Kauffman bracket polynomials of Conway-Coxeter Friezes

(joint work with Michihisa Wakui)

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Contents

Definition of Kauffman bracket polynomials of links

Let Λ be the Laurent polynomial ring $\mathbb{Z}[A, A^{-1}]$. For each link diagram D, Kauffman bracket polynomial $\langle D \rangle \in \Lambda$ is computed by applying the following rules repeatedly.

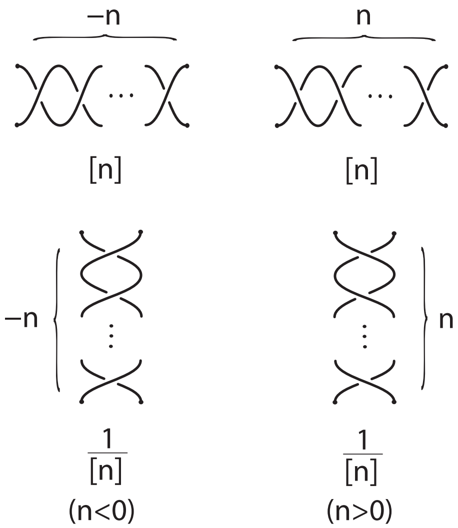
(KB1) $\langle$  $\rangle = A\langle\ )(\ \rangle + A^{-1}\langle$  $\rangle$

(KB2) $\langle\ D \coprod \bigcirc\ \rangle = \delta\langle D \rangle$, where $\delta = -A^2 - A^{-2}$.

(KB3) $\langle\ \bigcirc\ \rangle = 1$.

Rational tangles and continued fractions

For an integer n, we define by $[n]$, $\frac{1}{[n]}$ as follows:

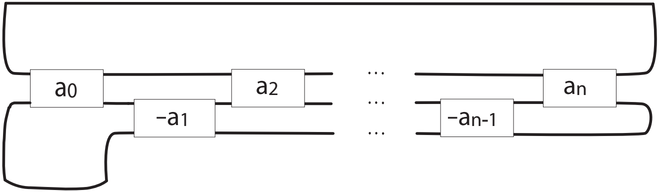


Rational tangles and continued fractions

We consider for the continued fraction expansion of an irreducible fraction $\frac{p}{q}$, i.e.

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

$L\left(\frac{p}{q}\right) :=$



Definition of Conway-Coxeter friezes

A Conway-Coxeter Frieze is an array of natural numbers, displayed on shifted lines such that the top and bottom lines are composed only of 1s and for each unit diamond:

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the determinant condition $ad - bc = 1$, namely

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad a, b, c, d > 0$$

.

Conway-Coxeter Frieze of type L^2R^2L

1 1 1 1 1 1 1 1 1 1 1 1

1

1

1

1

1

1

1 1 1 1 1 1 1 1 1 1 1 1

Conway-Coxeter Frieze of type L^2R^2L

1		1		1		1		1		1		1		1		1		1		1		1	
							?		1														
								1															
							1		?														
								1															
							?		1														
								1		?													
	1		1		1		1		1		1		1		1		1		1		1		1

Conway-Coxeter Frieze of type L^2R^2L

1		1		1		1		1		1		1		1		1		1		1		1	
							2		1														
						?		1		?													
							1		2														
						?		1		?													
							2		1														
						?		1		2													
	1		1		1		1		1		1		1		1		1		1		1		1

Conway-Coxeter Frieze of type L^2R^2L

1 1 1 1 1 1 1 1 1 1 1 1

? 2 1 ?

3 1 3

? 1 2 ?

3 1 3

? 2 1 ?

3 1 2

1 1 1 1 1 1 1 1 1 1 1 1

Conway-Coxeter Frieze of type L^2R^2L

1 1 1 1 1 1 1 1 1 1 1 1

2 2 1 4

? 3 1 3 ?

10 1 2 5

? 3 1 3 ?

5 2 1 7

? 3 1 2 ?

1 1 1 1 1 1 1 1 1 1 1

Conway-Coxeter Frieze of type L^2R^2L

⋯	1		1		1		1		1		1		1		1		1		1		1		1		1	⋯	
⋯		2		4		2		2		1		4		2		3		1		2		4		2		2	⋯
⋯	1		7		7		3		1		3		7		5		2		1		7		7		3		⋯
⋯		3		12		10		1		2		5		17		3		1		3		12		10		1	⋯
⋯	2		5		17		3		1		3		12		10		1		2		5		17		3		⋯
⋯		3		7		5		2		1		7		7		3		1		3		7		5		2	⋯
⋯	1		4		2		3		1		2		4		2		2		1		4		2		3		⋯
⋯		1		1		1		1		1		1		1		1		1		1		1		1		1	⋯

Remark

1973 $\Rightarrow$
J.H.Conway and H.S.M.Coxeter, , Triangulated polygons and frize pattern, Math. Gaz.57(1973), no.400, 87-94, no.401, 87–94.
2002 $\Rightarrow$
S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497-529 (electronic).

cluster of RLR^2LR-type

1		1		1		1	1	1	1	1	1	1
	x_1		?									
		x_2		?								
	x_3		?									
		x_4		?								
			x_5		?							
		x_6		?								
			x_7		?							
1		1		1		1	1	1	1	1	1	1

cluster of RLR^2LR-type

RLR^2LR-type:

1		1		1		1	1	1	1	1	1	1
	x_1		x_8									
		x_2		x_9								
	x_3		x_{10}									
		x_4		x_{11}								
			x_5		x_{12}							
		x_6		x_{13}								
			x_7		x_{14}							
1		1		1		1	1	1	1	1	1	1

$x_8 = \frac{x_2+1}{x_1}$, $x_9 = \frac{{x_2}^2x_4+x_1x_3+x_2x_4+x_2+1}{x_1x_3x_2}$, $x_{10} = \frac{x_2x_4+1}{x_3}$, $x_{11} = \frac{x_2x_4x_5+x_3+x_5}{x_3x_4}$,
$x_{12} = \frac{x_2x_4{x_5}^2x_7+x_2x_4x_5+x_3x_4x_6+x_3x_5x_7+{x_5}^2x_7+x_3+x_5}{x_3x_4x_6x_5}$, $x_{13} = \frac{x_5x_7+1}{x_6}$, $x_{14} = \frac{x_5x_7+x_6+1}{x_6x_7}$

cluster of RLR^2LR-type

$$x_{15} := \frac{x_1 x_3 + x_2 x_4 + 1}{x_2 x_3}$$

$$x_{17} := \frac{x_2^2 x_4 x_5 + (x_3 + (x_4+1) x_5) x_2 + (x_3 + x_5)(x_1 x_3 + 1)}{x_4 x_2 x_1 x_3}$$

$$x_{20} := \frac{x_7 (x_2 x_4 + 1) x_5^2 + (x_7 x_3 + (x_6+1)(x_2 x_4 + 1)) x_5 + x_3 (x_6 + 1)(x_4 x_6 + 1)}{x_3 x_4 x_5 x_6 x_7}$$

$$x_{16} := \frac{x_1 x_3^2 + x_1 x_3 x_5 + x_2 x_4 x_5 + x_3 + x_5}{x_2 x_3 x_4}$$

$$x_{18} := \frac{x_7 (x_1 x_3 + (x_2+1)(x_2 x_4 + 1)) x_5^2 + (x_1 x_3^2 x_7 + (x_2 x_7 + x_1 + x_7) x_3 + (x_2+1)(x_2 x_4+1)) x_5 + x_3 (x_4 x_6 + 1)(x_1 x_3 + x_2 + 1)}{x_1 x_2 x_3 x_4 x_5 x_6}$$

$$x_{19} := \frac{x_1 (x_5 x_7 + (x_6+1)(x_4 x_6 + 1)) x_3^2 + (x_1 x_5^2 x_7 + (x_1 x_6 + x_2 x_7 + x_1 + x_7) x_5 + (x_6+1)(x_4 x_6+1)(x_2+1)) x_3 + x_5 (x_5 x_7 + x_6 + 1)(x_2+1)(x_2 x_4 + 1)}{x_7 x_6 x_5 x_4 x_3 x_2 x_1}$$

$$x_{21} := \frac{(x_2 x_5 + x_3 x_6) x_4 + x_3 + x_5}{x_3 x_4 x_5}$$

cluster of RLR^2LR-type

$$x_{22} := \frac{x_3 + x_5}{x_4}$$

$$x_{24} := \frac{x_1 \left(x_4 x_6 + x_5 x_7 + 1\right) x_3^2 + \left(x_1 x_5^2 x_7 + \left(x_1 + x_7\right) x_5 + x_4 x_6 + 1\right) x_3 + x_5 \left(x_5 x_7 + 1\right) \left(x_2 x_4 + 1\right)}{x_2 x_3 x_4 x_5 x_6}$$

$$x_{27} := \frac{x_1 \left(x_4 x_6 + 1\right) x_3^2 + \left(\left(x_4 x_6 + 1\right) x_2 + x_5 x_1 + x_4 x_6 + 1\right) x_3 + x_5 \left(x_2 + 1\right) \left(x_2 x_4 + 1\right)}{x_1 x_2 x_3 x_4 x_5}$$

$$x_{23} := \frac{x_3 x_4 x_6 + x_3 x_5 x_7 + x_5^2 x_7 + x_3 + x_5}{x_6 x_5 x_4}$$

$$x_{25} := \frac{x_1 \left(x_5 x_7 + \left(x_6 + 1\right) \left(x_4 x_6 + 1\right)\right) x_3^2 + \left(x_1 x_5^2 x_7 + \left(x_1 x_6 + x_1 + x_7\right) x_5 + \left(x_6 + 1\right) \left(x_4 x_6 + 1\right)\right) x_3 + x_5 \left(x_5 x_7 + x_6 + 1\right) \left(x_2 x_4 + 1\right)}{x_2 x_3 x_4 x_5 x_6 x_7}$$

$$x_{26} := \frac{x_1 x_3^2 x_4 x_6 + x_1 x_3^2 + x_1 x_3 x_5 + x_2 x_4 x_5 + x_3 x_4 x_6 + x_3 + x_5}{x_2 x_3 x_4 x_5}$$

$$x_{28} := \frac{x_1 x_3 + x_2 + 1}{x_2 x_1}$$

cluster of RLR^2LR-type

$$x_{29} := \frac{x_4 x_6 + x_5 x_7 + 1}{x_5 x_6}$$

$$x_{31} := \frac{\left(x_5 x_7 + (x_6 + 1)(x_4 x_6 + 1)\right) x_3 + x_5 (x_5 x_7 + x_6 + 1)}{x_4 x_5 x_6 x_7}$$

$$x_{34} := \frac{x_1 x_3 + 1}{x_2}$$

$$x_{30} := \frac{x_4 x_6^2 + x_4 x_6 + x_5 x_7 + x_6 + 1}{x_7 x_5 x_6}$$

$$x_{32} := \frac{x_3 x_4 x_6 + x_3 + x_5}{x_5 x_4}$$

$$x_{33} := x_3$$

$$x_{35} := x_1$$

cluster of RLR^2LR-type

$$x_{36} := \frac{x_6 + 1}{x_7}$$

$$x_{38} := \frac{x_4 x_6 + 1}{x_5}$$

$$x_{41} := x_2$$

$$x_{37} := x_6$$

$$x_{39} := x_4$$

$$x_{40} := \frac{x_2 x_4 + 1}{x_3}$$

$$x_{42} := \frac{x_2 + 1}{x_1}$$

cluster of RLR^2LR-type

$$x_{43} := x_7$$

$$x_{45} := x_5$$

$$x_{48} := \frac{x_2^2 x_4 + x_1 x_3 + x_2 x_4 + x_2 + 1}{x_1 x_3 x_2}$$

$$x_{44} := \frac{x_5 x_7 + 1}{x_6}$$

$$x_{46} := \frac{x_2 x_4 x_5 + x_3 + x_5}{x_3 x_4}$$

$$x_{47} := \frac{x_2^2 x_4 x_5 + (x_3 + (x_4 + 1) x_5) x_2 + (x_3 + x_5)(x_1 x_3 + 1)}{x_4 x_2 x_1 x_3}$$

$$x_{49} := \frac{x_1 x_3 + x_2 x_4 + 1}{x_2 x_3}$$

cluster of RLR^2LR-type

$$x_{50} := \frac{x_5 x_7 + x_6 + 1}{x_6 x_7}$$

$$x_{52} := \frac{x_2 x_4 x_5^2 x_7 + x_2 x_4 x_5 + x_3 x_4 x_6 + x_3 x_5 x_7 + x_5^2 x_7 + x_3 + x_5}{x_3 x_4 x_6 x_5}$$

$$x_{55} := \frac{x_1 x_3^2 + x_1 x_3 x_5 + x_2 x_4 x_5 + x_3 + x_5}{x_2 x_3 x_4}$$

$$x_{51} := \frac{x_7 (x_2 x_4 + 1) x_5^2 + (x_7 x_3 + (x_6 + 1)(x_2 x_4 + 1)) x_5 + x_3 (x_6 + 1)(x_4 x_6 + 1)}{x_3 x_4 x_5 x_6 x_7}$$

$$x_{53} := \frac{x_7 (x_1 x_3 + (x_2 + 1)(x_2 x_4 + 1)) x_5^2 + (x_1 x_3^2 x_7 + (x_2 x_7 + x_1 + x_7) x_3 + (x_2 + 1)(x_2 x_4 + 1)) x_5 + x_3 (x_4 x_6 + 1)(x_1 x_3 + x_2 + 1)}{x_1 x_2 x_3 x_4 x_5 x_6}$$

$$x_{54} := \frac{x_1 (x_4 x_6 + x_5 x_7 + 1) x_3^2 + (x_1 x_5^2 x_7 + (x_1 + x_7) x_5 + x_4 x_6 + 1) x_3 + x_5 (x_5 x_7 + 1)(x_2 x_4 + 1)}{x_2 x_3 x_4 x_5 x_6}$$

$$x_{56} := \frac{x_3 + x_5}{x_4}$$

cluster of RLR^2LR-type

$$x_{57} := \frac{(x_2x_5+x_3x_6)\,x_4+x_3+x_5}{x_3x_4x_5}$$

$$x_{59} := \frac{x_1\left(x_5x_7+(x_6+1)(x_4x_6+1)\right)x_3^2+\left(x_1x_5^2x_7+(x_1x_6+x_2x_7+x_1+x_7)\,x_5+(x_6+1)(x_4x_6+1)(x_2+1)\right)x_3+x_5(x_5x_7+x_6+1)(x_2+1)(x_2x_4+1)}{x_7x_6x_5x_4x_3x_2x_1}$$

$$x_{62} := \frac{x_3x_4x_6+x_3x_5x_7+x_5^2x_7+x_3+x_5}{x_6x_5x_4}$$

$$x_{58} := \frac{x_1(x_4x_6+1)\,x_3^2+\left((x_4x_6+1)\,x_2+x_5x_1+x_4x_6+1\right)x_3+x_5(x_2+1)(x_2x_4+1)}{x_1x_2x_3x_4x_5}$$

$$x_{60} := \frac{x_1\left(x_5x_7+(x_6+1)(x_4x_6+1)\right)x_3^2+\left(x_1x_5^2x_7+(x_1x_6+x_1+x_7)\,x_5+(x_6+1)(x_4x_6+1)\right)x_3+x_5(x_5x_7+x_6+1)(x_2x_4+1)}{x_2x_3x_4x_5x_6x_7}$$

$$x_{61} := \frac{\left(x_5x_7+(x_6+1)(x_4x_6+1)\right)x_3+x_5(x_5x_7+x_6+1)}{x_4x_5x_6x_7}$$

$$x_{63} := \frac{x_4x_6+x_5x_7+1}{x_5x_6}$$

cluster of RLR^2LR-type

$$x_{64} := \frac{x_1 x_3 + x_2 + 1}{x_2 x_1}$$

$$x_{66} := \frac{x_1 x_3^2 x_4 x_6 + x_1 x_3^2 + x_1 x_3 x_5 + x_2 x_4 x_5 + x_3 x_4 x_6 + x_3 + x_5}{x_2 x_3 x_4 x_5}$$

$$x_{69} := \frac{x_4 x_6^2 + x_4 x_6 + x_5 x_7 + x_6 + 1}{x_7 x_5 x_6}$$

$$x_{65} := \frac{x_1 x_3 + 1}{x_2}$$

$$x_{67} := \frac{x_3 x_4 x_6 + x_3 + x_5}{x_5 x_4}$$

$$x_{68} := \frac{x_4 x_6 + 1}{x_5}$$

$$x_{70} := \frac{x_6 + 1}{x_7}$$

cluster of RLR^2LR-type

$x_{71} := x_1$

$x_{73} := x_3$

$x_{76} := x_6$

$x_{72} := x_2$

$x_{74} := x_4$

$x_{75} := x_5$

$x_{77} := x_7$

Main theorem (Recipe of making Kauffman bracket by using CCF)

For fractions $\frac{s}{M}, \frac{u}{M}, \frac{v}{M}, \frac{t}{M}$, associated Kauffman bracket polynomials $\langle L(\frac{s}{M})\rangle, \langle L(\frac{u}{M})\rangle, \langle L(\frac{v}{M})\rangle, \langle L(\frac{t}{M})\rangle$,
are determined by using "sin-curve" and "cos-curve through M in $\mathrm{CCF}(w)$ as follows :

Main theorem (Recipe of making Kauffman bracket by using CCF)

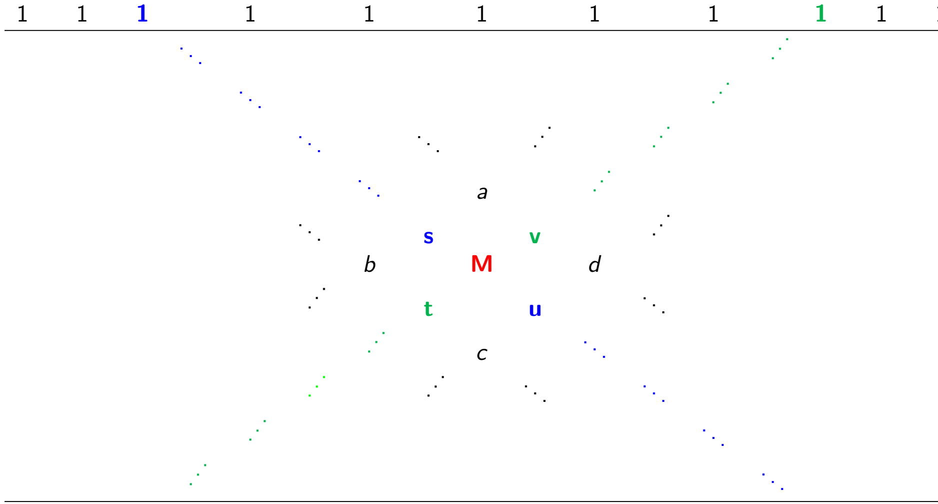
$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$$

$$\begin{array}{ccccccc}
 & & \ddots & & \iddots & & \\
 & \ddots & & a & & \iddots & \\
\ddots & & s & & v & & \iddots \\
 & b & & M & & d & \\
\iddots & & t & & u & & \ddots \\
 & \iddots & & c & & \ddots & \\
 & & \iddots & & \ddots & &
\end{array}$$

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$$

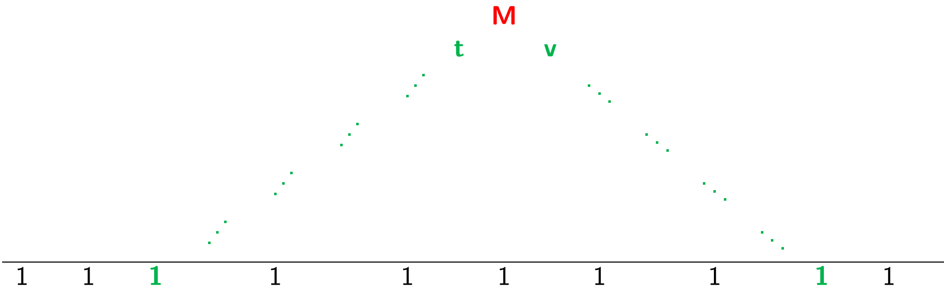
Main theorem (Recipe of making Kauffman bracket by using CCF)

(S) (C)



Main theorem (Recipe of making Kauffman bracket by using CCF)

(S) Picking up the "sinusoidal" part of the green, bend it at the maximum value "M":



Main theorem (Recipe of making Kauffman bracket by using CCF)

(S1)Put signature minus $-$ on segment from "M" to the left "1" (southwest-direction) and signature plus $+$ on segment from "M" to the right "1" (southeast-direction).
(S2) Connect three numbers in the above with signature- lines with the following rule. Draw a line segment so that the number on the vertex above the triangle is the two numbers on the bottom base and determine signature Extend the line segment so that positive and negative line segments are output one by one from the top vertex in accordance with the signs of the right end and the left end.
(S3) On each line segment, replace plus with weight $-A^4$ and minus with weight $-A^{-4}$.
(S4)Compute the product of weights on each path from "M" to the left "1" or to the right "1".
(S5)Sum the product of weights on each path from "M" to the left "1" or to the right "1".
Thus, a Laurent polynomial is obtained, which will be written as $\langle \Gamma(w) \rangle_S$

Main theorem (Recipe of making Kauffman bracket by using CCF)

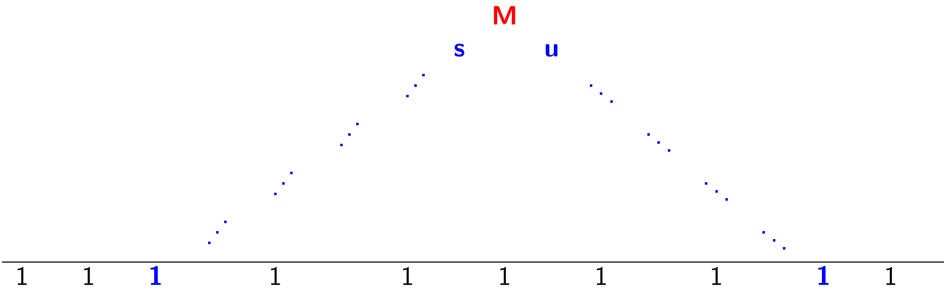
Claim1 $\langle \Gamma(w) \rangle_S$ coincides with $\langle L(\frac{s}{M}) \rangle$.
Claim2 Let $\langle \Gamma(w) \rangle_S^{\mathrm{numerate}}$ be sum of products of each path from "M" to right "1". Then the followings hold.

(1) $\langle \Gamma(w) \rangle_S \xrightarrow{A^4=-1} M$

(2) $)\langle \Gamma(w) \rangle_S^{\mathrm{numerate}} \xrightarrow{A^4=-1} s$

Main theorem (Recipe of making Kauffman bracket by using CCF)

(C) Picking up the "cosine curve" part of the blue, bend it at the maximum value "M":



Main theorem (Recipe of making Kauffman bracket by using CCF)

(C1)Put signature minus $-$ on segment from "M" to the left "1" (southwest-direction) and signature plus $+$ on segment from "M" to the right "1" (southeast-direction).
(C2) Connect three numbers in the above with signature- lines with the following rule. Draw a line segment so that the number on the vertex above the triangle is the two numbers on the bottom base and determine signature Extend the line segment so that positive and negative line segments are output one by one from the top vertex in accordance with the signs of the right end and the left end.
(C3) On each line segment, replace plus with weight $-A^4$ and minus with weight $-A^{-4}$.
(C4)Compute the product of weights on each path from "M" to the left "1" or to the right "1".
(C5)Sum the product of weights on each path from "M" to the left "1" or to the right "1".
Thus, a Laurent polynomial is obtained, which will be written as $\langle \Gamma(w) \rangle_C$

Main theorem (Recipe of making Kauffman bracket by using CCF)

Claim3 $\langle \Gamma(w) \rangle_C$ coincides with $\langle \frac{v}{M}) \rangle$.

Claim4 Let $\langle \Gamma(w) \rangle_C^{\mathrm{numerate}}$ be sum of products of each path from "M" to right "1". Then the followings hold.

(1) $\langle \Gamma(w) \rangle_C \xrightarrow{A^4=-1} M$

(2) $)\langle \Gamma(w) \rangle_C^{\text{numerate}} \xrightarrow{A^4=-1} v$

Periodicity of CCF

For Conway-Coxeter Frize $\mathrm{CCF}(w)$ of word w, if M is maximal integer which appears in $\mathrm{CCF}(w)$. We focus a diamond surrounding M in fundamental domain $\mathcal{D} - 1$ as follows:

$$\begin{array}{ccccc} & & s & & \\ & 7 & & b & \\ t & & 19 & & v \\ & c & & d & \\ & & u & & \end{array}$$

Periodicity of CCF

Then this satisfies the following relations :

$7 + d = b + c = 19$

$s + t = 7,$

$u + t = c,$

$v + u = d,$

$s + v = b,$

$7b - 17s = 1,$

$17t - 7c = 1,$

$17v - bd = 1,$

$cd - 17u = 1,$

$tv - su = 1,$

Periodicity of CCF

For CCF of type RL^2RL,

1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	
			1		2		4		3		1		2		3		2		3		1		2		4		3		1		2		3		2		3		1		2
				1		7		11		2		1		5		5		5		2		1		7		11		2		1		5		5		5		2		1	
			1		3		19		7		1		2		8		12		3		1		3		19		7		1		2		8		12		3		1		3
		1		2		8		12		3		1		3		19		7		1		2		8		12		3		1		3		19		7		1		2	
			1		5		5		5		2		1		7		11		2		1		5		5		5		2		1		7		11		2		1		5
		1		2		3		2		3		1		2		4		3		1		2		3		2		3		1		2		4		3		1		2	
	1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1

$\mathcal{D}(RL^2RL)-1$, $\mathcal{D}(RL^2RL)-2$, $\mathcal{D}(RL^2RL)-3$, $\mathcal{D}(RL^2RL)-4$

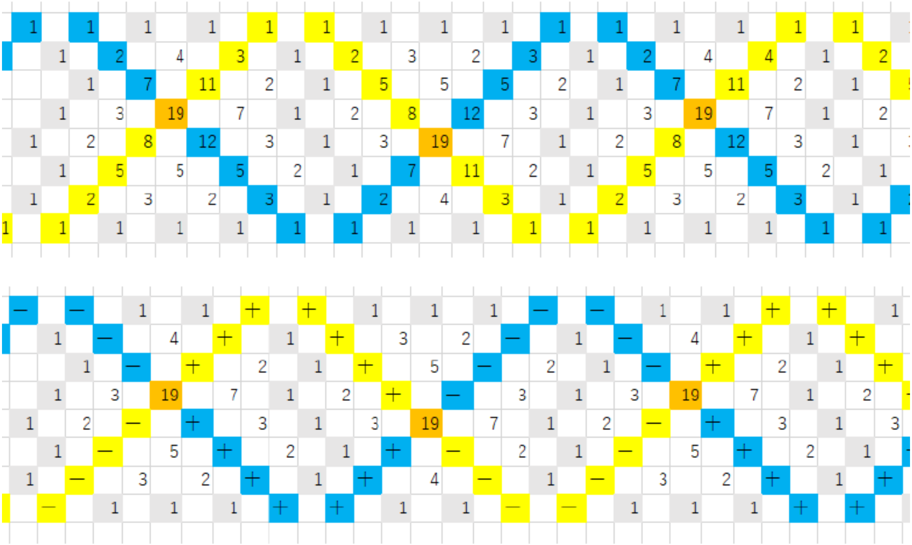
$\Rightarrow$

$\mathcal{D}(RL^2RL)-1=\mathcal{D}(RL^2RL)-3,\ \mathcal{D}(RL^2RL)-2=\mathcal{D}(RL^2RL)-4$

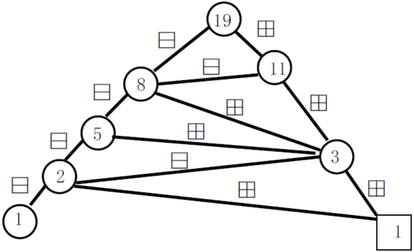
$$\mathcal{D}(RL^2RL)-2=\mathcal{D}(RL^2RL)-1$$

$$\mathcal{D}(RL^2RL)-4=\mathcal{D}(RL^2RL)-1$$

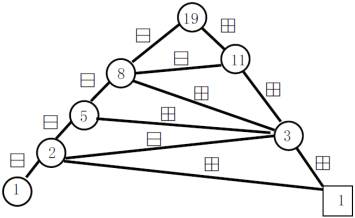
Constructing some Laurent polynomials on CCF's

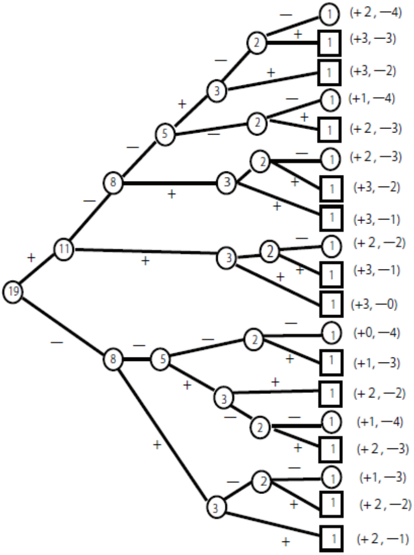


Constructing some Laurent polynomials on CCF's

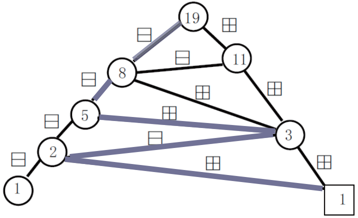


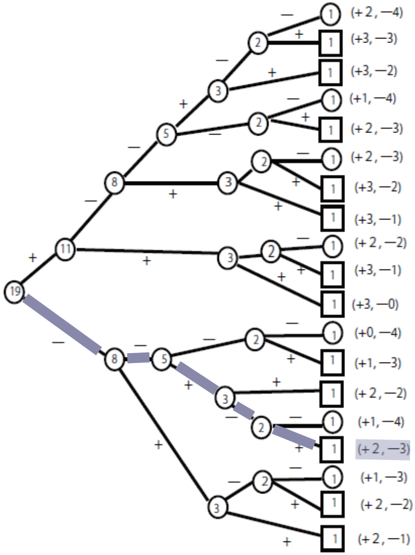
Constructing some Laurent polynomials on CCF's





Constructing some Laurent polynomials on CCF's





Constructing some Laurent polynomials on CCF's

For example,

Path γ: $19 \xrightarrow{-} 8 \xrightarrow{-} 5 \xrightarrow{+} 3 \xrightarrow{-} 2 \xrightarrow{+} \boxed{1}$

$\Rightarrow$

Path γ: $19 \xrightarrow{-A^{-4}} 8 \xrightarrow{-A^{-4}} 5 \xrightarrow{-A^{4}} 3 \xrightarrow{-A^{-4}} 2 \xrightarrow{-A^{4}} \boxed{1}$

Path γ-monomial :

$(-A^{-4}) \cdot (-A^{-4}) \cdot (-A^{4}) \cdot (-A^{-4}) \cdot (-A^{4}) = (-1)^{2+3} A^{(2-3)\cdot 4} = -A^{-4}$

In general, for corresponding signature $(+p, -q)$each path γ, associate a monomial as follows: $(-1)^{p+q} A^{4(p-q)}$

Definition of $\langle CCF \rangle$ associated to CCF's

path(CCF) := decreasing path from maximal 19 to ①, or $\boxed{1}$
path(CCF)$_{\text{numerate}}$:= decreasing path from maximal 19 to ①

$$\langle CCF \rangle := \sum_{\gamma \in \mathrm{path}(CCF)} (-1)^{p(\gamma)+q(\gamma)} A^{4(p(\gamma)-q(\gamma))}$$

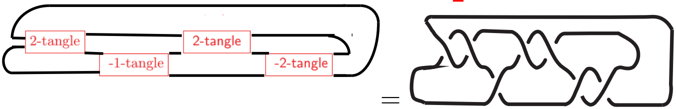
$$\langle CCF \rangle_{numerate} := \sum_{\gamma \in \mathrm{path}(CCF)_{\text{numerate}}} (-1)^{p(\gamma)+q(\gamma)} A^{4(p(\gamma)-q(\gamma))}$$

where $p(\gamma), q(\gamma)$ means number of $+$'s , $-$'s respectively in the path γ.
(cf. [Kogiso and Wakui1,2,2017])

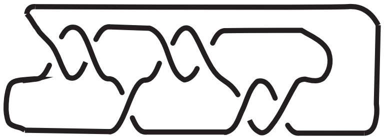
Example of $\langle CCF \rangle$

$\langle CCF(RL^2RL)\rangle = -A^{12} + 2A^8 - 3A^4 + 4 - 3A^{-4} + 3A^{-8} - 2A^{-12} + A^{-16}$

This $\langle CCF(RL^2RL)\rangle(A)$ coincides with Kauffman bracket polynomial of knot related to a fraction $\dfrac{7}{19} = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} \Leftrightarrow$



Example of $\langle CCF \rangle$

$\langle$  $\rangle$

$$= -A^{12} + 2A^{8} - 3A^{4} + 4 - 3A^{-4} + 3A^{-8} - 2A^{-12} + A^{-16} = \langle CCF(RL^2RL)\rangle(A)$$

Example of $\langle CCF \rangle$

$$\begin{cases} \langle CCF(RL^2RL)\rangle = -A^{12}+2A^8-3A^4+4-3A^{-4}+3A^{-8}-2A^{-12}+A^{-16} \\ \langle CCF(RL^2RL)\rangle_{\text{numerate}} = 1-A^{-4}+2A^{-8}-2A^{-12}+A^{-16} \end{cases}$$

$\Rightarrow^{\text{subsutitute } A^4=-1}$

$$\begin{cases} \langle CCF(RL^2RL)\rangle \xrightarrow{A^4=-1} 19 = \text{denominator of the fraction } \frac{7}{19} \\ \langle CCF(RL^2RL)\rangle_{\text{numerate}} \xrightarrow{A^4=-1} 7 = \text{numerator of the fraction } \frac{7}{19} \end{cases}$$

namely,

$$\frac{\langle CCF(RL^2RL)\rangle_{\text{numerate}}}{\langle CCF(RL^2RL)\rangle} = \frac{1-A^{-4}+2A^{-8}-2A^{-12}+A^{-16}}{-A^{12}+2A^8-3A^4+4-3A^{-4}+3A^{-8}-2A^{-12}+A^{-16}}$$

$$= \frac{-A^{16}+A^{12}-2A^8+2A^4-1}{A^{28}-2A^{24}+3A^{20}-4A^{16}+3A^{12}-3A^8+2A^4-1}$$

$$\xrightarrow{A^4=-1} \frac{7}{19}$$

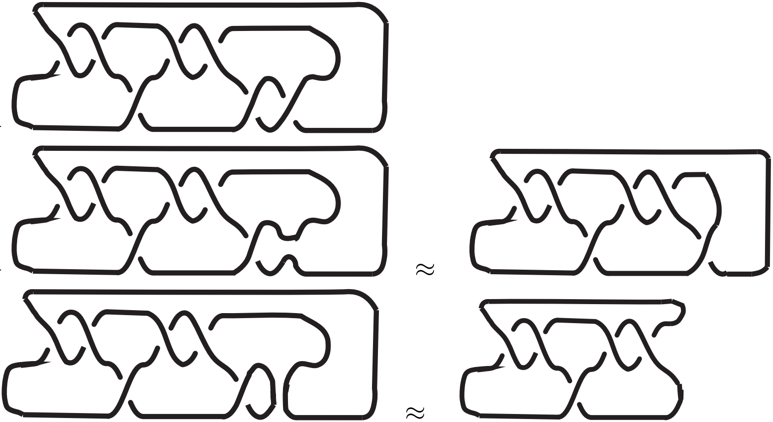
CCF's and links

Here we note that
$\frac{7}{19} = \frac{4}{11} \sharp \frac{3}{8}$

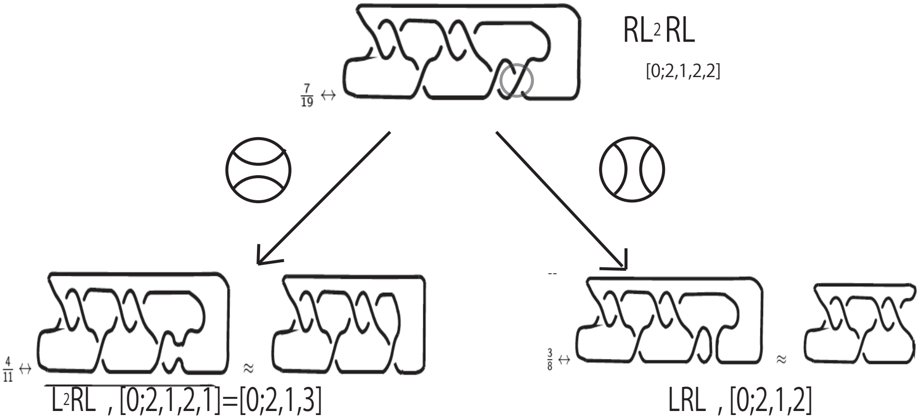
$\frac{7}{19} \leftrightarrow$

$\frac{4}{11} \leftrightarrow$ $\approx$

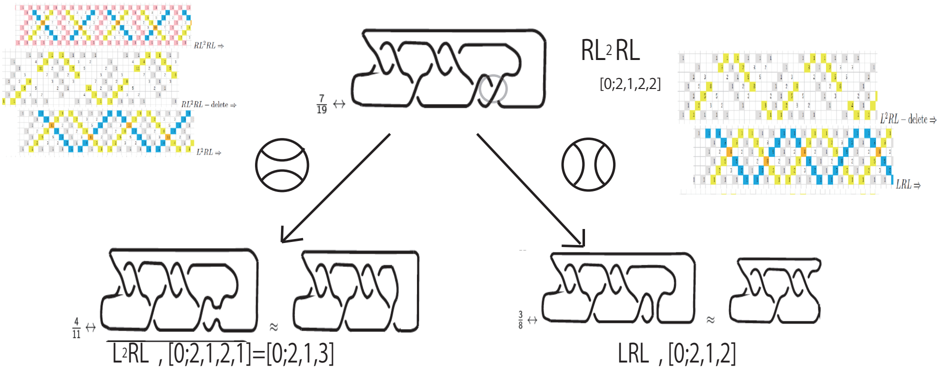
$\frac{3}{8} \leftrightarrow$ $\approx$



CCF's and links



CCF’s and links



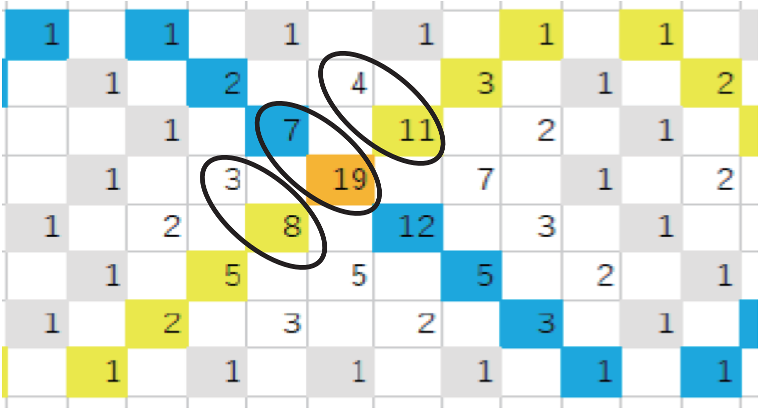
Corresponding CCF to YAT

Method of calculating Kauffman bracket polynomials associated fractions by using Yamada's ancestor triangles=YAT

$\Rightarrow$

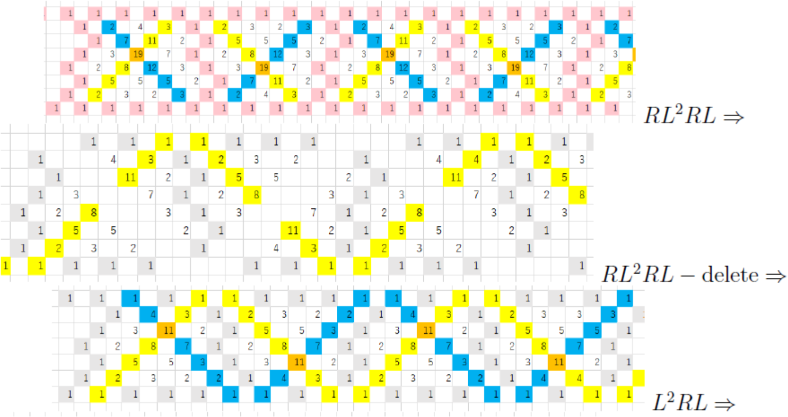
YAT fits this model very much!

Corresponding CCF to YAT



7/19=4/11 # 3/8

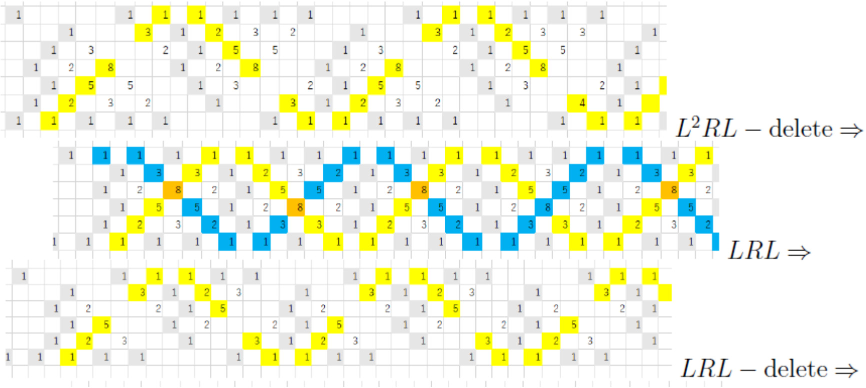
Corresponding CCF to YAT



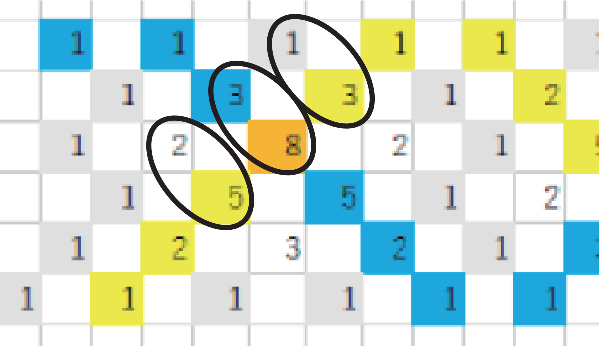
Corresponding CCF to YAT

$4/11 = 1/3 \# 3/8$

Corresponding CCF to YAT

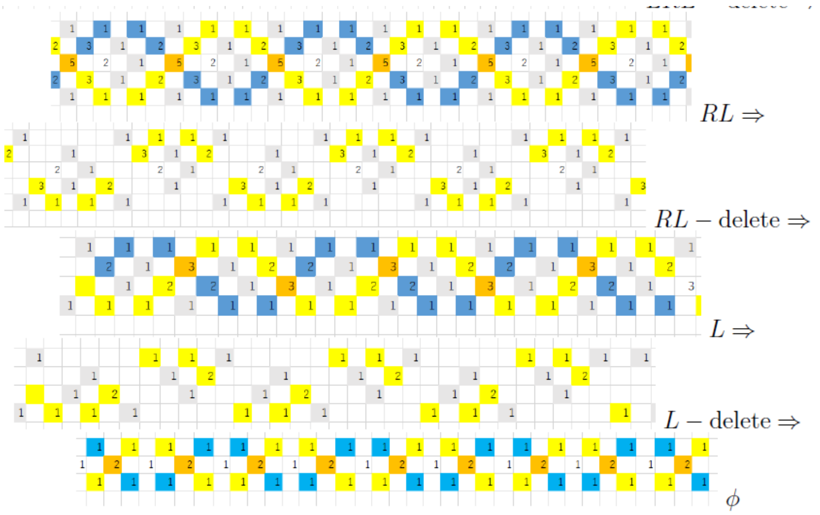


Corresponding CCF to YAT

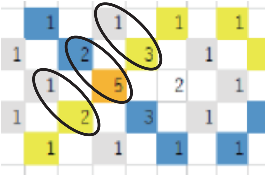


3/8=1/3 # 2/5

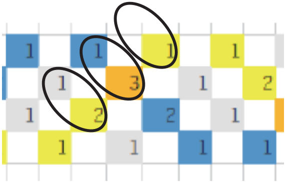
Corresponding CCF to YAT



Corresponding CCF to YAT

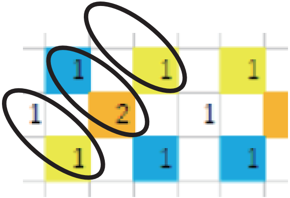


2/5=1/3 # 1/2



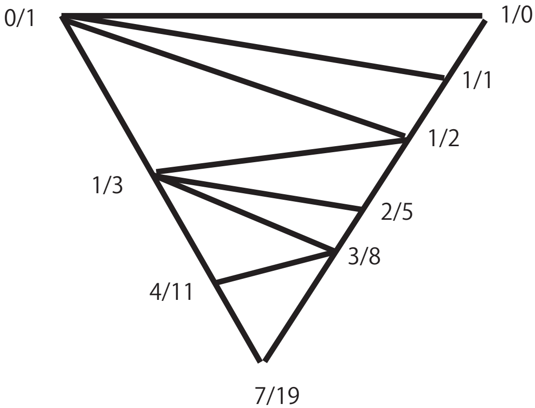
1/3=0/1 # 1/2

Corresponding CCF to YAT

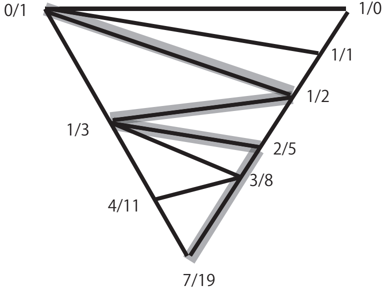


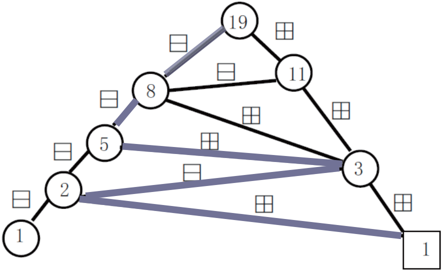
1/2=0/1 # 1/1

CCF's , Markov tree and Yamada's ancestor triangles



CCF's , Markov tree and Yamada's ancestor triangles





CCF's , Markov tree and Yamada's ancestor triangles

Theorem(S.Yamada, 1996)
For $\Lambda = \mathbb{Z}[A, A^{-1}]$,
the map $v \circ \phi : \mathbb{Q}_+ \cup \{\infty\} \longrightarrow \Lambda$ satisfies

$$v(\phi(\frac{p}{q})) = -A^4 v(\phi(\frac{s}{t})) - A^{-4} v(\phi(\frac{u}{v}))$$

where $\frac{p}{q} = \frac{s}{t} \sharp \frac{u}{v}$ (Farey sum with $sv - tu = -1,\ p = s + u,\ q = t + v$).

(**This is rewrite version in Kogiso and Wakui, 2017**)

CCF's , Markov tree and Yamada's ancestor triangles

Theorem(Kogiso and Wakui, 2017)
For Conway-Coxeter Frieze with LR-words w,

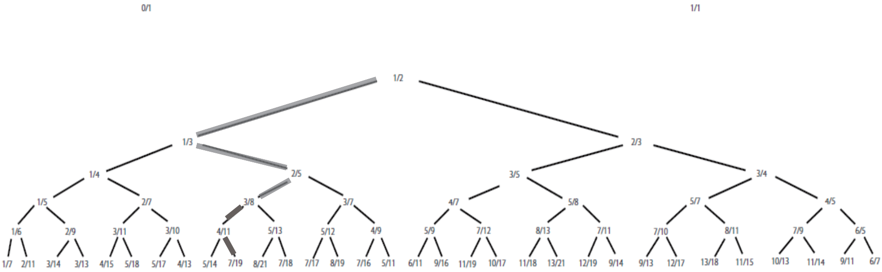
(1) if $w = R^m L^n w'$ for shorter word w',
$$\langle \Gamma(w) \rangle = -A^4 \langle \Gamma(R^{m-1} L^n w') \rangle - A^{-4} \langle \Gamma(L^{n-1} w') \rangle$$

,

(2) if $w = L^m R^n w'$ for shorter word w',
$$\langle \Gamma(w) \rangle = -A^4 \langle \Gamma(R^{n-1} w') \rangle - A^{-4} \langle \Gamma(L^{m-1} R^n w') \rangle$$

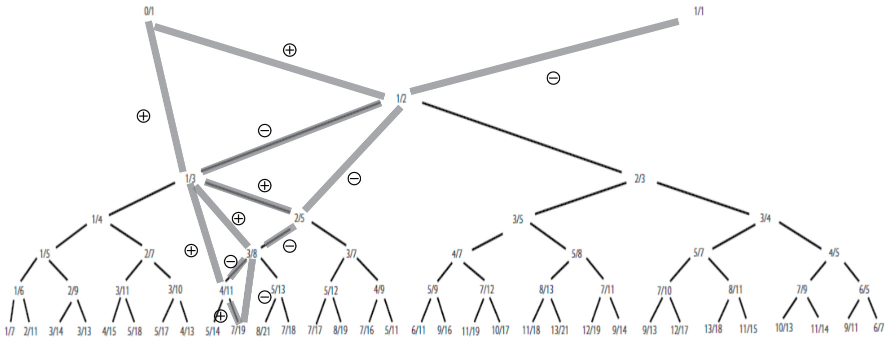
,

CCF's , Markov tree and Yamada's ancestor triangles



1/2-->1/3-->2/5-->3/8-->4/11-->7/19 L-->R-->L-->L-->R =>RL2RL

CCF and Farey tree



$\oplus = \; -A^4 \qquad \ominus = \; -A^{-4}$

CCF's , Markov tree and Yamada's ancestor triangles

Questions on Yamada's ancestor triangles

(1) Why does the following equation hold?

When $p < q$, $v(\phi(\frac{p}{q})(A^{-1}) = v(\phi(\frac{q-p}{q}))$

(2)Why do fractions with the same denominator of the same generation in Markov tree have the same Laurent polynomial or its-dual Laurent polynomial ($A \mapsto A^{-1}$)? For example,

fraction	*word*	Kauffman Bracket polynomial$v(\phi(\frac{p}{q})$
$\frac{5}{17}$	LR^2L^2	$-A^{12}+2A^8-3A^4-3A^{-4}+3A^{-8}-A^{-12}+A^{-16}+3$
$\frac{7}{17}$	L^2R^2L	$-A^{12}+2A^8-3A^4-3A^{-4}+3A^{-8}-A^{-12}+A^{-16}+3$
$\frac{10}{17}$	R^2L^2R	$A^{16}-A^{12}+3A^8-3A^4-3A^{-4}+2A^{-8}-A^{-12}+3$
$\frac{12}{17}$	RL^2R^2	$A^{16}-A^{12}+3A^8-3A^4-3A^{-4}+2A^{-8}-A^{-12}+3$

$v(\phi(\frac{5}{17})) = v(\phi(\frac{7}{17}))$, $v(\phi(\frac{10}{17})) = v(\phi(\frac{12}{17})) = v(\phi(\frac{5}{17}))'A^{-1})$?

CCF's , Markov tree and Yamada's ancestor triangles

We can answer these questions by using CCF's!!

CCF's , Markov tree and Yamada's ancestor triangles

$\frac{p}{q}$	$\frac{5}{17}$	$\frac{7}{17}$	$\frac{10}{17}$	$\frac{12}{17}$
conti.frac. exp.	$[0, 3, 2, 2]$	$[0, 2, 2, 3]$	$[0, 1, 1, 2, 3]$	$[0, 1, 2, 2, 2]$
$w(\frac{p}{q})$	LR^2L^2	L^2R^2L	R^2L^2R	RL^2R^2
	w	$r(w)$	$i(w)$	$r(w)$

CCF's , Markov tree and Yamada's ancestor triangles

$$\frac{5}{17} \qquad\qquad\qquad\qquad \frac{7}{17}$$

$$[0;3,2,2] \overset{\text{odd vertical sym.}}{\longleftrightarrow} [0;2,2,3]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$[0;3,2,1,1] \qquad\qquad\qquad\qquad [0;2,2,2,1]$$

$$\updownarrow \quad \Leftarrow \text{even vertical sym.} \Rightarrow \quad \updownarrow$$

$$[0;1,1,2,3] \qquad\qquad\qquad\qquad [0;1,2,2,2]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$[0;1,1,2,2,1] \overset{\text{odd vertical sym.}}{\longleftrightarrow} [0;1,2,2,1,1]$$

$$\frac{10}{17} \qquad\qquad\qquad\qquad \frac{12}{17}$$

CCF's , Markov tree and Yamada's ancestor triangles

For $\frac{5}{17}, \frac{7}{17}, \frac{10}{17}, \frac{12}{17}$

$$\begin{array}{ccccccccc} & & & & 7 & & & & \\ & & 12 & & & & 10 & & \\ 5 & & & & 17 & & & & 3 \\ & & 7 & & & & 5 & & \\ & & & & 2 & & & & \end{array} \Rightarrow$$

	1		1		1		1		1		1		1		1		1		1		1		1		1		1				
				4		2		3		1		2		4		2		2		1		4									
					7		5		2		1		7		7		3		1		3		7								
						17		3		1		3		12		10		1		2		5		17							
							10		1		2		5		17		3		1		3		12		10						
								3		1		3		7		5		2		1		7		7		3					
									2		1		4		2		3		1		2		4		2		2				
1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	

$\Rightarrow RL^2R^2$-type

CCF's , Markov tree and Yamada's ancestor triangles

For $RL^2R^2 \leftrightarrow LR^2L^2, R^2L^2R, R^2L^2R$type, these Kauffman bracket polynomials are the followings:

$LR^2L^2 \Rightarrow v(\phi(\frac{5}{17})) = -A^{12} + 2A^8 - 3A^4 + 3 - 3A^{-4} + 3A^{-8} - A^{-12} + A^{-16}$
$L^2R^2L \Rightarrow v(\phi(\frac{7}{17})) = v(\phi(\frac{5}{17}))$
$R^2L^2R \Rightarrow v(\phi(\frac{10}{17})) = v(\phi(\frac{5}{17}))(A^{-1})$
$RL^2R^2 \Rightarrow v(\phi(\frac{12}{17})) = v(\phi(\frac{5}{17}))(A^{-1})$

Kauffman Bracket of fractions with denominator 17

In the Conway-Coxeter Frieze that satisfies the above conditions, make the numbers surrounding 17 as follows :

		s		
	a		b	
t		17		v
	c		d	
		u		

Kauffman Bracket of fractions with denominator 17

Then this satisfies the following relations ：

$a + d = b + c = 17$

$s + t = a,$

$u + t = c,$

$v + u = d,$

$s + v = b,$

$ab - 17s = 1,$

$17t - ac = 1,$

$17v - bd = 1,$

$cd - 17u = 1,$

$tv - su = 1,$

The greatest common divisor of diagonally adjacent integers must be 1.

Kauffman Bracket of fractions with denominator 17

Then

$$\begin{array}{ccccc} & & s & & \\ & a & & b & \\ t & & 17 & & v \\ & c & & d & \\ & & u & & \end{array} = \text{CCF、its vertical-symm , horizontal-symm}$$

CCF	CCF
CCF	CCF

also appear.
The representative forms are arranged as follows.

Kauffman Bracket of fractions with denominator 17

0
1 1
1 17 1 $\Rightarrow \frac{1}{17}, \ \frac{16}{17}$
16 16
15

1
2 9
1 17 8 $\Rightarrow \frac{2}{17}, \ \frac{8}{17}, \ \frac{9}{17}, \frac{15}{17}$
8 15
7

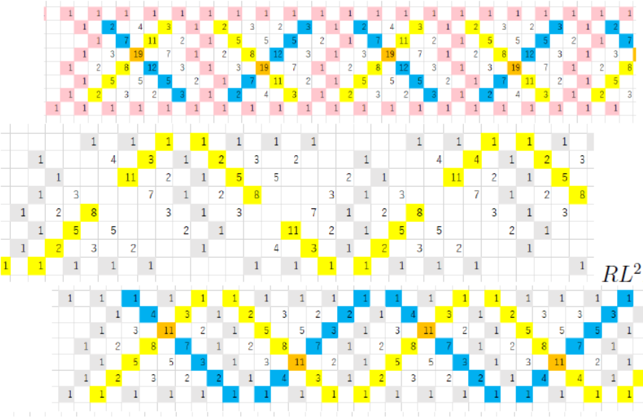
1
3 6
2 17 5 $\Rightarrow \frac{3}{17}, \ \frac{6}{17}, \ \frac{11}{17}, \frac{14}{17}$
11 14
9

Kauffman Bracket of fractions with denominator 17

$$
\begin{array}{ccccc}
 & & 3 & & \\
 & 4 & & 13 & \\
1 & & 17 & & 10 \\
 & 4 & & 13 & \\
 & & 3 & &
\end{array}
\Rightarrow \frac{4}{17},\ \frac{13}{17},
$$

$$
\begin{array}{ccccc}
 & & 2 & & \\
 & 5 & & 7 & \\
3 & & 17 & & 5 \\
 & 10 & & 12 & \\
 & & 7 & &
\end{array}
\Rightarrow \frac{5}{17},\ \frac{7}{17},\ \frac{10}{17},\ \frac{12}{17}
$$

deleting and inserting of CCF's

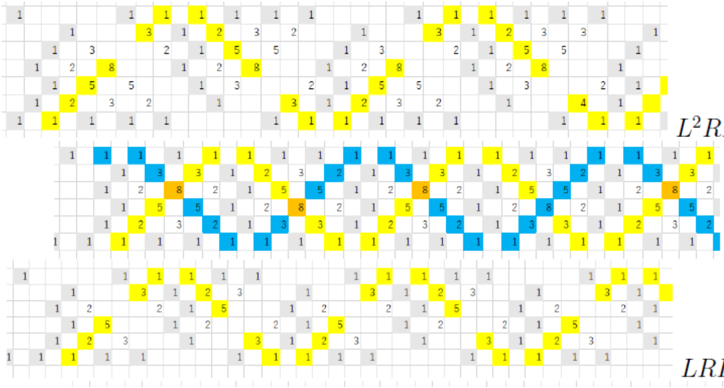


$RL^2RL \Rightarrow$

$RL^2RL - \text{delete} \Rightarrow$

$L^2RL \Rightarrow$

deleting and inserting of CCF's

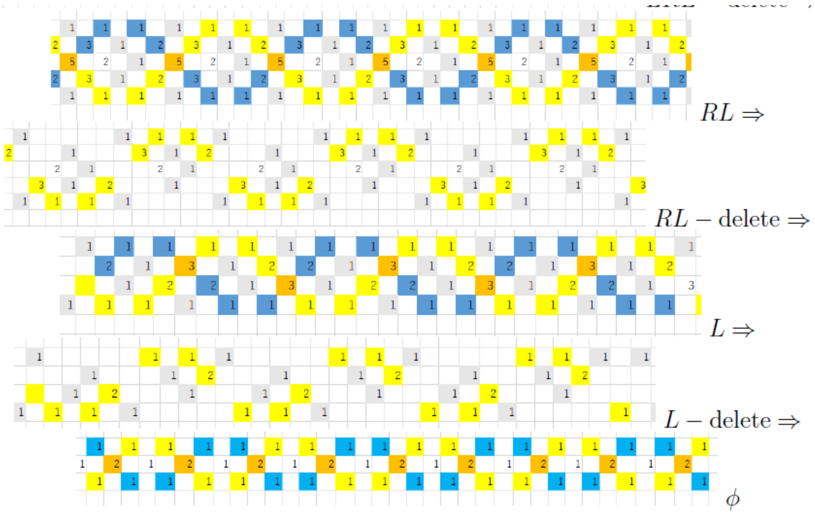


L^2RL − delete ⇒

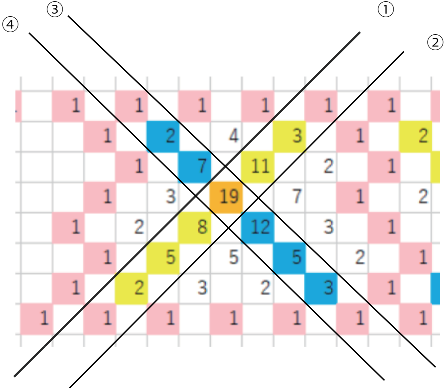
LRL ⇒

LRL − delete ⇒

deleting and inserting of CCF's

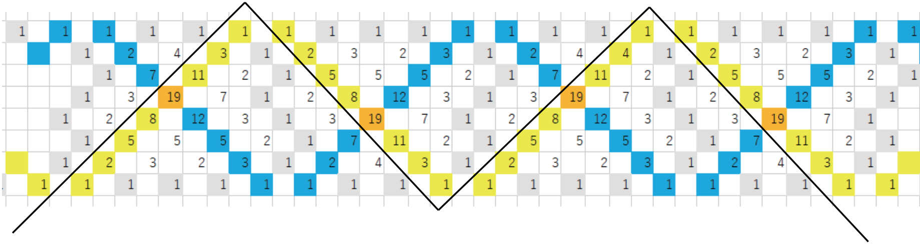


Inserting CCF



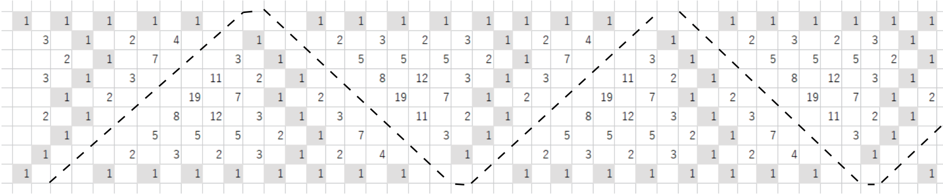
cutting and inserting lines ①, ②, ③, ④

Inserting CCF ①



①

Inserting CCF ①



Inserting CCF ①

1		1		1		1		1						1		1		1		1		1		1		1		1						1		1		1		1		1		1	
	3		1		2		4				1				2		3		2		3		1		2		4				1				2		3		2		3		1		2
		2		1		7				3		1				5		5		5		2		1		7				3		1				5		5		5		2		1	
	3		1		3				11		2		1				8		12		3		1		3				11		2		1				8		12		3		1		3
		1		2				19		7		1		2				19		7		1		2				19		7		1		2				19		7		1		2	
	2		1				8		12		3		1		3				11		2		1				8		12		3		1		3				11		2		1		
		1				5		5		5		2		1		7				3		1				5		5		5		2		1		7				3		1			
	1				2		3		2		3		1		2		4				1				2		3		2		3		1		2		4				1				2
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Inserting CCF ①

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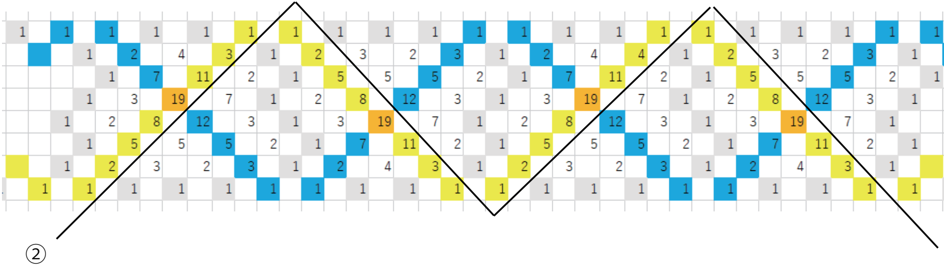
$\Rightarrow RL^2RL^2$

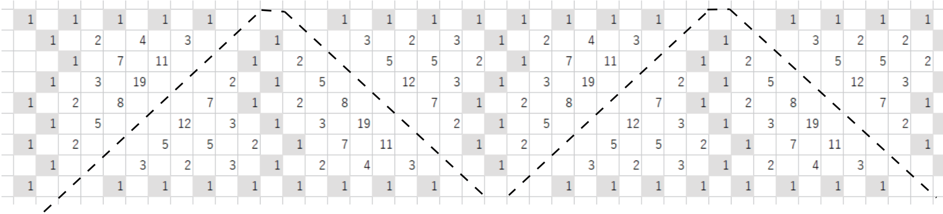
$$\frac{7}{26} = \frac{4}{15}\sharp\frac{3}{11}$$

$$\frac{19}{26} = \frac{8}{11}\sharp\frac{11}{15}$$

Inserting CCF ②



Inserting CCF ②



Inserting CCF ②

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Inserting CCF ②

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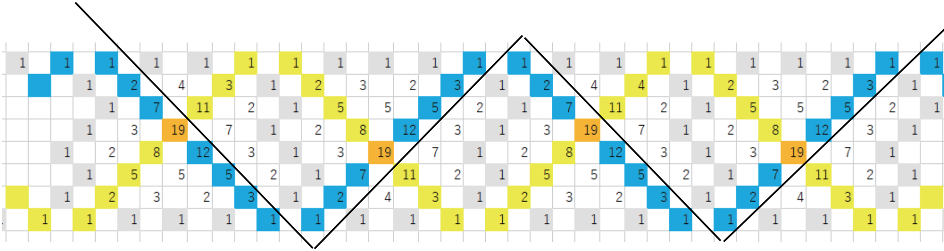
$\Rightarrow RL^2RLR$

$$\frac{19}{31} = \frac{11}{18} \sharp \frac{8}{13}$$

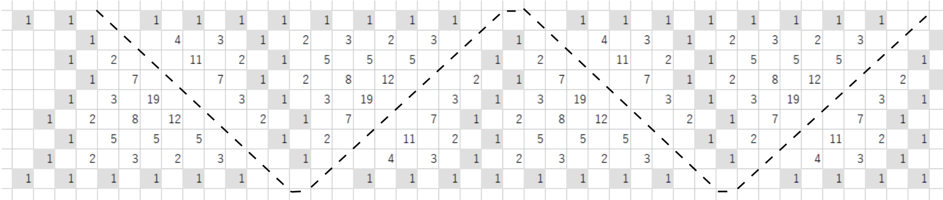
$$\frac{12}{31} = \frac{5}{13} \sharp \frac{7}{18}$$

Inserting CCF ③

③



Inserting CCF ③



Inserting CCF ③

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Inserting CCF ③

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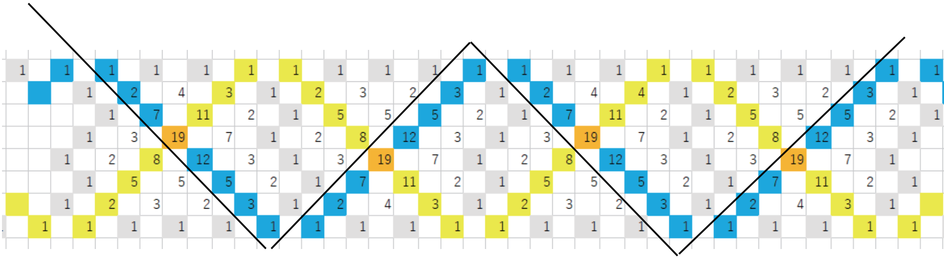
$\Rightarrow LRL^2RL$

$$\frac{19}{30} = \frac{12}{19} \sharp \frac{7}{11}$$

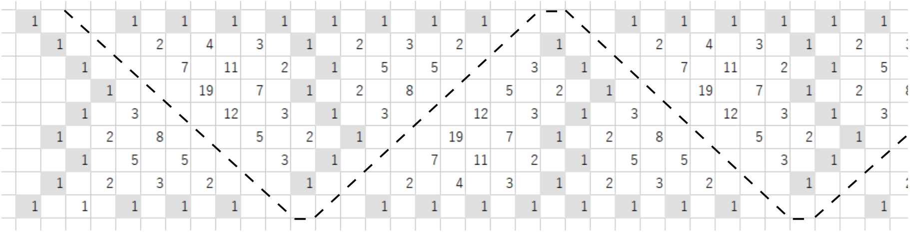
$$\frac{11}{30} = \frac{4}{11} \sharp \frac{7}{19}$$

Inserting CCF ④

④



Inserting CCF ④



Inserting CCF ④

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	1		2		3		2				1				2		4		3		1		2		3		2				1			
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Inserting CCF ④

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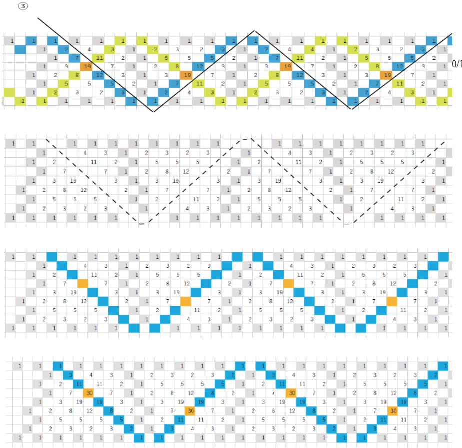
1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	

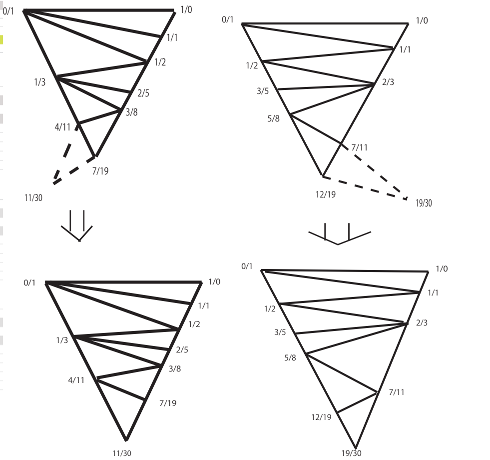
$\Rightarrow R^2L^2RL$

$$\frac{19}{27} = \frac{7}{10} \sharp \frac{12}{17}$$

$$\frac{8}{31} = \frac{5}{17} \sharp \frac{3}{10}$$

Inserting CCF ③ and YAT(11/30)



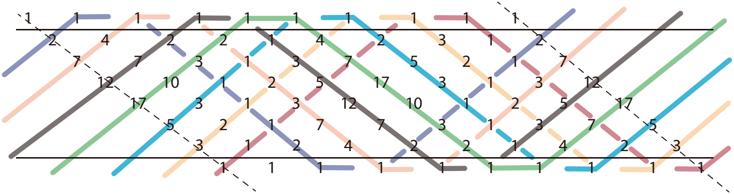


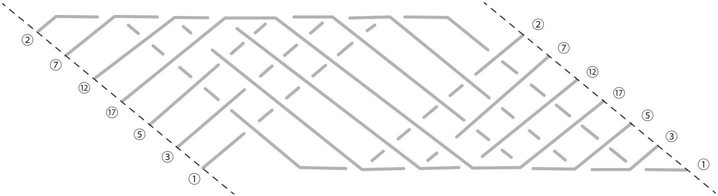
$\Rightarrow LRL^2RL$

$\frac{19}{30} = \frac{12}{19} \sharp \frac{7}{11}$

$\frac{11}{30} = \frac{4}{11} \sharp \frac{7}{19}$

A braid from a CCF L^2R^2L





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