$\bar{\mu}$ INVARIANT OF NANOPHRASES

YUKA KOTORII
TOKYO INSTITUTE OF TECHNOLOGY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

1. Introduction

A word will be a sequence of symbols, called letters, belonging to a given set $\mathcal{A}$, called alphabet. Turaev developed the theory of words based on the analogy with curves on the plane, knots in the 3-sphere, virtual knot, etc. in [6, 7, 8, 9].

A Gauss word is a sequence of letters with the condition that any letter appearing in the sequence does so exactly twice. A Gauss word can be obtained from an oriented virtual knot diagram introduced by Kauffman in [3]. Given a diagram, label the real crossings and pick a base point on the curve somewhere away from any of the real crossings. Starting from the base point, we follow the curve and read off the labels of the crossings as we pass through them. When we return to the base point, we will have a sequence of letters in which each label of a real crossing appears exactly twice. Thus this sequence is a Gauss word. It is natural to introduce combinatorial moves on Gauss words to generate equivalence relation, which we call homotopy, based on Reidemeister moves on a knot diagram.

We then assign to each crossing some data that it is under or over and so on. This leads us to the notion of nanoword by Turaev in [9]. By introducing refined combinatorial moves on nanowords, the notion of homotopy can be refined also. From this viewpoint, homotopy of Gauss words is the simplest kind of nanoword homotopy. In fact, homotopic nanowords in any refinement are homotopic as Gauss words.

The theory of nanowords can be naturally generalized to the theory of nanophrases as the knot theory does to the link theory. The purpose of this paper is to develop weaker homotopy theory on nanophrases, called M-homotopy, which is an analogue of Milnor's link homotopy [4, 5]. By a link homotopy we mean a deformotion of one link onto another, during which each component of the link is allowed to cross itself, but no two components are allowed to intersect. Milnor introduced an invariant under link homotopy called $\bar{\mu}$ in [4, 5]. We introduce a self crossing move on nanophrases and the associated M-homotopy allowing self crossings. The main result stated in Theorem 3.1 is to define an M-homotopy invariant of nanophrases corresponding to virtual links as an extension of Milnor's $\bar{\mu}$ invariant.

2. Nanowords and nanophrases

In this section, following Turaev [6, 7, 8, 9], we review formal definitions of words, phrases and so on.

2.1. **Words and phrases.** An *alphabet* is a finite set and its element is called a *letter*. For any positive integer m, let $\hat{m}$ denote the set $\{1, 2, \ldots, m\}$. A *word* on an alphabet $\mathcal{A}$ of length m is a map

$$w : \hat{m} \to \mathcal{A}.$$

Informally, we can think of a word w on $\mathcal{A}$ as a finite sequence of letters in $\mathcal{A}$ and we will usually write words in this way. For example, ABA is a word of length 3 on

$\{A, B, C\}$, where 1 is mapped to A, 2 to B and 3 to A. By convention, the empty word of length 0 on any alphabet is written by $\emptyset$.

An n-component *phrase* on an alphabet $\mathcal{A}$ is a sequence of n words on $\mathcal{A}$. We write phrases as a sequence of words, separated by '$|$'. For example, $A|BCA|CD$ is a 3-component phrase on $\{A, B, C, D\}$. There is a unique phrase with 0-components which we denote by $\emptyset_P$. In this paper, we will regard words as 1-component phrases.

2.2. **Nanowords and nanophrases.** Let α be a finite set. An α-alphabet is an alphabet $\mathcal{A}$ together with an associated map from $\mathcal{A}$ to α. This map is called a *projection*. The image of any $A \in \mathcal{A}$ in α will be denoted by $|A|$. An *isomorphism* of α-alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$ is a bijection f from $\mathcal{A}_1$ to $\mathcal{A}_2$ such that $|f(A)|$ is equal to $|A|$ for any letter A in $\mathcal{A}_1$.

A *Gauss word* on an alphabet $\mathcal{A}$ is a word on $\mathcal{A}$ such that every letter in $\mathcal{A}$ appears exactly twice. Similarly, a Gauss phrase on $\mathcal{A}$ is a phrase on $\mathcal{A}$ such that the concatenation of the words appearing in the phrase is a Gauss word on $\mathcal{A}$. By definition, a 1-component Gauss phrase is a Gauss word.

A *nanowrod* over α is a pair $(\mathcal{A}, w)$ where $\mathcal{A}$ is an α-alphabet and w is a Gauss word on $\mathcal{A}$. An n-component *nanophrase* over α is a pair $(\mathcal{A}, p)$ where $\mathcal{A}$ is an α-alphabet and p is an n-component Gauss phrase on $\mathcal{A}$.

Two nanophrases over α, $(\mathcal{A}_1, p_1)$ and $(\mathcal{A}_2, p_2)$, are *isomorphic* if there exists a bijection f from $\mathcal{A}_1$ to $\mathcal{A}_2$ such that f applied letterwise to the ith component of p_1 gives the ith component of p_2 for all i.

Rather than writing $(\mathcal{A}, p)$, we will simply use p to indicate a nanophrase. When we write a nanophrase in this way, we do not forget the set $\mathcal{A}$ of letters and the projection : $\mathcal{A} \to \alpha$.

2.3. **Equivalence relations on nanophrases.** Fix α and then let τ be an involution on α (that is, $\tau(\tau(a))$ is equal to a for all $a \in \alpha$). Let S be a subset of $\alpha \times \alpha \times \alpha$. We call the triple (α, τ, S) a *homotopy* data.

Fixing a homotopy data (α, τ, S), we define three *homotopy moves* on nanophrases over α as follows. In the moves on nanophrases, the lower cases x, y, z and t represent arbitrary sequences of letters, possibly including one or more '$|$', so that the phrase on each side of the move is a nanophrase. The moves are

move H1: for any $|A|$,
$(\mathcal{A}, xAAy) \longleftrightarrow (\mathcal{A} - \{A\}, xy)$

move H2: if $\tau(|A|) = |B|$,
$(\mathcal{A}, xAByBAz) \longleftrightarrow (\mathcal{A} - \{A, B\}, xyz)$

move H3: if $(|A|, |B|, |C|) \in S$,
$(\mathcal{A}, xAByACzBCt) \longleftrightarrow (\mathcal{A}, xBAyCAzCBt)$.

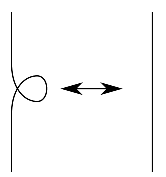

Figure 1. H1

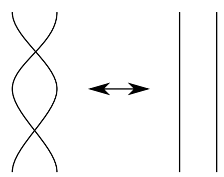

Figure 2. H2

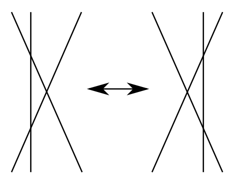

Figure 3. H3

The *homotopy* is an equivalence relation on nanophrases over α generated by isomorphisms and three homotopy moves. The homotopy depends on the choice of the homotopy data (α, τ, S), so different choices of homotopy data give different equivalence relations. As none of the moves add or remove components, the number of components of a nanophrase is an invariant under any kind of homotopy.

In [7], Turaev defined a *shift move* on nanophrases. Let ν be an involution on α which is independent of τ. Let p be an n-component nanophrase over α. A *shift* move on the ith component of p is a move which gives a new nanophrase p' as follows. If the ith component of p is empty or contains a single letter, p' is p. If not, the ith component of p has the form Ax. Then the ith component of p' is xA and for all j not equal to i, the jth component of p' is the same as the jth component of p. Furthermore, if we write $|A|_p$ for $|A|$ in p and $|A|_{p'}$ for $|A|$ in p', then $|A|_{p'}$ equals $\nu(|A|_p)$ when x contains the letter A and otherwise, $|A|_{p'}$ equals $|A|_p$.

We call the equivalence relation generated by isomorphisms, homotopy moves and shift moves also a *homotopy* of nanophrases over α. We call the first homotopy without shift moves an *open homotopy* and the second homotopy simply a *homotopy*. The homotopy depends on the triple (α, τ, S) and ν.

Let α_v be the set $\{a_+, a_-, b_+, b_-\}$ and τ_v the involution on α_v which sends a_+ to b_- and a_- to b_+. Let S_v be the set

$$S_v = \left\{ \begin{array}{l} (a_+, a_+, a_+), (a_+, a_+, a_-), (a_+, a_-, a_-), \\ (a_-, a_-, a_-), (a_-, a_-, a_+), (a_-, a_+, a_+), \\ (b_+, b_+, b_+), (b_+, b_+, b_-), (b_+, b_-, b_-), \\ (b_-, b_-, b_-), (b_-, b_-, b_+), (b_-, b_+, b_+) \end{array} \right\}.$$

Let ν_v be the involution on α_v where a_+ is mapped to b_+ and a_- to b_-.

In [7], Turaev proved

Theorem 2.1 (Turaev [7]). *Under the homotopy defined by (α_v, τ_v, S_v) and ν_v, the set of homotopy classes of nanophrases over α_v is in bijective correspondence with ordered virtual links (namely, virtual links where the components are ordered and the equivalence of ordered virtual links respects the order).*

3. $\bar{\mu}$ INVARIANTS OF NANOPHRASES

In this section, we introduce a self crossing move on nanophrases and a new weak homotopy, which we call the M-homotopy, by allowing self crossing moves. Consider the set of equivalence classes of nanophrases corresponding to ordered virtual links, that is equivalence classes of nanophrases defined by (α_v, τ_v, S_v) and ν_v. We define a M-homotopy invariant of nanophrases corresponding to ordered virtual links. This invariant is an extension of $\bar{\mu}$ invariants introduced by Milnor in [4, 5].

3.1. Self crossing and M-homotopy.

Let (α, τ, S) be any homotopy data and ν an involution on α independent of τ. We introduce the *self crossing move* on nanophrases over α. Let σ be an involution on α which is independent of τ and ν. Let p be an n-component nanophrase over α. A *self crossing move* on the kth component of p is a move which gives a new nanophrase p' as follows. If there is a letter A in $\mathcal{A}$ which appears exactly twice in the kth component of p, then the kth component of p has the form $xAyAz$. Then the kth component of p' also has the form $xAyAz$. Furthermore, writing $|A|_p$ for $|A|$ in p and $|A|_{p'}$ for $|A|$ in p', we have the identity $|A|_{p'} = \sigma(|A|_p)$.

We define the *open M-homotopy* to be the equivalence relation on nanophrases over α generated by isomorphisms, three homotopy moves with respect to (α, τ, S) and self crossing moves with respect to σ. We also define the *M-homotopy* to be the equivalence relation of nanophrases over α generated by isomorphisms, three homotopy moves with respect to (α, τ, S), self crossing moves with respect to σ and shift moves with respect to ν.

3.2. **Definition of $\bar{\mu}$.** From now on, we work only on the homotopy defined by (α_v, τ_v, S_v), ν_v and σ_v which sends a_+ to a_- and b_+ to b_-.

Let $(\mathcal{A}, p)$ be an n-component nanophrase whose Gauss phrase p is represented by $w_1|w_2|\cdots|w_n$. We will write p for $(\mathcal{A}, p)$ for simplicity. Let w_i be $A_{i1}A_{i2}\cdots A_{im_i}$, where A_{ij}'s are letters in $\mathcal{A}$. To each w_i, we define a word on $\mathcal{A}\cup\mathcal{A}^{-1}$ by

$$w_i^\varepsilon = A_{i1}^{\varepsilon_{i1}} A_{i2}^{\varepsilon_{i2}} \cdots A_{im_i}^{\varepsilon_{im_i}},$$

where ε_{ij} is determined as follows. Since p is a nanophrase, any letter appears exactly twice in a Gauss phrase $w_1|w_2|\cdots|w_n$. Let A denote the letter represented by A_{ij}. Then there exist integers k and l ($(k,l) \neq (i,j)$) such that A_{kl} represents A. In other words, the other A appears as the lth letter on the kth component. If $i < k$ and $|A| = b_+$ or $i > k$ and $|A| = a_+$, then $\varepsilon_{ij} = 1$. If $i < k$ and $|A| = a_-$ or $i > k$ and $|A| = b_-$, then $\varepsilon_{ij} = -1$. Otherwise, $\varepsilon_{ij} = 0$. Namely, if the letter A appears exactly once in the ith component, and if A_{ij} appears earlier (or latter) than the other A and $|A| = b_+$(or a_+), then ε_{ij} is 1. If A appears exactly once in the ith component, and if A_{ij} appears earlier (or latter) than the other A and $|A| = a_-$(or b_-), then ε_{ij} is -1. For other cases, let ε_{ij} be zero. In the following, we use the convention that $A^0 = \emptyset$ and $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\cdots A_1^{-1}$. We note that $AA^{-1} \neq \emptyset$.

Let $\mathcal{L}$ denote the set of words on $A \cup A^{-1}$. Then we define a sequence of maps ρ^q ($q = 2, 3, \cdots$) from $\mathcal{L}$ to itself by induction on q.

$$\begin{aligned}
\rho^2(A_{ij}^{\pm}) &= A_{ij}^{\pm}\\
\rho^q(A_{ij}^{\pm}) &= \rho^{q-1}(x_{ij}^{-1})A_{ij}^{\pm}\rho^{q-1}(x_{ij}),\ q \geq 3\\
\rho^q(\emptyset) &= \emptyset \text{ for all } q \geq 2,
\end{aligned}$$

where

$$x_{ij} = A_{k1}^{\varepsilon_{k1}} A_{k2}^{\varepsilon_{k2}} \cdots A_{kl}^{\varepsilon_{kl}}.$$

Here, k and l are derived from i and j as in the above. We naturally extend ρ^q to $\mathcal{L}$. We will concern exclusively with $\rho^q(w_i^\varepsilon)$.

Recall that $p = w_1|w_2|\cdots|w_n$ is an n-component nanophrase. We call the index i of a component w_i the order of w_i. Let M denote a finite set $\{a_1, \ldots, a_n\}$. Let $\mathcal{M}$ denote the set of words on $M \cup M^{-1}$. Using a nanophrase property of p, we define a map η from $\mathcal{L}$ to $\mathcal{M}$ as follows. For any letter A in $\mathcal{A}$, let $\eta(A)$ be a_k and $\eta(A^{-1})$ a_k^{-1}, where k is determined by the following rule. If $|A| = b+$ or $a-$, then k is the order of the component in p in which the second A occurs, and if $|A| = a+$ or $b-$, then k is the order of the component in p in which the first A occurs. To see what k is for a letter in w_i, let us recall the definition of w_i^ε. The letter A in w_i survives in w_i^ε only when either $|A| = b_+$ or a_- and the other A appears in latter component, or $|A| = a_+$ or b_- and the other appears in former component. Thus, if A is represented by A_{ij} and $\varepsilon_{ij} \neq 0$, then k is the number appeared in the definition of ε_{ij}, namely, the order of the component in which the other A occurs.

We define a map φ from $\mathcal{M}$ to $\mathbb{Z}[[\kappa_1, \kappa_2, \ldots, \kappa_n]]$ by

$$\begin{aligned}
\varphi(a_h) &= 1 + \kappa_h,\\
\varphi(a_h^{-1}) &= 1 - \kappa_h + \kappa_h^2 - \kappa_h^3 + \cdots,
\end{aligned}$$

where $\mathbb{Z}[[\kappa_1, \kappa_2, \ldots, \kappa_n]]$ is the ring of formal power series on non-commuting variables $\kappa_1, \kappa_2, \ldots, \kappa_n$.

We consider $\varphi \circ \eta(\rho^q(w_i^\varepsilon))$ in $\mathbb{Z}[[\kappa_1, \kappa_2, \ldots, \kappa_n]]$. Since $\varphi \circ \eta(\rho^q(w_i^\varepsilon))$ agrees with $\varphi \circ \eta(\rho^r(w_i^\varepsilon))$ for any $r \geq q$ up to degree q, the coefficient of a term $\kappa_{c_1}\kappa_{c_2}\ldots\kappa_{c_u}$ in $\varphi \circ \eta(\rho^q(w_i^\varepsilon))$ converges as $q \to \infty$. Thus we have a well-defined expansion,

$$\lim_{q\to\infty} \varphi \circ \eta(\rho^q(w_i^\varepsilon)) = 1 + \sum \mu(p; c_1, c_2, \ldots, c_u, i)\kappa_{c_1}\kappa_{c_2}\cdots\kappa_{c_u}.$$

where $c_1, c_2, \dots, c_u, i$ is a sequence of integers between 1 and n. We here note that the integers in the sequence are not necessarily mutually different.

Let $\Delta(p; c_1, c_2, \dots, c_u, i)$ denote the greatest common divisor of $\mu(p; d_1, d_2, \dots, d_t)$, where the sequence $d_1, d_2, \dots, d_t$ $(2 \leq t \leq u)$ ranges over all sequences obtained by eliminating at least one of $c_1, c_2, \dots, c_u, i$, and permuting the remaining indices cyclically. We also define $\Delta(p; c_1, i) = 0$. Let $\bar{\mu}(p; c_1, c_2, \dots, c_u, i)$ denote the residue class of $\mu(p; c_1, c_2, \dots, c_u, i)$ modulo $\Delta(p; c_1, c_2, \dots, c_u, i)$.

The main theorem of this paper is as follows.

Theorem 3.1. *Let p be an n-component nanophrase. Let $c_1, c_2, \dots, c_u, i$ be a sequence of integers between 1 and n such that $c_1, c_2, \dots, c_u, i$ are pairwise distinct. Then $\bar{\mu}(p; c_1, c_2, \dots, c_u, i)$ is an invariant under M-homotopy of nanophrases with respect to (α_v, τ_v, S_v), ν_v and σ_v.*

Example 3.2. Let $p = ABCD|ECFA|DFBE$ where $|A| = |E| = b+$, $|B| = b-$, $|C| = |F| = a-$ and $|D| = a+$. This corresponds to the Borromean rings illustrated in Fig. 4. Then since $w_1^\varepsilon = AC^{-1}$,

$$\rho^2(w_1^\varepsilon) = AC^{-1}$$
$$\rho^3(w_1^\varepsilon) = FE^{-1}AEF^{-1}E^{-1}C^{-1}E.$$

Thus we have

$$\eta(\rho^2(w_1^\varepsilon)) = a_2 a_2^{-1}$$
$$\eta(\rho^3(w_1^\varepsilon)) = a_3 a_3^{-1} a_2 a_3 a_3^{-1} a_3^{-1} a_2^{-1} a_3.$$

Therefore $\mu(2,1) = \mu(3,1) = 0$ and $\mu(2,3,1) = -1$. Similarly $\mu(1,2) = \mu(1,3) = \mu(2,3) = \mu(3,2) = 0$ and so $\Delta(2,3,1) = 0$. Hence $\bar{\mu}(2,3,1) \equiv -1 \pmod 0$.

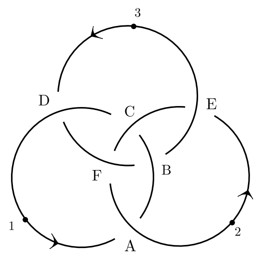

FIGURE 4

References

[1] T. Fukunaga, *Homotopy classification of nanophrases with less than or equal to four letters*, arXiv:0904.3478. (2009).

[2] A. Gibson, *Factorization of homotopies of nanophrases*, arXiv:0910.5281v2 (2009).

[3] L. H. Kauffman, *Virtual knot theory*, European J. Combin. **20** (1999), no. 7, 663–690. MR 1721925 (2000i:57011)

[4] J. Milnor, *Link groups*, Ann. of Math. (2) **59** (1954), 177–195. MR 0071020 (17,70e)

[5] ———, *Isotopy of links. Algebraic geometry and topology*, (1957), 280–306.

[6] V. Turaev, *Curves on surfaces, charts, and words*, Geom. Dedicata **116** (2005), 203–236. MR 2195447 (2007b:57031)

[7] ———, *Knots and words*, Int. Math. Res. Not. (2006), Art. ID 84098, 23. MR 2276346 (2007k:57017)

[8] ———, *Lectures on topology of words*, Jpn. J. Math. **2** (2007), no. 1, 1–39. MR 2295606 (2008i:57024)

[9] ———, *Topology of words*, Proc. Lond. Math. Soc. (3) **95** (2007), no. 2, 360–412. MR 2352565 (2008i:57025)