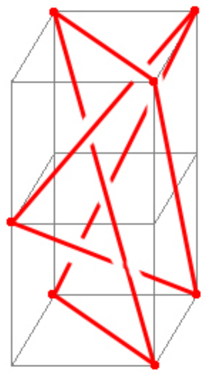

Figure 8: A trefoil

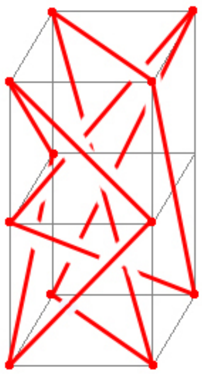

Figure 9: A split union of a trefoil and a trivial knot

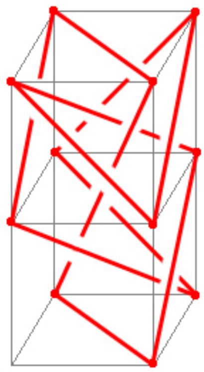

Figure 10: A figure-eight knot

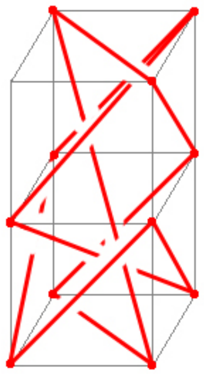

Figure 11: A $(4,2)$-torus knot

Question.

The set of links with $h(L) = 2$ has not been completely determined yet. Is that all of those listed above?

References

[1] C. Adams, B. Brennan, D. Greilsheimer and A. Woo: Stick numbers and composition of knots and links. J. Knot Theory Ramifications 6 (1997), 149-161.

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[3] G. T. Jin: Polygon indices and superbridge indices of torus knots and links. J. Knot Theory Ramifications 6 (1997), 281-289.

[4] M. S. Shimabara: Topological Ramsey theorem for complete bipartite graphs, J. Combin. Theory Ser. B 62 (1994) 164-179.

[5] R.Ghrist: Branched two-manifolds supporting all links. Topology 36 (1997), 423-448.