非自明な平面グラフの不変量について

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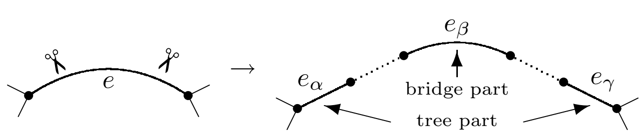
1 Introduction

1.1 The TB decomposition of graphs

A graph is a pair $G = (V, E)$, where $V = V(G)$ is a non-empty finite set of elements, and $E = E(G)$ is a finite set of unordered pairs of elements of $V(G)$. The elements of $V(G)$ are called vertices, and those of $E(G)$ are called edges. We assume that two edges adjacent to the vertex of degree 2 are amalgamated into one edge. A connected acyclic graph with degree 1 or 3 is called tree. The vertex of degree 1 in a tree is called leaf. An edge of G is called bridge. Let $G = (V(G), E(G))$ be a connected planar cubic graph. The natural inclusion of $\mathbb{R}^2$ into $\mathbb{R}^2 \cup \infty \simeq \mathbb{S}^2$ induces that the graph G is considered as being embedded in $\mathbb{S}^2$. We assume that G has no multiple edges, no loops, and no bridges. Since every vertex is incident to three edges, the numbers of $V(G)$ and $E(G)$ are $2n$ and $3n$ for an integer $n \geq 1$. Then, from the Euler's formulae, G has $n + 2$ faces. Let $\mathcal{E}$ be a 1-factor of G. Namely, $\mathcal{E}$ is a 1-regular spanning subgraph of G. $\mathcal{E}$ is identified with the subset of $E(G)$ such that $\mathcal{E}$ consists of exactly n edges which are not adjacent. We denote the pair $(G, \mathcal{E})$ by $G_{\mathcal{E}}$. Let $\mathcal{F}$ be a subset of $E(G)$ such that $\mathcal{F} \cap \mathcal{E} = \emptyset$ and $G - \mathcal{F}$ is a spanning tree of G. The number of $\mathcal{F}$ is $n + 1$. We denote the triple $(G, \mathcal{E}, \mathcal{F})$ by $G_{\mathcal{E},\mathcal{F}}$. For an edge $e \in \mathcal{F}$, we divide e into three edges $e_\alpha, e_\beta, e_\gamma$.



(1.1)

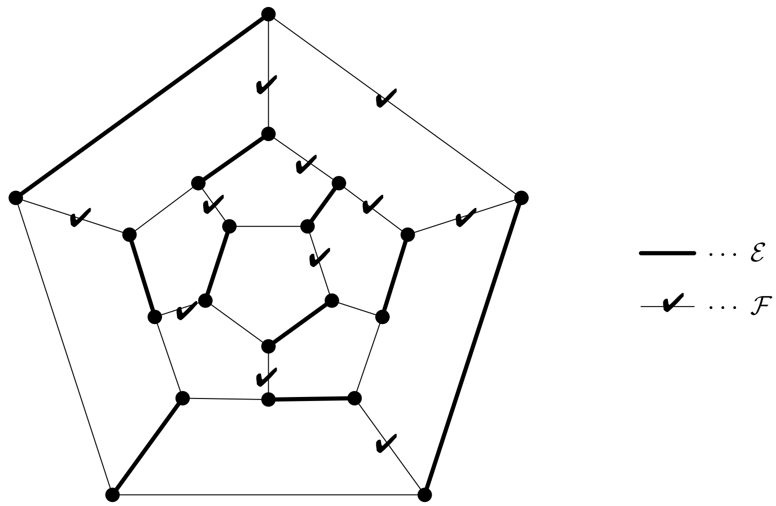
Then, e_β turns to be a bridge consisting of two vertices and one edge. By dividing all edges of $\mathcal{F}$ into three edges, we obtain one tree and $n + 1$ bridges. The tree is referred to as $T = T_{G_{\mathcal{E},\mathcal{F}}}$, and the set of these bridges as $B = B_{G_{\mathcal{E},\mathcal{F}}}$. We call this decomposition of G into (T, B) as the *TB decomposition* of G associated with $\mathcal{E}$ and $\mathcal{F}$, which is denoted by $G_{\mathcal{E},\mathcal{F}} = (T_{G_{\mathcal{E},\mathcal{F}}}, B_{G_{\mathcal{E},\mathcal{F}}})$, or simply $G = (T, B)$ if we are not confused.

Proposition 1.1. *There exists a one-to-one correspondence between the TB decomposition of G and the non-intersecting subsets $\mathcal{E}, \mathcal{F} \subset E(G)$ such that $\mathcal{E}$ is a 1-factor of G and $G - \mathcal{F}$ is a spanning tree of G.*

An edge (z, z') of a tree T is *colliding* if there exists a pair of edges (x, z) and (y, z) of T such that vertices x and y are leaves of T.

For our convenience, $B_{G_{\mathcal{E},\mathcal{F}}}$ is described as the union of the shape of $\cap$, and the ends of $B_{G_{\mathcal{E},\mathcal{F}}}$ are numbered from left to right. The leaves of $T_{G_{\mathcal{E},\mathcal{F}}}$ are numbered from left to right so that the ith end of $B_{G_{\mathcal{E},\mathcal{F}}}$ is consistent with the ith leaf of $T_{G_{\mathcal{E},\mathcal{F}}}$ $(i = 1, \ldots, 2(n+1))$. We give an example of a TB decomposition of the dodecahedron D.

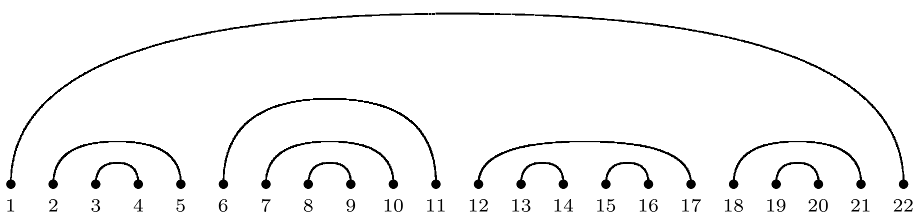
$D_{\mathcal{E},\mathcal{F}} =$



(1.2)

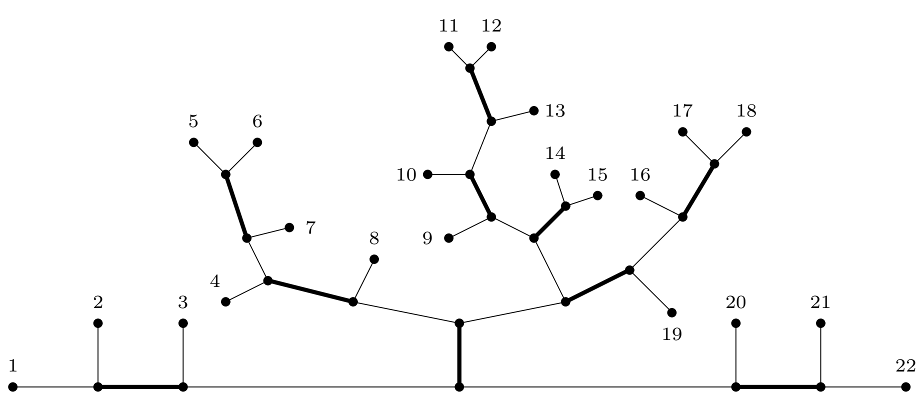
By dividing edges of $\mathcal{F}$ into a tree and bridges, we obtain another form of the TB decomposition $D_{\mathcal{E},\mathcal{F}} = (T_{D_{\mathcal{E},\mathcal{F}}}, B_{D_{\mathcal{E},\mathcal{F}}})$ of D. Connecting the ith vertex of $B_{D_{\mathcal{E},\mathcal{F}}}$ and that of $T_{D_{\mathcal{E},\mathcal{F}}}$ $(i = 1, \ldots, 22)$, we obtain the dodecahedron D. The bold edge between the first leaf and the second leaf is colliding. And so are the edges between the 5th and the 6th, the 11th and the 12th, the 14th and the 15th, the 17th and the 18th, the 21th and the 22th.

$B_{D_{\mathcal{E},\mathcal{F}}} =$



(1.3)

$T_{D_{\mathcal{E},\mathcal{F}}} =$



(1.4)

1.2 Linear skein theory

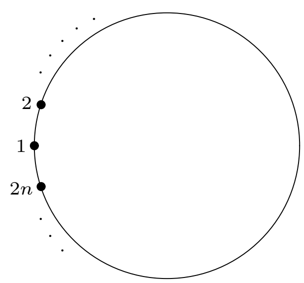
Let A be a non-zero fixed complex number. Let F be an oriented surface with $2m$ points specified in the boundary of F. A link diagram in F consists of finite closed curves and m arcs in F, and the end points of arcs are connected to the specified $2m$ points. The linear skein $\mathcal{S}(F)$ is the vector space over $\mathbb{C}$ consisting of formal linear sum of link diagrams in F quotiented by the relations

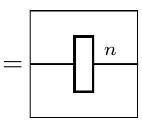
(1) $\emptyset = 1$,

(2) $D \cup (\text{a trivial closed curve}) = (-A^{-2} - A^{2})D$, (1.5)

(3) 

Note that a link diagram L in $\mathbb{S}^2$ is the element of $\mathcal{S}(\mathbb{S}^2) \simeq \mathbb{C}$, which is equal to $(-A^2 - A^{-2})^{-1}$ times of the Kauffman bracket $\langle L \rangle$. Let D^2_{2n} is a disk with $2n$ points specified in the boundary of the disk D^2. The $2n$ points are numbered clockwise. We denote the linear skein of D^2_{2n} by $\mathcal{D}_{2n} = \mathcal{S}(D^2_{2n})$. Let TL_n be the *nth Temperley-Lieb algebra*, and let $f^{(n)} \in TL_n$ be the nth Jones-Wenzl idempotent. We represent $f^{(n)}$ by a blank rectangle.

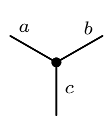
$D^2_{2n} =$ 

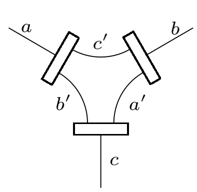
$f^{(n)} =$ 

(1.6)

1.3 Weights and Weighted graphs

A triple (a, b, c) of non-negative integers is *admissible* if $a + b + c$ is even, $a \leq b + c$, $b \leq c + a$, and $c \leq a + b$. For an integer $r \geq 3$, the admissible triple (a, b, c) is called *r-admissible* if $a + b + c \leq 2(r - 2)$. Let $e_x, e_y, e_z \in E(G)$ are three distinct edges adjacent to one vertex $v \in V(G)$. A mapping w from $E(G)$ to non-negative integers is a *weight(or a r-weight)* of G if the triple $(w(e_x), w(e_y), w(e_z))$ is admissible(or r-admissible) for any $v \in V(G)$. A *weighted graph* G_w is a graph G with a weight w of G. We call $w(e)$ a weight of a edge e. For a weighted graph G_w, we associate an edge e with $w(e)$ copies of arcs and insert the $w(e)$th Jones-Wenzl projection $f^{(w(e))}$ in the arcs. The arcs in the neighborhood of a vertex are arranged as the following figure:

$G_w =$ 

$\rightarrow$ 

$= \langle G_w \rangle$ (1.7)

where non-negative integers (a', b', c') are defined by

$$a' = \frac{-a + b + c}{2}, \quad b' = \frac{a - b + c}{2}, \quad c' = \frac{a + b - c}{2}. \tag{1.8}$$

Since the graph G is embedded in $\mathbb{S}^2$, the obtained diagram is an element of $\mathcal{S}(\mathbb{S}^2) \simeq \mathbb{C}$, denoted by $\langle G_w \rangle$. Finally, we define a graph invariant G associated with the weight w by $\langle G_w \rangle$.

$$G \xrightarrow{w} \langle G_w \rangle \in \mathbb{C} \tag{1.9}$$

We define two weights of G, which play important roles in this article. Let w_2 be the weight of G such that $w_2(e) = 2$ for any $e \in E(G)$. For a 1-factor $\mathcal{E}$ of G, Then there exists the weight $w_{\mathcal{E}}$ of $G_{\mathcal{E}}$ such that

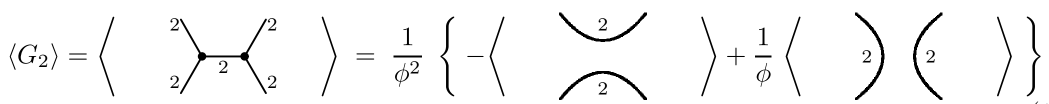
$$w_{\mathcal{E}}(e) = \begin{cases} 2 & \text{if } e \in \mathcal{E}, \\ 1 & \text{if } e \notin \mathcal{E}. \end{cases} \tag{1.10}$$

We denote the weighted graph G associated with w_2 and $w_{\mathcal{E}}$ by G_2 and $G_{\mathcal{E}}$, respectively.

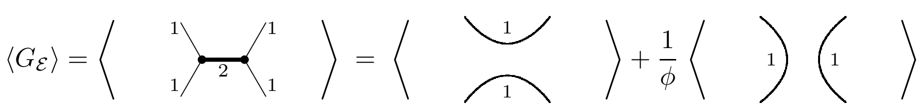
1.4 State sum invariants for weighted graphs

For a non-negative integer n, we define $\Delta_n = (-1)^n(A^{2(n+1)} - A^{-2(n+1)})/(A^2 - A^{-2})$. In what follows, we put $A = \exp(\frac{2\pi\sqrt{-1}}{20})$. Then we have $\Delta_1 = -\phi, \Delta_2 = \phi, \Delta_3 = -1$ and $\Delta_4 = 0$, where ϕ is knows as the golden ratio $\phi = 1.6180339887\dots$.

With respect to the specialization of $A = \exp(\frac{2\pi\sqrt{-1}}{20})$, $\langle G_2 \rangle$ and $\langle G_{\mathcal{E}} \rangle$ have the following expressions:



(1.11)



(1.12)

Proposition 1.2. *We have* $\langle G_2 \rangle = (-1)^{|G-\mathcal{E}|+|\mathcal{E}|}\phi^{-2|\mathcal{E}|}\langle G_{\mathcal{E}} \rangle$, *where* $|G-\mathcal{E}|$ *is the number of trivial closed curves of* $G - \mathcal{E}$.

Corollary 1.3. $\langle G_2 \rangle$ *is not zero if and only if* $\langle G_{\mathcal{E}} \rangle$ *is not zero for any* 1*-factor* $\mathcal{E}$.

1.5 A bilinear form on $\mathcal{D}_{2(n+1)}$

We define a bilinear form $\langle\ ,\ \rangle : \mathcal{D}_{2(n+1)} \times \mathcal{D}_{2(n+1)} \to \mathbb{C}$ as follows. For an element $x \in \mathcal{D}_{2(n+1)}$, let $\overline{x}$ be the mirror image of x with respect to the boundary of D^2. The mirror image $\overline{x}$ is considered as the element of $\overline{\mathcal{D}}_{2(n+1)} = \mathcal{S}(\overline{D^2_{2(n+1)}})$, where $\overline{D^2_{2(n+1)}}$ is the disk complementary to $D^2_{2(n+1)}$ in $\mathbb{S}^2$, with the same specified $2(n+1)$ points on the boundary of $\overline{D^2}$. Let x, y be elements of $\mathcal{D}_{2(n+1)}$. Then $\langle x, y \rangle$ is defined by connecting the ith vertex of x to that of $\overline{y}$ ($i = 1, \dots, 2(n+1)$), which belongs to $\mathcal{S}(\mathbb{S}^2) \simeq \mathbb{C}$.

By using the bilinear form $\langle , \rangle$, we describe $\langle G_{\mathcal{E}} \rangle$ from the view point of the TB decomposition of G. We define a weight of $T = T_{G_{\mathcal{E}},\mathcal{F}}$ induced by $w_{\mathcal{E}}$. Consider T as the element of $\mathcal{D}_{2(n+1)}$ by fixing the ith leaf of T to the specified ith points on the boundary of $\mathcal{D}_{2(n+1)}$, and replacing the edges and the vertices of T by arcs and the diagram (1.7). We denote it by $T_{\mathcal{E}} \in \mathcal{D}_{2(n+1)}$. In a similar way, consider $B = B_{G_{\mathcal{E},\mathcal{F}}}$ as the element of $\overline{\mathcal{D}}_{2(n+1)}$, denoted by $B_1 \in \overline{\mathcal{D}}_{2(n+1)}$. The mirror image with respect to the boundary of D^2 induces the mirror image of the element of $\overline{\mathcal{D}}_{2(n+1)}$. We denote the induced mirror image $\overline{\mathcal{D}}_{2(n+1)} \to \mathcal{D}_{2(n+1)}$ by the same symbol$\bar{\ }$. Then, $\langle G_{\mathcal{E}} \rangle$ is described by the following lemma.

Lemma 1.4. *Let $G_{\mathcal{E},\mathcal{F}} = (T, B)$ is a TB decomposition of G associated with $\mathcal{E}, \mathcal{F}$. Then, we obtain*

$$\langle G_{\mathcal{E}} \rangle = \langle T_{\mathcal{E}}, \overline{B}_1 \rangle. \tag{1.13}$$

By means of the bilinear form $\langle\ ,\ \rangle$, the element of $\mathcal{D}_{2(n+1)}$ corresponds to the element of the dual of $\mathcal{D}_{2(n+1)}$, denoted by $\mathcal{D}^*_{2(n+1)}$.

As the previous argument, a pair of T and its weight w induces an element of $\mathcal{D}^*_{2(n+1)}$, which is denoted by T_w. From the linear skein theory at a root of unity, we obtain the following propositions.

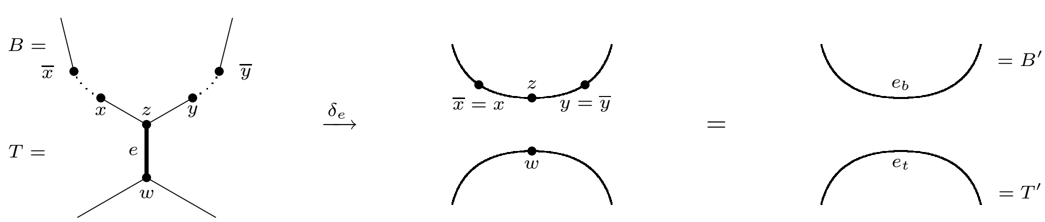
Proposition 1.5. *Let w, w' be 5-weights of T. Then $\langle T_w, T_{w'} \rangle = 0$ if and only if $w \neq w'$.*

Proposition 1.6. *$\{T_w \mid w$ is a 5-weight$\}$ is an orthogonal basis of $\mathcal{D}^*_{2(n+1)}$.*

2 Main theorem

Theorem 2.1. *Let G be a connected planar cubic graph. We assume that G has no multiple edges, no loops, and no bridges. Then, we obtain that $\langle G_2 \rangle$ is not zero.*

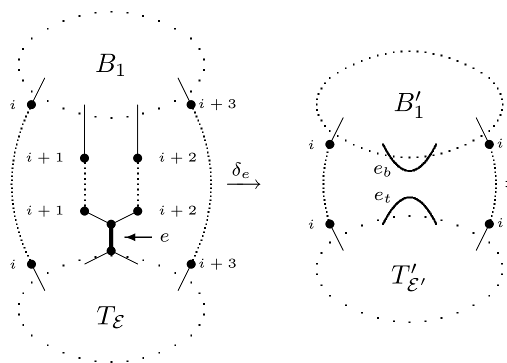
Before we prove the theorem, we introduce a notion of a deletion operation to use induction. Let $G = (T, B)$ be a TB decomposition of G. Let $e = (z, w)$ be a colliding edge of T, and let (x, z) and (y, z) be the edges adjacent to e at the vertex z such that x and y are the leaves of T. Let $\overline{x}$ and $\overline{y}$ be the vertices of B corresponding to the mirror image of x and y, respectively. Define a tree T' by $T' = (V(T \setminus \{x, y, z\}), E(T \setminus \{e, (x, z), (y, z)\})$. Consider the graph consisting of three vertices x, z, y and two edges $(x, z), (z, y)$, and define a set of bridges B' from its graph and B by connecting x to $\overline{x}$, and y to $\overline{y}$. We denote the edge of T' containing w by e_t, and the edge of B' containing z by e_b.



(2.1)

This operation induces a graph $G' = G - e$ and a TB decomposition (T', B') of G' associated with $\mathcal{E}'$ and $\mathcal{F}'$, where $\mathcal{E}'$ is defined by $\mathcal{E} \setminus \{e\}$, and $\mathcal{F}'$ is defined by removing two edges from $\mathcal{F}$. The two edges are those of containing $x \cdots \overline{x}$, and containing $y \cdots \overline{y}$. We call it a *deletion operation* δ_e of e. Note that B is obtained by cutting e_b of B' into two edges.

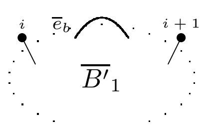
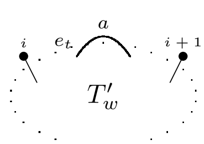
Outline of the proof of the theorem. It is sufficient to prove that $\langle G_{\mathcal{E}} \rangle \neq 0$. Let (T, B) be a TB decomposition of G, and let e be a colliding edge of T. We may assume that G has no triangle and there exists two arcs of B such that their ends are $(i, i+1)$ and $(i+2, i+3)$. We apply the deletion operation δ_e to $G = (T, B)$, and obtain an induced TB decomposition $(T', B') = (T'_{G'_{\mathcal{E}',\mathcal{F}'}}, B'_{G'_{\mathcal{E}',\mathcal{F}'}})$ of G'. For these TB decompositions of G and G', $\langle G_{\mathcal{E}} \rangle$ and $\langle G'_{\mathcal{E}'} \rangle$ are given by $\langle G_{\mathcal{E}} \rangle = \langle T_{\mathcal{E}}, \overline{B}_1 \rangle$ and $\langle G'_{\mathcal{E}'} \rangle = \langle T'_{\mathcal{E}}, \overline{B'}_1 \rangle$. They are described by

$\langle G_{\mathcal{E}} \rangle = \langle T_{\mathcal{E}}, \overline{B}_1 \rangle =$  $= \langle T'_{\mathcal{E}'}, \overline{B'}_1 \rangle = \langle G'_{\mathcal{E}'} \rangle.$ (2.2)

We prove the theorem by the induction on the number of edges. By the assumption of the induction, we have $\langle G'_{\mathcal{E}'} \rangle = \langle T'_{\mathcal{E}'}, \overline{B'}_1 \rangle \neq 0$. Since $\overline{B'}_1$ belongs to $\mathcal{D}^*_{2n}$, $\overline{B'}_1$ is expressed by the linear combination of T'_w where w denotes the 5-admissible weight:

$$\overline{B'}_1 = \sum_w C_w T'_w. \tag{2.3}$$

Its graphical presentation is given by

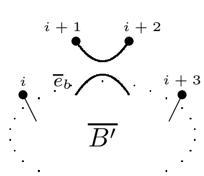
 $= \sum_w C_w$  , (2.4)

where $a = w(e_t)$ is the weight of e_t. By Proposition 1.5, $\langle G'_{\mathcal{E}'} \rangle$ is calculated as

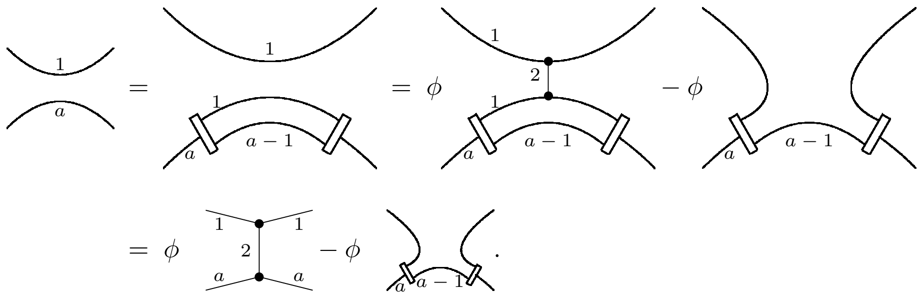
$$\langle G'_{\mathcal{E}'} \rangle = \langle T'_{\mathcal{E}'}, B'_1 \rangle = \langle T'_{\mathcal{E}'}, \sum_w C_w T'_w \rangle = C_{\mathcal{E}'} \langle T'_{\mathcal{E}'}, T'_{\mathcal{E}'} \rangle \neq 0. \tag{2.5}$$

Hence we have $C_{\mathcal{E}'} \neq 0$.

Adding one arc to $\overline{B'}_1$ right above $\overline{e}_b$, we consider an element $\widetilde{B'}_1 \in \mathcal{D}^*_{2(n+1)}$.

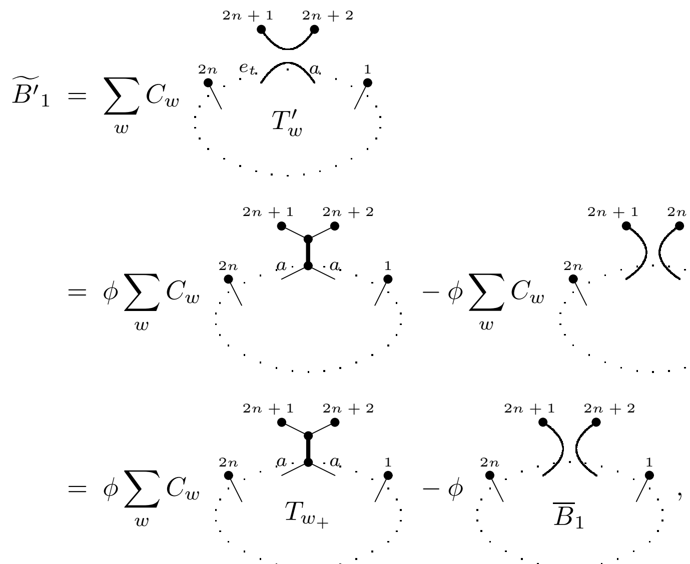
$\widetilde{B'}_1 =$  (2.6)

Recall the following transformation from (1.12):



(2.7)

Then, $\widetilde{B'}_1$ is transformed by



(2.8)

(2.9)

(2.10)

where the weight w of T' is naturally extended to that of T. The third equality is due to (2.4) since cutting e_b of $\overline{B'}$ is equivalent that we cut the uppermost arc in e_t for $a \geq 1$. The pairing of the added arc of $\widetilde{B'}_1$ and e of $T_{\mathcal{E}}$ causes the bilinear form $\langle T_{\mathcal{E}}, \widetilde{B'}_1\rangle$ to be 0. On the other hand,

$$\langle T_{\mathcal{E}}, \widetilde{B'}_1\rangle = \langle T_{\mathcal{E}}, \phi \sum_{w_+} C_{w_+} T_{w_+} - \phi\, \overline{B}_1\rangle = \phi\, C_{\mathcal{E}'}\langle T_{\mathcal{E}}, T_{\mathcal{E}}\rangle - \phi\, \langle T_{\mathcal{E}}, \overline{B}_1\rangle. \tag{2.11}$$

Therefore we obtain

$$0 = \phi\, C_{\mathcal{E}'}\langle T_{\mathcal{E}}, T_{\mathcal{E}}\rangle - \phi\, \langle G_{\mathcal{E}}\rangle. \tag{2.12}$$

This leads $\langle G_{\mathcal{E}}\rangle \neq 0$. We complete the proof of the theorem. □

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