The algebraic Gordian distance of Seifert matrices

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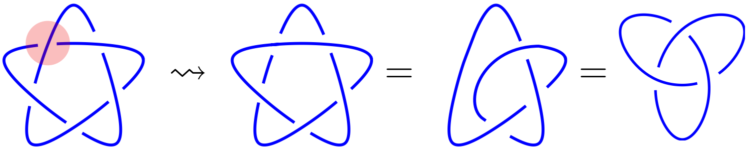
The Gordian distance

K, K': two knots in S^3.

Definition 1 (Gordian distance)

$$d_G(K, K') := \min\{\#\text{ of crossing changes from } K \text{ to } K'\}.$$



$$d_G(3_1, 5_1) = 1$$

Definition 2 (unknotting number)

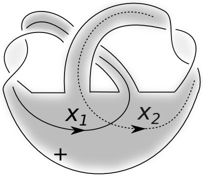
$$u(K) := d_G(K, O).$$

Seifert matrix

Definition 3 (Seifert matrix)

$V := (v_{ij})$, where $v_{ij} = \mathrm{lk}(x_i, x_j^+)$.

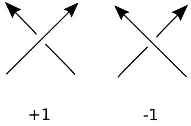
Seifert surface for 3_1 and its Seifert matrix



$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{cc} x_1^+ & x_2^+ \\ \end{array}$$

$$\begin{array}{c} x_1 \\ x_2 \end{array} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$\mathrm{lk}(x_i, x_j^+)$



Fact 4

V: a square integer matrix.

$$\det(V - V^T) = 1 \implies V \textit{ is a Seifert matrix.}$$

S-equivalence

V: a $2n \times 2n$ Seifert matrix.

Definition 5 (congruence)

V is *congruent* to V' $\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}P \in \mathrm{GL}_{2n}(\mathbb{Z})$, s.t. $V' = PVP^T$.

Definition 6 (enlargement/reduction)

V' is an *enlargement* of V, (V is a *reduction* of V')

$$\overset{\text{def}}{\Longleftrightarrow} V' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 \\ 0 & x & M \\ 0 & N^T & V \end{pmatrix},$$

- $x \in \mathbb{Z}$.
- M, N: row vectors.

Definition 7 (S-equivalence)

congruence, enlargement, reduction.

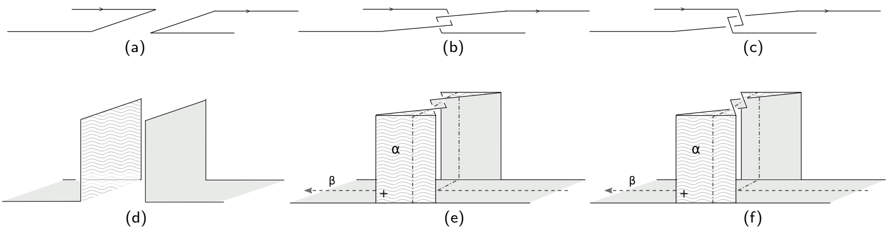
The algebraic unknotting operation

V: a Seifert matrix.

Definition 8 (algebraic unknotting operation, Murakami 1990)

$$V \xleftrightarrow{\text{algebraic unknotting operation}} \begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix},$$

- $\varepsilon = \pm 1$.
- $x \in \mathbb{Z}$.
- M, N: row vectors.



The algebraic Gordian distance

- V, V': two Seifert matrices.
- $[\cdot]$: S-equivalence class.

Definition 9 (Algebraic Gordian distance, Murakami 1990)

$$d_G^a([V],[V']) = \min_{\substack{W \in [V] \\ W' \in [V']}} \# \text{ of algebraic unknotting operations from } W \text{ to } W'.$$

- $[K] := [V]$, where V is a Seifert matrix of K.

Theorem 10 (Murakami 1990)

$$d_G(K,K') \geq d_G^a([K],[K']).$$

The Blanchfield pairing

- $\tilde{X}(K)$: infinite cyclic cover of $S^3 \setminus K$.
- $\Lambda := \mathbb{Z}[t, t^{-1}]$.
- V: a $2n \times 2n$ Seifert matrix of K.
- Alexander module $A_V := \Lambda^{2n}/(tV - V^T)\Lambda^{2n} \cong H_1(\tilde{X}(K); \mathbb{Z})$.
- A_V: Λ-module, where t acts on $\tilde{X}(K)$ as the deck transformation.
- $Q(\Lambda)$: quotient field of Λ.

Theorem 11 (Blanchfield 1959)

$$^{\exists}\beta_V : A_V \times A_V \longrightarrow Q(\Lambda)/\Lambda.$$

- $\beta_V(ax, by) = a\bar{b}\beta_V(x, y)$.
- $\bar{b} := b|_{t=t^{-1}}$.

Calculate the Blanchfield pairing

- V: a Seifert matrix of K.
- $A_V := H_1(\tilde{X}(K); \mathbb{Z})$.

Definition 12 (Blanchfield pairing)

$$\beta_V : A_V \times A_V \longrightarrow Q(\Lambda)/\Lambda$$

$$([x], [y]) \longmapsto \sum_{i \in \mathbb{Z}} \frac{\text{int}(X, t^i y) t^i}{p}$$

where $\partial_2 X = px$ for a 2-chain X, $p \in \Lambda$.

The Blanchfield pairing of a knot (Seifert matrix)

- $tV - V^T$: presentation matrix of A_V
  w.r.t. generators $\{a_1, a_2, \ldots, a_{2n}\}$.

Theorem 13 (Kearton 1975, Trotter 1973)

$$\begin{aligned}\beta_V : A_V \times A_V &\longrightarrow Q(\Lambda)/\Lambda \\ (a_i, a_j) &\longmapsto (i,j)\text{-entry of } (t-1)(tV - V^T)^{-1} \pmod{\Lambda}.\end{aligned}$$

Fact 14 (Trotter 1973)

$$(A_V, \beta_V) \cong (A_{V'}, \beta_{V'}) \Longleftrightarrow V \text{ is } S\text{-equivalent to } V'.$$

The Blanchfield pairing for $d_G^a([V],[V'])=1$

- V, V': two Seifert matrices.
- $\Delta_V := \det(t^{\frac{1}{2}}V - t^{-\frac{1}{2}}V^T)$.

Theorem 15 (C.)

$$d_G^a([V],[V'])=1 \quad \Longrightarrow \begin{cases} {}^\exists a \in A_V \quad s.t.\ \beta_V(a,a) \equiv \pm\dfrac{\Delta_{V'}}{\Delta_V} \pmod{\Lambda}, \\ {}^\exists a' \in A_{V'} \quad s.t.\ \beta_{V'}(a',a') \equiv \pm\dfrac{\Delta_V}{\Delta_{V'}} \pmod{\Lambda}. \end{cases}$$

Proof.

$$d_G^a([V],[V']) = 1 \implies {}^\exists W \in [V],\ {}^\exists W' \in [V'] \text{ s.t. } W = \begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N^T & W' \end{pmatrix}$$

$$\implies tW - W^T = \begin{pmatrix} \varepsilon(t-1) & -1 & 0 \\ t & x(t-1) & tM - N \\ 0 & tN^T - M^T & tW' - W'^T \end{pmatrix}.$$

$\beta_V(a_1, a_1) =$ the $(1,1)$-entry of matrix $(t-1)(tW - W^T)^{-1}$

$$\implies \beta_V(a_1, a_1) \equiv (t-1)\frac{\begin{vmatrix} x(t-1) & tM - N \\ tN^T - M^T & tW' - W'^T \end{vmatrix}}{|tW - W^T|}. \quad (1)$$

$$|tW - W^T| = \varepsilon(t-1)\begin{vmatrix} x(t-1) & tM - N \\ tN^T - M^T & tW' - W'^T \end{vmatrix} + t|tW' - W'^T|. \quad (2)$$

Substitute (2) from (1) $\implies \beta_V(a, a) \equiv \frac{\varepsilon(\Delta_V - \Delta_{V'})}{\Delta_V} \equiv -\varepsilon \frac{\Delta_{V'}}{\Delta_V} \pmod{\Lambda}$. □

Algebraic unknotting number one

Definition 16 (Algebraic unknotting number)

$$u_a([V]) := d_G^a([V],[O]).$$

Corollary 17 (C.)

$u_a([V]) = d_G^a([V],[V']) = 1 \implies {}^{\exists}c \in \Lambda \ s.t. \ \pm\Delta_{V'} \equiv c\bar{c} \pmod{\Delta_V}.$

Δ_K: the Alexander polynomial of K.

Theorem 18 (Kawauchi 2012)

$u(K) = d_G(K,K') = 1 \implies {}^{\exists}c \in \Lambda \ s.t. \ \pm\Delta_{K'} \equiv c\bar{c} \pmod{\Delta_K}.$

Corollary 18 (C.)

$u_a([V]) = d_G^a([V],[V']) = 1 \;\Rightarrow\; {}^{\exists}c \in \Lambda \; s.t. \; \pm\Delta_{V'} \equiv c\bar{c} \pmod{\Delta_V}.$

Proof.

$u_a([V]) = d_G^a([V],[V']) = 1 \Rightarrow^{\exists} a, {}^{\exists} h \in A_V$ s.t.

$$\beta_V(a,a) \equiv \pm\frac{\Delta_{V'}}{\Delta_V} \pmod{\Lambda} \quad \text{and} \quad \beta_V(h,h) \equiv \pm\frac{1}{\Delta_V} \pmod{\Lambda}.$$

Lemma 19 (Murakami 1990)

$u_a([V]) = 1 \Longrightarrow A_V$ *is generated by* g *s.t.* $\beta_V(g,g) \equiv \pm\frac{1}{\Delta_V} \pmod{\Lambda}.$

$\Longrightarrow^{\exists} c \in \Lambda$ s.t. $a = cg$.

$$\Longrightarrow \pm\frac{\Delta_{V'}}{\Delta_V} \equiv \beta_V(cg,cg) \equiv c\bar{c}\beta_V(g,g) \equiv \frac{c\bar{c}}{\Delta_V} \pmod{\Lambda}.$$

$\Longrightarrow \pm\Delta_{V'} = c\bar{c} \pmod{\Delta_V}.$ □

Alexander polynomial and algebraic Gordian distance

Corollary 20

- $u_a([V]) = 1$.
- $\Delta_V = h(t + t^{-1}) + 1 - 2h$. $|h|$: *prime or* 1.
- $\Delta_{V'} \equiv d \pmod{\Delta_V}$, *where* $d \in \mathbb{Z}$.
- $h^2x^2 + y^2 + (2h-1)xy = \pm d$ *does not have an integer solution* x, y.

$\Longrightarrow d_G^a([V], [V']) > 1$.

Proof.

Suppose $d_G^a([V], [V']) = 1 \Longrightarrow {}^{\exists}c \in \Lambda$ s.t. $c\bar{c} \equiv \pm\Delta_{V'} \equiv \pm d \pmod{\Delta_V}$.

Let $c =: \sum_{-n \le i \le m} a_i t^i \Longrightarrow c\bar{c} = a_{-n}a_m t^{m+n} + \cdots + a_m a_{-n} t^{-(m+n)}$.

$\Longrightarrow h|a_{-n}a_m \Longrightarrow h|a_{-n}$ or $h|a_m$. $\Longrightarrow$ can reduce m or n.

Repeat until $c \equiv pt^{j+1} + qt^j \pmod{\Delta_V}$, $p, q \in \mathbb{Z}$.

$\Longrightarrow (p^2 + q^2) + pq(t + t^{-1}) \equiv \pm d \pmod{\Delta_V} \Longrightarrow h|p$ or $h|q$.

Assume $p = hx \Longrightarrow h^2x^2 + q^2 + (2h-1)xq = \pm d$. □

Algebraic Unknotting number one

V: 2×2 Seifert matrix.

Proposition 21 (C.)

$\det V = D \in \{1, 2, 3, 5\} \implies u_a([V]) = 1.$

Proof.

V or $-V$ is positive definite. Assume V is positive definite.

$\implies V$ is congruent to $\begin{pmatrix} a & b+1 \\ b & c \end{pmatrix}$, $0 < 2b + 1 \leq \min(a, c)$

$\implies ac - b(b+1) = D \implies ac = D$

$\implies V = \begin{pmatrix} D & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -D & -1 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & D \end{pmatrix}$ or $\begin{pmatrix} -1 & -1 \\ 0 & -D \end{pmatrix}$. □

Corollary 22 (C.)

$\Delta_V = ht + ht^{-1} + 1 - 2h,\ h \in \{1, 2, 3, 5\} \implies u_a([V]) = 1.$

The Alexander polynomial distance

Δ_K: the Alexander polynomial of K.

Definition 23 (Alexander polynomial distance)

$$\rho(\Delta, \Delta') := \min_{\substack{K, K' \\ \Delta_K = \Delta \\ \Delta_{K'} = \Delta'}} d_G(K, K').$$

Fact 24 (Kawauchi 2012)

$$\Delta \neq \Delta' \Longrightarrow \rho(\Delta, \Delta') = 1 \textit{ or } 2.$$

Lemma 25 (C.)

$$d_G^a([K], [K']) \geq \rho(\Delta_K, \Delta_{K'}).$$

$\therefore d_G(K, K') \geq d_G^a([K], [K']) \geq \rho(\Delta_K, \Delta_{K'})$.

The Alexander polynomial distance

Question (Jong)

Find Δ and Δ' s.t. $\rho(\Delta, \Delta') = 2$.

- $\Delta_{3_1} = \quad t + t^{-1} - 1.$
- $\Delta_{4_1} = -t - t^{-1} + 3.$

Question (Nakanishi)

$\rho(\Delta_{3_1}, \Delta_{4_1}) = 2\,?$ Yes. (Kawauchi 2012)

Theorem 26 (Kawauchi 2012)

$A_h := h(t + t^{-1}) - 2h + 1$. *p: prime.* $p \nmid h$, $p \nmid m$

$$\Longrightarrow \rho(A_h, A_{h+mp^{2s+1}}) = 2.$$

Corollary 27 (C.)

$\Delta \equiv 2 + 4m \pmod{t - 1 + t^{-1}} \Longrightarrow \rho(t - 1 + t^{-1}, \Delta) = 2.$

Proof.

Recall:

Corollary 20

- $u_a([V]) = 1$
- $\Delta_V = h(t + t^{-1}) + 1 - 2h.$ $|h|$: *prime or* 1 $\Delta_{V'} \equiv d \pmod{\Delta_V}$
- $h^2x^2 + y^2 + (2h - 1)xy = \pm d$ *does not have an integer solution* x, y

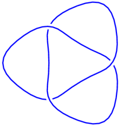
$\Longrightarrow d_G^a([V], [V']) > 1.$

Corollary 22

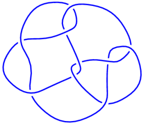
$\Delta_V = ht + ht^{-1} + 1 - 2h,\ h \in \{1, 2, 3, 5\} \Longrightarrow u_a([V]) = 1.$

$x^2 + y^2 + xy = 2 + 4m$ does not have an integer solution x, y. □

Example



$\Delta_{3_1} = t + t^{-1} - 1$



$\Delta_{9_{25}} = -3t^2 - 3t^{-2} + 12t + 12t^{-1} - 17$

$$^{\forall}K_1, ^{\forall}K_2, \Delta_{K_1} = \Delta_{3_1}, \Delta_{K_2} = \Delta_{9_{25}}$$

$$\Longrightarrow d_G(K_1, K_2) \geq d_G^a([K_1],[K_2]) \geq \rho(\Delta_{3_1}, \Delta_{9_{25}}) = 2.$$

Calculation of algebraic Gordian distances

$$u_a([9_{25}]) = u_a([3_1]) = 1 \Longrightarrow d_G^a([3_1],[9_{25}]) \leq u_a([3_1]) + u_a([9_{25}]) = 2$$

$$\Longrightarrow d_G^a([3_1],[9_{25}]) = 2.$$

Calculation of Gordian distances

Fact: $d_G(3_1, 9_{25}) \leq 2 \quad \Longrightarrow \quad d_G(3_1, 9_{25}) = 2.$