Minimal coloring numbers of minimal diagrams for $\mathbb{Z}$-colorable links

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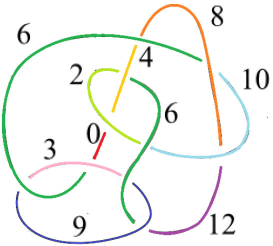
$\mathbb{Z}$-coloring

Let L be a link, and D a diagram of L.

$\mathbb{Z}$-coloring

A map $\gamma : \{\text{arcs of } D\} \to \mathbb{Z}$ is called a $\mathbb{Z}$-coloring on D if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of D with the over arc a and the under arcs b and c.
A $\mathbb{Z}$-coloring which assigns the same color to all the arcs of the diagram is called the trivial $\mathbb{Z}$-coloring.
L is $\mathbb{Z}$-colorable if $\exists$ a diagram of L with a non-trivial $\mathbb{Z}$-coloring.



Let L be a $\mathbb{Z}$-colorable link.

Minimal coloring number

$mincol_{\mathbb{Z}}(L)$
$:= \min\{\#\mathrm{Im}(\gamma) \mid \gamma :$ non-tri. $\mathbb{Z}$-coloring on a diagram of $L\}$

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Theorem [Ichihara-M. (2017 JKTR)]

[1] If L is non-splittable, then $mincol_{\mathbb{Z}}(L) \geq 4$.

[2] Let L be a non-splittable $\mathbb{Z}$-colorable link. If there exists a "simple" $\mathbb{Z}$-coloring on a diagram of L, then $mincol_{\mathbb{Z}}(L) = 4$.

[3] If a non-splittable link L admits a $\mathbb{Z}$-coloring C such that $\#\mathrm{Im}(C) = 5$, then $mincol_{\mathbb{Z}}(L) = 4$.

Theorem [M. (2017 preprint)]

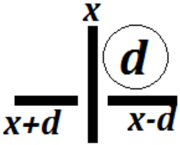
The minimal coloring number of any non-splittable $\mathbb{Z}$-colorable link is equal to 4.

This result is also proved by Meiqiao Zhang, Xian'an Jin and Qingying Deng almost independently.
[M. Zhang, X. Jin, and Q. Deng, The Minimal Coloring Number Of Any Non-splittable Z-colorable Link Is Four, to appear in J. Knot Theory Ramifications, online ready.]

Outline of proof of Theorem

L : a $\mathbb{Z}$-colorable link

d-diff crossing : the over arc is colored by x and the under arcs are colored by $x-d$ and $x+d$



Note.

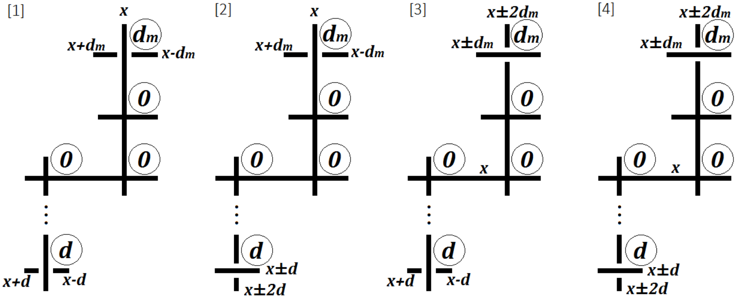
$\mathbb{Z}$-coloring γ with only d-diff crossings and 0-diff crossings

$\Rightarrow \gamma$ is a simple $\mathbb{Z}$-coloring

D : a diagram of L with non-simple $\mathbb{Z}$-coloring

d_m : the maximum of the set $\{0, d_1, d_2, \cdots, d_m\}$ s.t. D has d_i-diff crossings for $i = 1, 2, 3, \cdots$

We can find a path on D from a d_m-diff crossing to a d-diff crossing passing only 0-diff crossings with $0 < d < d_m$.
Such a path is one of the 4 types [1], [2], [3] and [4] illustrated in the following figures.



In the following, for a path from a d_m-diff crossing, we will modify the diagram and the coloring to eliminate the d_m-diff crossing.

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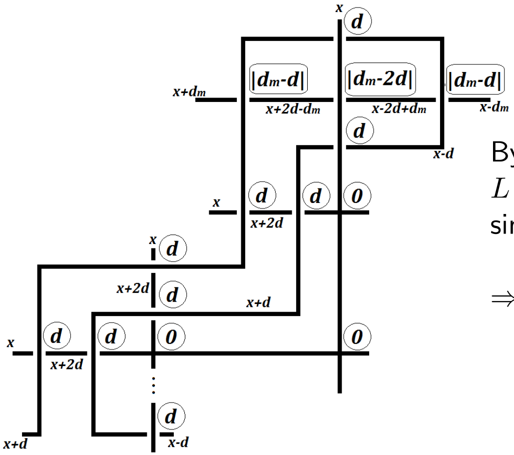
Type [1]



By the induction for d_m, L has a diagram with a simple $\mathbb{Z}$-coloring

$\Rightarrow mincol_{\mathbb{Z}}(L) = 4$

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The obtained diagrams with 4 colors in the proofs
$\rightarrow$ often very complicated

Minimal coloring number of a diagram

D :a diagram of $\mathbb{Z}$-colorable link
$mincol_{\mathbb{Z}}(D) := \min\{\#\mathrm{Im}(\gamma) \mid \gamma$: non-tri. $\mathbb{Z}$-coloring on $D\}$

$\rightarrow$We consider the minimal coloring numbers of minimal diagrams for $\mathbb{Z}$-colorable links.

Main theorem

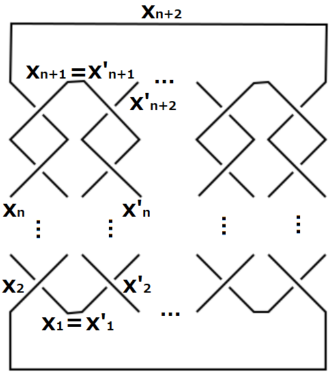
[1] For an even integer $n \geq 2$, the pretzel link $P(n, -n, \cdots, n, -n)$ with at least 4 strands has a minimal diagram admitting only $\mathbb{Z}$-colorings with $n+2$ colors.

[2] For an integer $n \geq 2$, the pretzel link $P(-n, n+1, n(n+1))$ has a minimal diagram admitting only $\mathbb{Z}$-colorings with n^2+n+3 colors.

[3] For even integer $n > 2$ and non-zero integer p, the torus link $T(pn, n)$ has a minimal diagram admitting a $\mathbb{Z}$-coloring with only four colors.

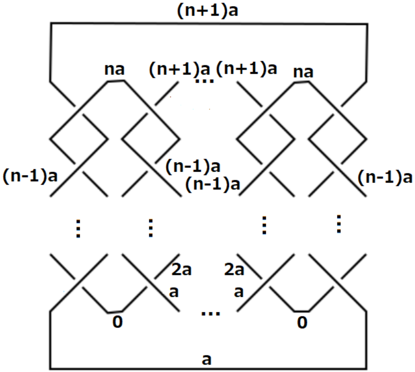
Outline of Proof of Main theorem [1]

$P := P(n, -n, \cdots, n, -n)$
D : the diagram of P as follows



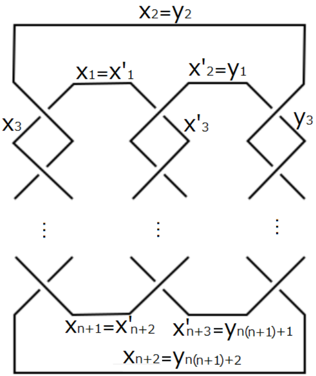
This diagram D is a minimal diagram of the link.

γ : a non-trivial $\mathbb{Z}$-coloring γ on D
We assume $\min\{\text{colors}\} = 0$.



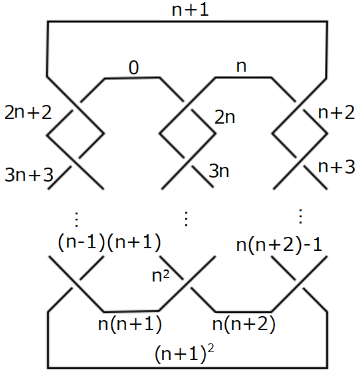
As shown in the figure, the colors of γ are $0, a, 2a, \cdots, (n+1)a$.
That is, the minimal coloring number of D is equal to $n+2$.

Outline of Proof of Main theorem [2]
D : the diagram as follows.



D is a minimal diagram of $P(-n, n+1, n(n+1))$.

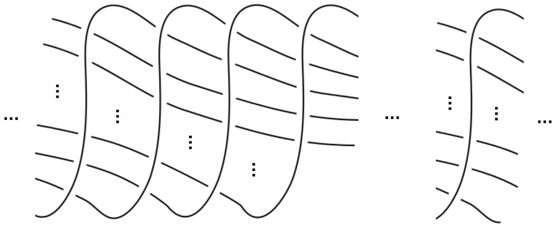
γ : a non-trivial $\mathbb{Z}$-coloring γ on D
We assume $\min\{\text{colors}\} = 0$.



Therefore, there are only mutually distinct colors
$\{0, n, n+1, n+2, n+3, \cdots, n+n(n+1), n+n(n+1)+1 = (n+1)^2\}$,
and so, the minimal coloring number of D is
$2 + n(n+1) + 1 = n^2 + n + 3$.

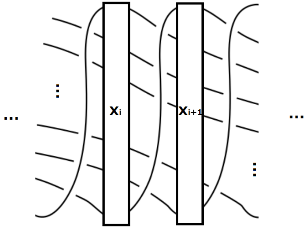
Outline of Proof of Main theorem [3]

D : the standard diagram of $T(pn, n)$

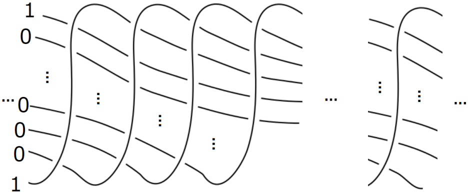


D is a minimal diagram of $T(pn, n)$.

There are pn such sets of parallel arcs in D.



We fix the colors of X_1 as ${}^t(1, 0, \cdots, 0, 1)$.



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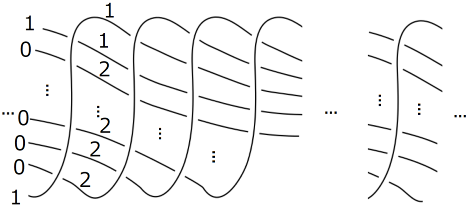

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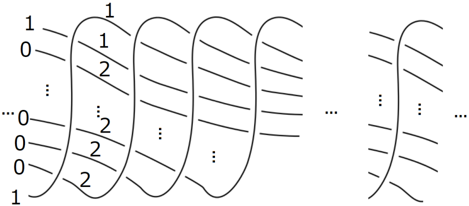

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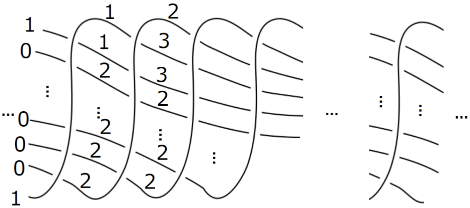

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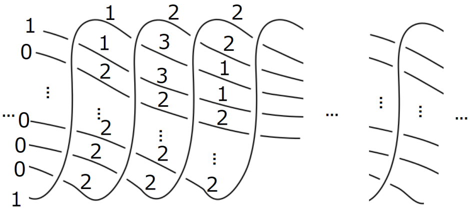

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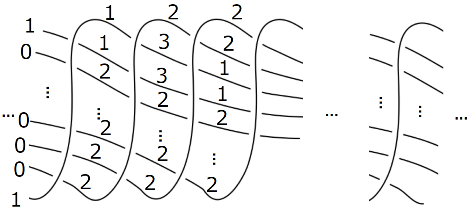

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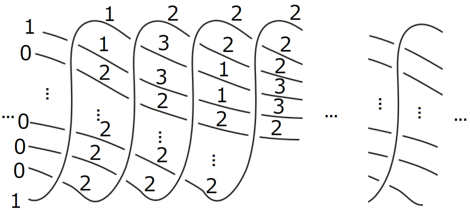

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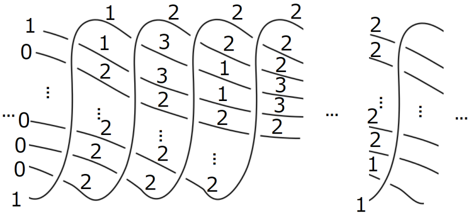

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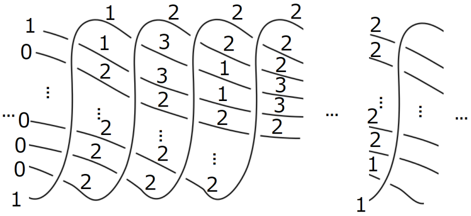

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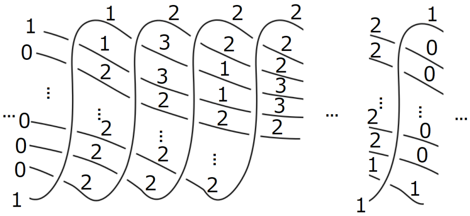

It concludes that the colors of this $\mathbb{Z}$-coloring is $\{0, 1, 2, 3\}$. $\square$

Thank you
for your attention.