# A relation between quandle coloring and biquandle coloring

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(conj. of) group → quandle → biquandle

generalize    generalize

(conj. of) group $\overset{\text{gen.}}{\subset}$ quandle $\overset{\text{gen.}}{\subset}$ biquandle

| representation | coloring | coloring |
|---|---|---|
| $\mathrm{Hom}(\pi_L, G)$ | $\mathrm{Hom}(Q(L), Q)$ | $\mathrm{Hom}(BQ(L), X)$ |

$\mathrm{Hom}(\pi_L, G) \curvearrowright \mathrm{Hom}(Q(L), Q)$: refined invs.

$\mathrm{Hom}(Q(L), Q) \curvearrowright \mathrm{Hom}(BQ(L), X)$: refined invs?

(conj. of) group $\overset{\text{gen.}}{\subset}$ quandle $\overset{\text{gen.}}{\subset}$ biquandle

representation | coloring | coloring

$\mathrm{Hom}(\pi_L, G)$ $\curvearrowright$ $\mathrm{Hom}(Q(L), Q)$ $\curvearrowright$ $\mathrm{Hom}(BQ(L), X)$

refined | refined?

Fact: Yes, for virtual knots.

(conj. of) group $\overset{\text{gen.}}{\subset}$ quandle $\overset{\text{gen.}}{\subset}$ biquandle

representation

coloring $\overset{1:1}{\longleftrightarrow}$ coloring

$\mathrm{Hom}(\pi_L, G)$ $\curvearrowright$ refined $\mathrm{Hom}(Q(L), Q)$ $\curvearrowright$ refined? $\mathrm{Hom}(BQ(L), X)$

Fact: Yes, for virtual knots.

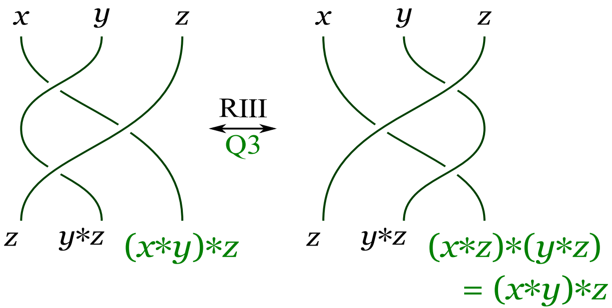
This talk: NO, for classical/surface links.

# Contents

$Q$ : a set $\qquad *: Q \times Q \to Q$ : a map

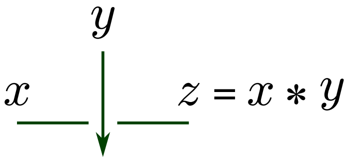
$Q = (Q, *)$ is a **quandle**

$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \text{(Q1) } x * x = x, \\ \text{(Q2) the map } s_x : Q \ni a \mapsto a * x \in Q \text{ is a bij.,} \\ \text{(Q3) } (x * y) * z = (x * z) * (y * z), \\ \qquad\qquad\qquad \text{for } {}^{\forall} x, y, z \in Q. \end{cases}$$



$L$ : a link $\qquad$ $D$ : a diag. of $L$

· A $Q$-**coloring** is a map $\{\text{arcs of } D\} \to Q$ such that




at each crossing.

<u>Ex</u> A **conjugation quandle**:

$$Q = G : \text{a group}, \qquad x * y = y^{-1}xy.$$

$$\rightsquigarrow \mathrm{Col}_Q(D) \overset{1:1}{\longleftrightarrow} \mathrm{Hom}_{\mathrm{gp}}(\pi_L, G).$$

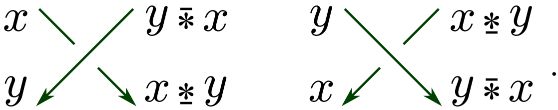
- $\#\mathrm{Col}_Q(D)$ is an inv. of $L$.

$X$ : a set $\qquad \underline{*}, \overline{*} : X \times X \to X$ : maps

$X = (X, \underline{*}, \overline{*})$ is a **biquandle**

$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \text{(B1) } x \,\underline{*}\, x = x \,\overline{*}\, x, \\ \text{(B2) the maps } \bullet \,\underline{*}\, x, \bullet \,\overline{*}\, x : X \to X, \\ \qquad X \times X \ni (a, b) \mapsto (b \,\overline{*}\, a, a \,\underline{*}\, b) \in X \times X \text{ are bijs.}, \\ \text{(B3) } (x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) = (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z), \\ \qquad (x \,\underline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) = (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z), \\ \qquad (x \,\overline{*}\, y) \,\overline{*}\, (z \,\overline{*}\, y) = (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z) \\ \qquad\qquad\qquad\qquad \text{for } {}^\forall x, y, z \in X. \end{cases}$$

· An **$X$-coloring** is a map $\{\text{semi-arcs of } D\} \to X$ s.t.




· $(Q, *)$: a quandle

$$\Rightarrow \begin{cases} X = (Q, \underline{*} = *, \overline{*} = \mathrm{pr}_1) : \text{a biquandle.} \\ \mathrm{Col}_Q(D) \overset{1:1}{\longleftrightarrow} \mathrm{Col}_X(D). \end{cases}$$

$\rightsquigarrow$ Biquandle is a generalization of quandle.

- $\#\mathrm{Col}_X(D)$ is an inv. of $L$.

# Main result

· $X = (X, \underline{*}, \overline{*})$: a biquandle

$$\mathcal{Q}(X) := (X, *); \qquad x * y := (x \,\underline{*}\, y) \,\overline{*}^{-1}\, y.$$

- $\mathcal{Q}(X)$ is a quandle (structure rack).

**Theorem A**

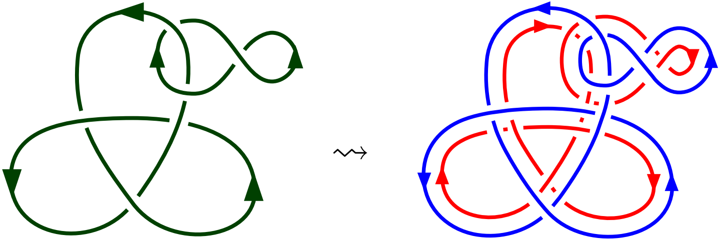
There exists a one-to-one correspondence

$$\mathrm{Col}_X(D) \overset{1:1}{\longleftrightarrow} \mathrm{Col}_{\mathcal{Q}(X)}(D).$$

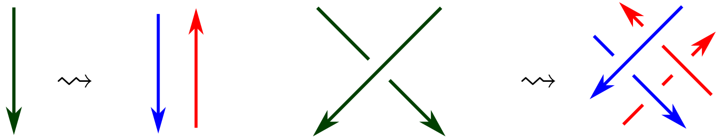
**Corollary (cf. Soloviev (2000))**

$X$-coloring number $= \mathcal{Q}(X)$-coloring number.

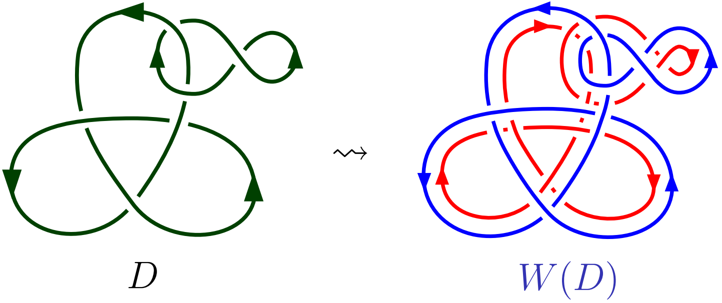
· The **double** $W(D)$ of $D$:

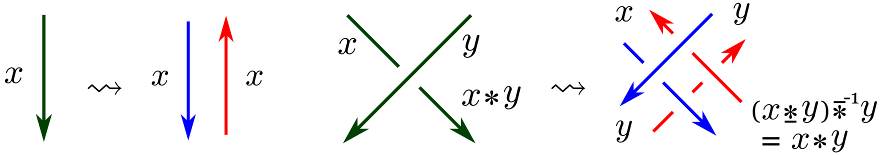


$D$ $\qquad$ $W(D)$

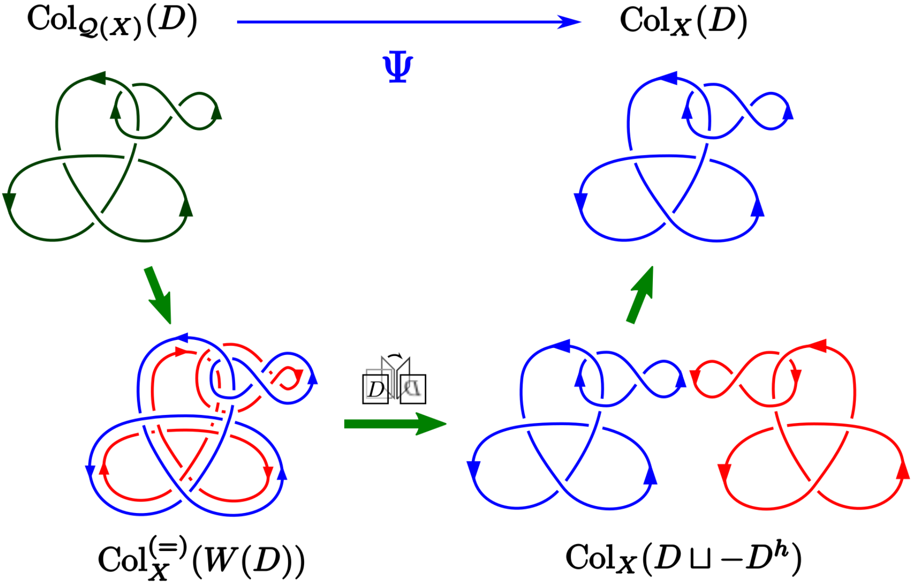


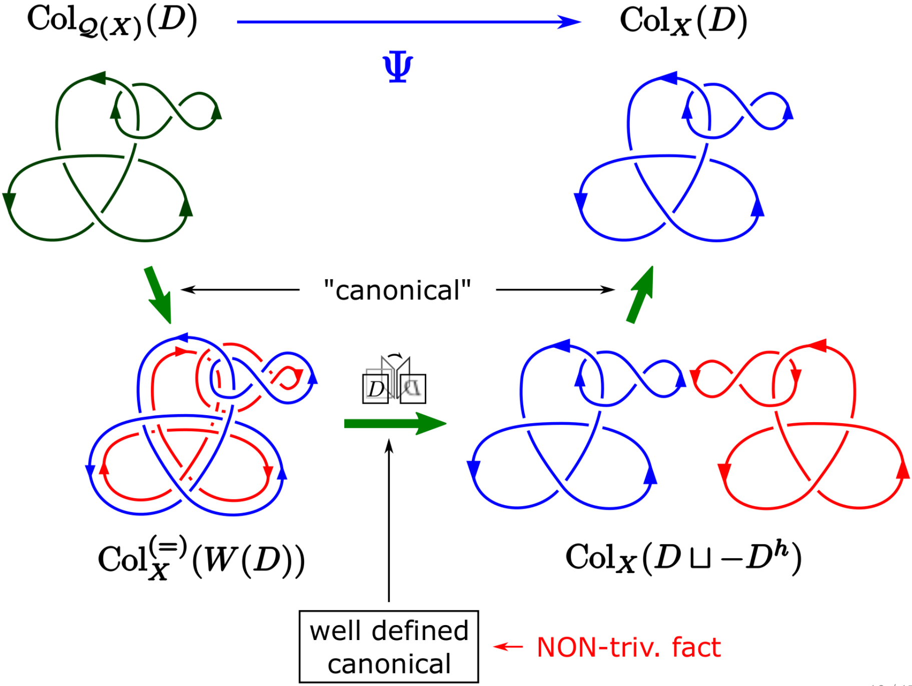
· The **double** $W(D)$ of $D$:

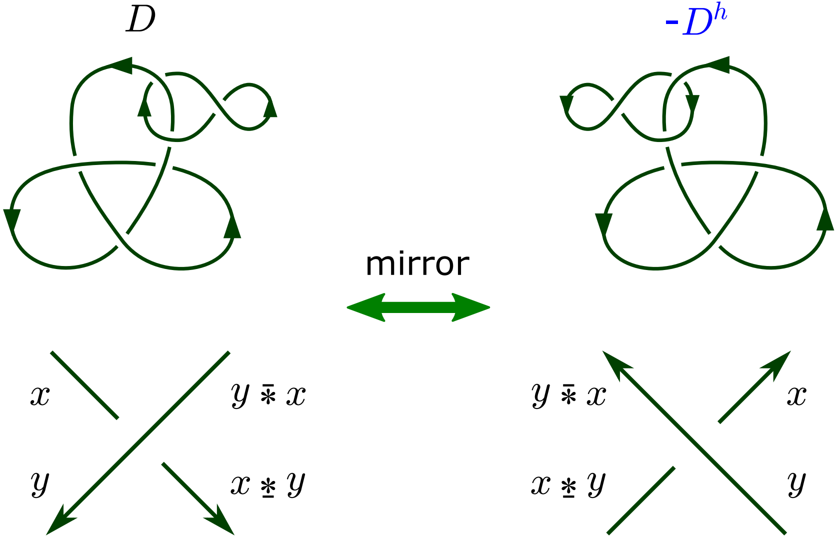


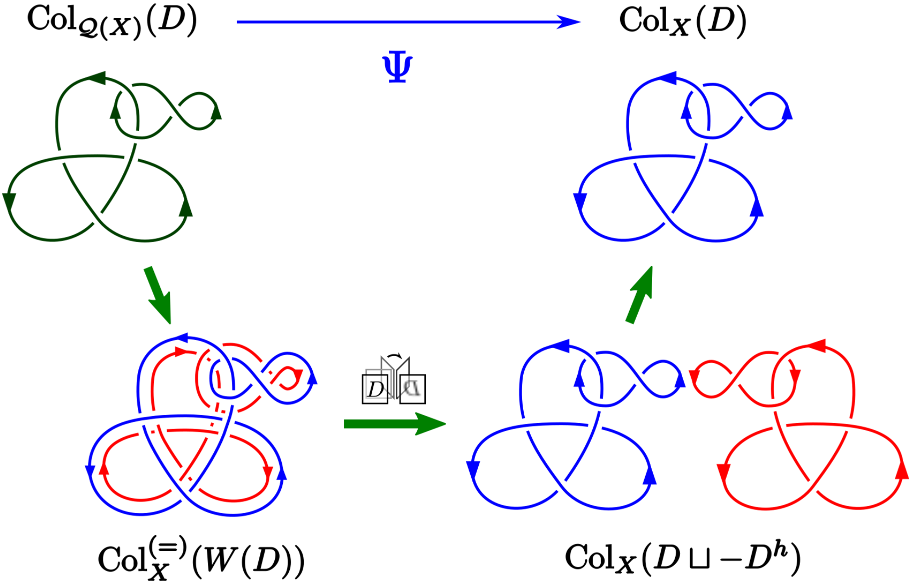

- $\mathrm{Col}_{\mathcal{Q}(X)}(D) \overset{1:1}{\longleftrightarrow} \mathrm{Col}_X^{(=)}(W(D)) \subset \mathrm{Col}_X(W(D))$


$\mathrm{Col}_{\mathcal{Q}(X)}(D)$
$\Psi$
$\mathrm{Col}_X(D)$
$D$
$\mathrm{Col}_X^{(=)}(W(D))$
$\mathrm{Col}_X(D \sqcup -D^h)$

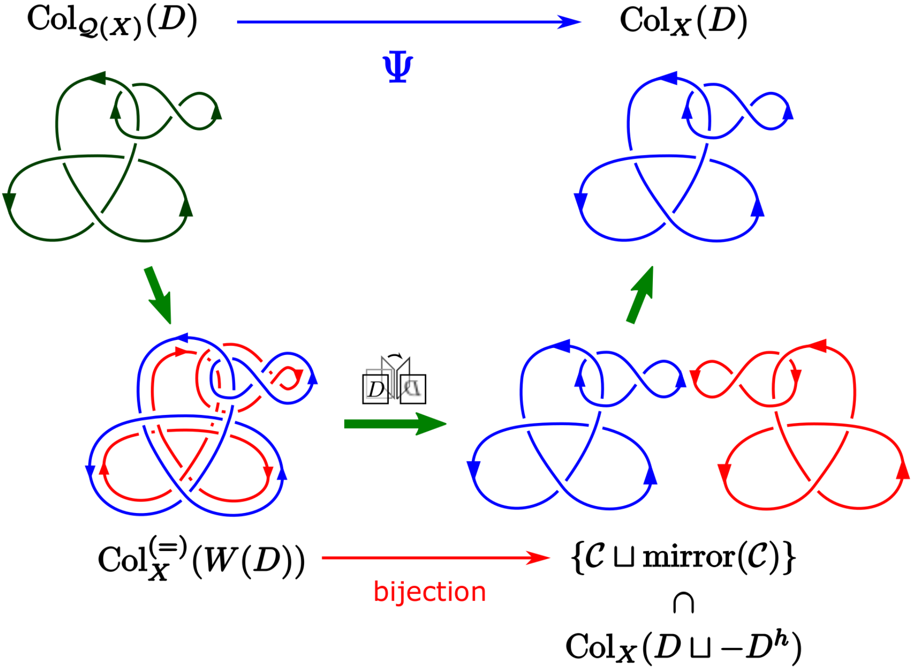

$\mathrm{Col}_{\mathcal{Q}(X)}(D)$
$\Psi$
$\mathrm{Col}_X(D)$
"canonical"
$D$
$\mathrm{Col}_X^{(=)}(W(D))$
$\mathrm{Col}_X(D \sqcup -D^h)$
well defined
canonical
← NON-triv. fact

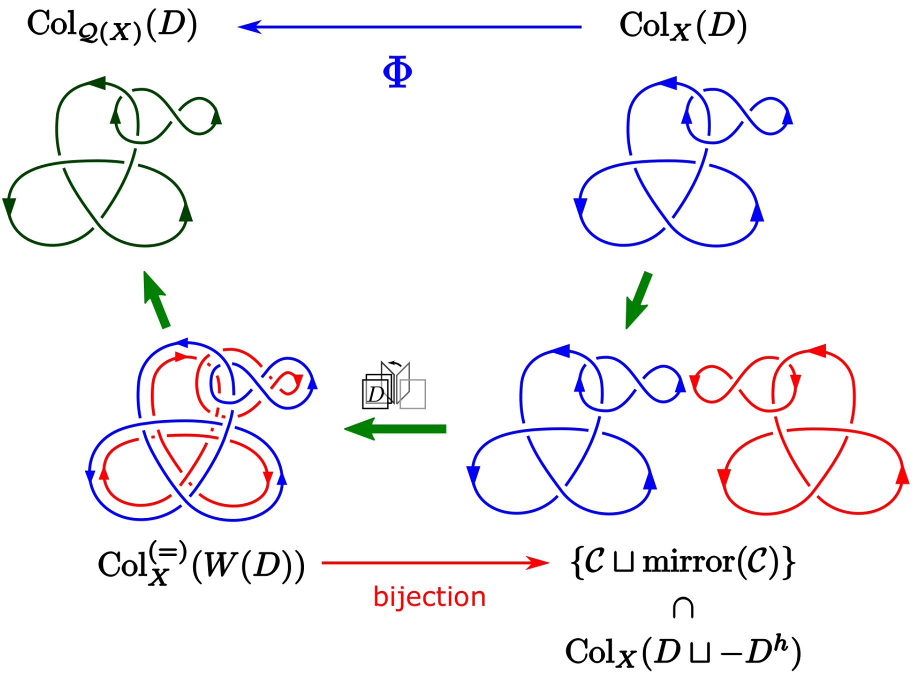



$$\rightsquigarrow \quad \mathrm{Col}_X(D) \quad \overset{1:1}{\longleftrightarrow} \quad \mathrm{Col}_X(-D^h)$$



$\mathrm{Col}_{\mathcal{Q}(X)}(D)$
$\mathrm{Col}_X(D)$
$\Psi$
$D$
$\mathrm{Col}_X^{(=)}(W(D))$
$\mathrm{Col}_X(D \sqcup -D^h)$

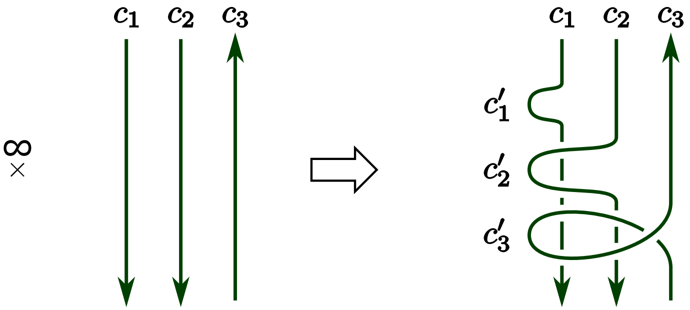

$\mathrm{Col}_{\mathcal{Q}(X)}(D)$
$\Psi$
$\mathrm{Col}_X(D)$
$D$
$\mathrm{Col}_X^{(=)}(W(D))$
bijection
$\{\mathcal{C} \sqcup \mathrm{mirror}(\mathcal{C})\}$
$\cap$
$\mathrm{Col}_X(D \sqcup -D^h)$



$\mathrm{Col}_{\mathcal{Q}(X)}(D)$
$\mathrm{Col}_X(D)$
$\Phi$
$D$
$\mathrm{Col}_X^{(=)}(W(D))$
bijection
$\{\mathcal{C} \sqcup \mathrm{mirror}(\mathcal{C})\}$
$\cap$
$\mathrm{Col}_X(D \sqcup -D^h)$

# Another definition of $\Phi : \mathrm{Col}_X(D) \to \mathrm{Col}_{\mathcal{Q}(X)}(D)$

$$\mathrm{Col}_X(D) \ni \mathcal{C} : x_i \mapsto c_i$$



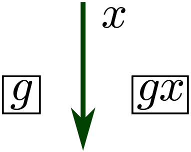

$$\Phi'(\mathcal{C})(x_i) := c'_i. \quad \rightsquigarrow \quad \Phi = \Phi'.$$

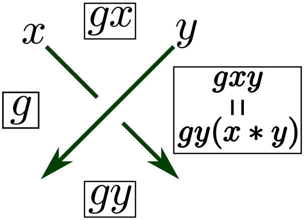
## · The associated group

$$\mathrm{As}(Q) := \langle Q \mid xy = y(x * y) \rangle_{\mathrm{grp}}.$$

· A **shadow** $(Q, \mathrm{As}(Q))$**-coloring** is a pair $(\mathcal{A}, \mathcal{R})$ s.t

$$\begin{cases} \mathcal{A} : \text{a } Q\text{-coloring} \\ \mathcal{R} : \{\text{regions}\} \to \mathrm{As}(Q) \text{ s.t.} \end{cases}$$

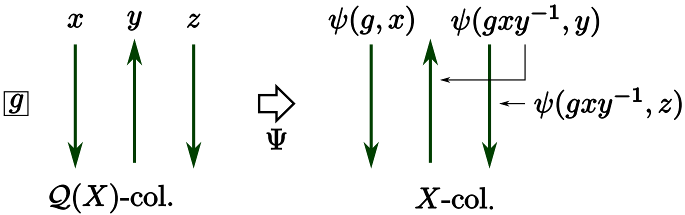



· $\mathcal{A}$: fixed $\Rightarrow \exists! \mathcal{R}$ s.t. $\mathcal{R}(\infty) = e$.

## Theorem A'

There exists a map $\psi : \mathrm{As}(\mathcal{Q}(X)) \times \mathcal{Q}(X) \to X$ which induces the bijection $\Psi : \mathrm{Col}_{\mathcal{Q}(X)}(D) \to \mathrm{Col}_X(D)$.




## Theorem B

$\phi : X \times X \to A$ a biquandle 2-cocycle

$\Rightarrow \psi^*\phi : \mathrm{As}(\mathcal{Q}(X)) \times \mathcal{Q}(X) \times \mathcal{Q}(X) \to A$

is a shadow quandle 2-cocycle and

$$\Phi_\phi(D) = \Phi_{\psi^*\phi}^{\infty=e}(D).$$

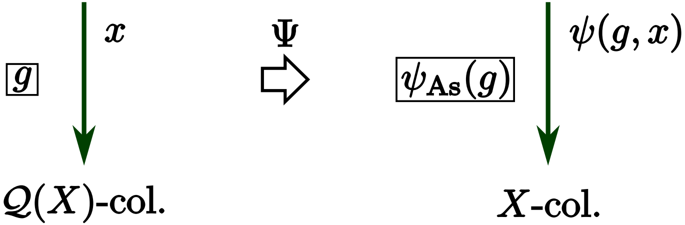
## Theorem A"

There exist one-to-one correspondences

$$\psi_{\mathrm{As}} : \mathrm{As}(\mathcal{Q}(X)) \to \mathrm{As}(X),$$

$$\tilde{\psi} = (\psi_{\mathrm{As}}, \psi) : \mathrm{As}(\mathcal{Q}(X)) \times \mathcal{Q}(X) \to \mathrm{As}(X) \times X,$$

where $\tilde{\psi}$ induces the bij. $\Psi : \mathrm{Col}_{\mathcal{Q}(X)}(D) \to \mathrm{Col}_X(D)$.




"quandle col. + shadow $\overset{1:1}{\longleftrightarrow}$ biquandle col. + shadow"

$$\begin{array}{ccc} \pi_1(B\mathcal{Q}(X)) & & \pi_1(BX) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{As}(\mathcal{Q}(X)) & \xrightarrow[1:1]{\psi_{\mathrm{As}}} & \mathrm{As}(X) \end{array}$$

Lem $\widetilde{B\mathcal{Q}(X)} \cong \widetilde{BX}$ as $\square$-sets.

$$\rightsquigarrow \begin{cases} \pi_n(B\mathcal{Q}(X)) \cong \pi_n(BX) \\ \pi_n(B^Q\mathcal{Q}(X)) \cong \pi_n(B^{BQ}X) \end{cases} \text{for } n \geq 2.$$

## Theorem C

$$\Xi_X(D) = \tilde{\psi}_* \Xi_{\mathcal{Q}(X)}(D).$$