On adequacy
and the crossing number of satellite knots

Adrián Jiménez Pascual
The University of Tokyo

Tokyo Woman's Christian University

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Agenda

Preliminaries

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Link adequacy

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Link parallels

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Cable knots

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Main result

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Summary

Definition (*Satellite knot*)

P: knot in ST. (*Pattern*)
C: knot in $\mathbb{S}^3$ with framing 0. (*Companion*)
$e : ST \hookrightarrow N(C)$: faithful embedding.
Then eP is called a *satellite knot* (of C). From here on $eP =: Sat(P, C)$.

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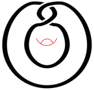
P   C

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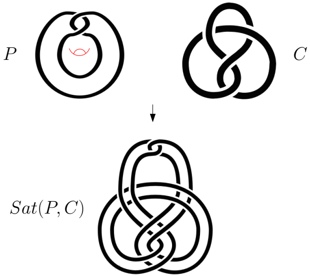

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GOAL

Problem 1.65 (*Kirby, 1995*)

Is the crossing number $cr(K)$ of a knot K additive with respect to connected sum, that is, is the equality $cr(K_1 \# K_2) = cr(K_1) + cr(K_2)$ true?

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- $cr(K_1 \# \ldots \# K_n) \geq \frac{cr(K_1) + \ldots + cr(K_n)}{152}$. *(Lackenby, 2011)*

Link adequacy

Definition

A *state* of a link is a function

$$s : \{c_1, c_2, ..., c_n\} \to \{-1, 1\}.$$

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The *Kauffman bracket* of a link with diagram D can be written as:

$$\langle D \rangle = \sum_s \left(A^{\sum_{i=1}^n s(i)} (-A^{-2} - A^2)^{|sD|-1} \right).$$

- s_+ is the state for which $\sum_{i=1}^{n} s_+(i) = n$
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D is *plus-adequate* if $|s_+ D| > |sD|$ for all s with $\sum_{i=1}^{n} s(i) = n - 2$.
D is *minus-adequate* if $|s_- D| > |sD|$ for all s with $\sum_{i=1}^{n} s(i) = -n + 2$.
D is *adequate* if *plus-adequate* and *minus-adequate*.

Lemma 1 (*Lickorish*)

Let D be a link diagram with n crossings.

1. $M_{\langle D \rangle} \leq n + 2|s_+ D| - 2$, with equality if D is plus-adequate,
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Corollary 1 (*Lickorish*)

If D is adequate:

$$B(\langle D\rangle) = M_{\langle D\rangle} - m_{\langle D\rangle} = 2n + 2|s_+ D| + 2|s_- D| - 4.$$

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Lemma 2 (*Lickorish*)

Let D be a connected link diagram with n crossings.

$$|s_+ D| + |s_- D| \leq n + 2,$$

with equality if D alternating.

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Theorem 1 (*Lickorish*)

Let D be a connected, n-crossing diagram of an oriented link L.

1. $B(J(L)) \leq n$,
2. if D is alternating and reduced, $B(J(L)) = n$.

Link parallels

Definition

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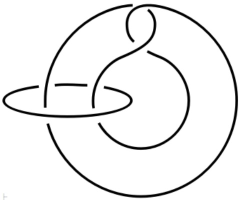

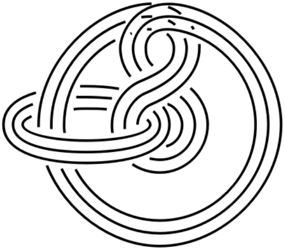

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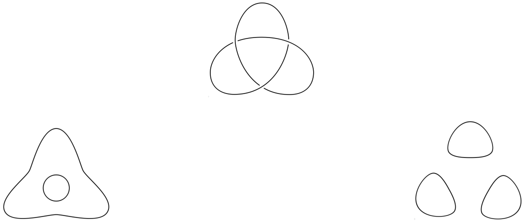

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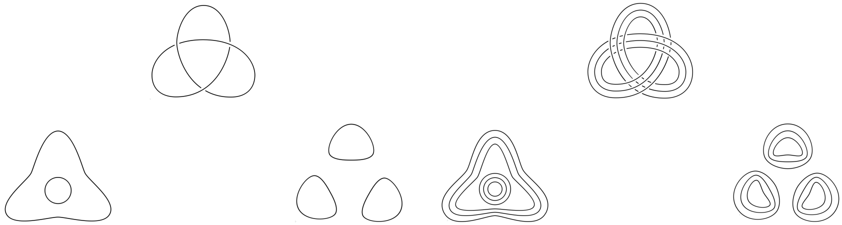

We construct the parallel versions of the given results.

Lemma 1-*bis* (*JP*)

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Corollary 1-*bis*

If D is adequate:

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If L is alternating:

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Proof uses *Theorem 2* and *Lemma 2*.

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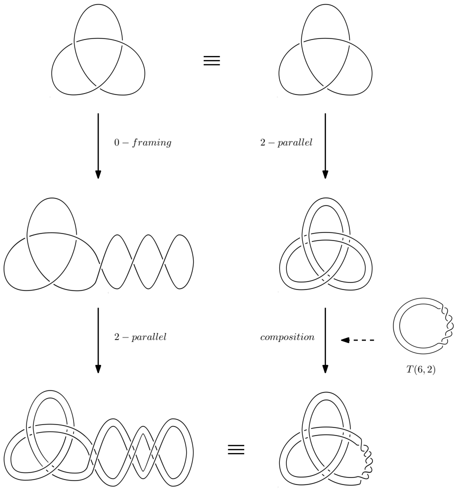

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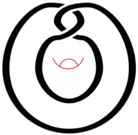
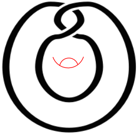
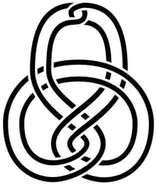
P   C

P   C 

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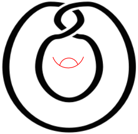

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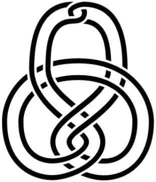

Theorem 3

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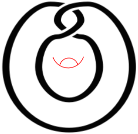
P 

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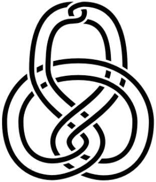

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Corollary 3

In particular,

$$B(J(Sat(P, C))) \geq B(\beta_M J(C; M)).$$

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Problem 1.65 (*Kirby, 1995*)

Is the crossing number $cr(K)$ of a knot K additive with respect to connected sum, that is, is the equality $cr(K_1 \# K_2) = cr(K_1) + cr(K_2)$ true?

- *Murasugi* proved it is true for adequate knots. (Also *Kauffman* and *Thistlethwaite*)

Problem 1.67 (*Kirby, 1995*)

Is the crossing number of a satellite knot bigger than that of its companion?

Remarks: *"Surely the answer is yes, so the problem indicates the difficulties of proving statements about the crossing number."*

- It is true for adequate knots.

*(Arxiv: 1712.05635)

On adequacy and the crossing number of satellite knots

Adrián Jiménez Pascual
The University of Tokyo

Tokyo Woman's Christian University

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