Dijkgraaf-Witten invariants of cusped hyperbolic 3-manifolds

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1 Introduction

In 1990 Dijkgraaf and Witten [4] introduced a topological invariant of closed oriented 3-manifolds using a finite group and its 3-cocycle. Let M be a closed oriented 3-manifold, G a finite group and $\alpha \in Z^3(BG, U(1))$. Then the Dijkgraaf-Witten invariant $Z(M)$ (we abbreviate it to the DW invariant in this paper) is defined as follows:

$$Z(M) = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M), G)} \langle \gamma^*[\alpha], [M] \rangle \in \mathbb{C}.$$

The topological invariance of $Z(M)$ is obvious from the definition and it is also evident that $Z(M)$ is a homotopy invariant since M only appears at the fundamental group and the fundamental class in the definition of $Z(M)$.

Dijkgraaf and Witten reformulated the invariant by using a triangulation of M in the following way. Let K be a triangulation of M. Then the fundamental class of M is described by the sum of the tetrahedra of K and $\gamma \in \mathrm{Hom}(\pi_1(M), G)$ is represented by assigning an element of G to each edge of K. $Z(M)$ is described as follows:

$$Z(M) = \frac{1}{|G|^a} \sum_{\varphi \in \mathrm{Col}(K)} \prod_{\text{tetrahedron}} \alpha(g, h, k)^{\pm 1},$$

where a is the number of the vertices of K and $g, h, k \in G$ are colors of edges of a tetrahedron of K.

The Turaev-Viro invariant [6] is well known as a state sum invariant of compact 3-manifolds constructed by using a triangulation. Due to the above construction of $Z(M)$ by using a triangulation, we can view the DW invariant as the "Turaev-Viro type" invariant. However, one of the big difference between the constructions of these two invariants is that an orientation of each edge of a triangulation of M is essential in constructing the DW invariant, meanwhile it is unnecessary in constructing the Turaev-Viro invariant. This difference makes us expect that the DW invariants may distinguish some pairs of 3-manifolds with the same Turaev-Viro invariants.

2 Definition of the Dijkgraaf-Witten invariant

First we review the group cohomology briefly. Let G be a finite group. The n-cochain group $C^n(G, U(1))$ is defined as follows:

$$C^n(G,U(1)) = \begin{cases} U(1) & (n=0) \\ \{\alpha : \overbrace{G \times \cdots \times G}^{n} \to U(1)\} & (n \geq 1). \end{cases}$$

The group operation of $C^n(G, U(1))$ is a multiplication of maps induced by the multiplication of $U(1)$ and then $C^n(G, U(1))$ is a multiplicative abelian group since $U(1)$ is so. The n-coboundary map $\delta^n : C^n(G, U(1)) \to C^{n+1}(G, U(1))$ is defined by

$$(\delta^0 a)(g) = 1 \quad (a \in U(1),\ g \in G),$$

$$(\delta^n \alpha)(g_1, \dots, g_{n+1}) = \alpha(g_2, \dots, g_{n+1}) \times \prod_{i=1}^{n} \alpha(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})^{(-1)^i}$$
$$\times \alpha(g_1, \dots, g_n)^{(-1)^{n+1}},$$
$$(\alpha \in C^n(G, U(1)),\ g_1, \dots, g_{n+1} \in G,\ n \geq 1).$$

Then we can confirm by the above definition that $\{(C^n(G, U(1)), \delta^n)\}_{n=0}^{\infty}$ is a cochain complex. Hence the n-cocycle group $Z^n(G, U(1))$ and the n-th cohomology group $H^n(G, U(1))$ are defined as usual.
$\alpha \in C^n(G, U(1))$ is said to be normalized if for any $g_1, \dots, g_n \in G$, α satisfies $\alpha(1, g_2, \dots, g_n) = \alpha(g_1, 1, g_3, \dots, g_n) = \cdots = \alpha(g_1, \dots, g_{n-1}, 1) = 1 \in U(1)$.
We assume that any n-cochain is normalized. As we only consider 3-cocycles in the following of this paper, we restate the cocycle condition for a 3-cocycle α.

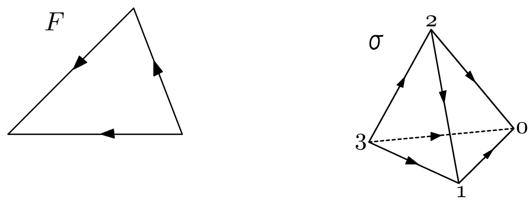


Figure 1: A local order for a face and for a tetrahedron.

$$\alpha(h,k,l)\alpha(g,hk,l)\alpha(g,h,k) = \alpha(gh,k,l)\alpha(g,h,kl) \quad (g,h,k,l \in G).$$

The cocycle condition takes a important role in the proof of the invariance of the DW invariant.

Next we explain a local order and a coloring. Fix a triangulation K of M. Give an orientation to each edge of K such that for any 2-face F of K, the orientations of the three edges of F are not cyclic (as the left side of Figure 1). Then each tetrahedron σ of K has one of each vertex incident to i outgoing edges of σ and to $(3-i)$ incoming edges of σ for $i = 0,1,2,3$ (as the right side of Figure 1). Hence a total order on the set of the vertices of each tetrahedron σ by the number of outgoing edges of σ is induced by the choice of the orientations. We call a choice of the orientations of edges of K *a local order of* K (or *a branching of* K).

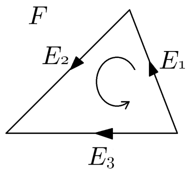
Fix a triangulation K of M. Give an orientation to each edge (which is unnecessarily a local order) or each face of K. *A coloring* φ of K is a map $\varphi : \{\text{oriented edges of } K\} \to G$ such that for any oriented 2-face F, $\varphi(E_3)^{\epsilon_3}\varphi(E_2)^{\epsilon_2}\varphi(E_1)^{\epsilon_1} = \varphi(E_2)^{\epsilon_2}\varphi(E_1)^{\epsilon_1}\varphi(E_3)^{\epsilon_3} = \varphi(E_1)^{\epsilon_1}\varphi(E_3)^{\epsilon_3}\varphi(E_2)^{\epsilon_2} = 1$, where E_1, E_2, and E_3 are the oriented edges of F and

$$\epsilon_i = \begin{cases} 1 & \text{the orientation of } E_i \text{ agrees with that of } \partial F \\ -1 & \text{otherwise.} \end{cases}$$

The above condition for a coloring φ is required because a coloring φ originally comes from $\gamma \in \mathrm{Hom}(\pi_1(M), G)$. $\mathrm{Col}(K)$ denotes the set of the colorings of K.

An orientation of each edge of K is required in settling a local order of K and a coloring of K, which is one of the big difference from the construction of the Turaev-Viro invariant.

Now we define the DW invariant. Let M be a closed oriented 3-manifold,



$\epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = -1.$

Figure 2: The sign of edges.

G a finite group and $\alpha \in Z^3(G, U(1))$. Fix a triangulation K of M with a local order. Put a coloring φ of K, and then some element $\varphi(E)$ of G is assigned to each oriented edge E of each tetrahedron σ. We call $\varphi(E)$ the color of E and such a tetrahedron σ the colored tetrahedron, denoted by (σ, φ).

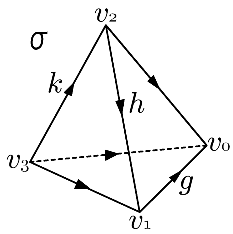


Figure 3: A colored tetrahedron.

Let v_0, v_1, v_2, v_3 be the vertices of σ with $v_0 < v_1 < v_2 < v_3$ by the local order (v_i is incident to i outgoing edges of σ). Put $\varphi(\langle v_0 v_1 \rangle) = g$, $\varphi(\langle v_1 v_2 \rangle) = h$, $\varphi(\langle v_2 v_3 \rangle) = k$. Correspond $\alpha(g, h, k)^\epsilon \in U(1)$ to the colored tetrahedron (σ, φ). We call $W(\sigma, \varphi) = \alpha(g, h, k)^\epsilon$ the symbol of the colored tetrahedron (σ, φ). As well as $g, h, k \in G$, ϵ is determined by the local order. As previously stated, a local order determines a total order on the set of the vertices of each tetrahedron σ. Furthermore the total order determined by the local order settles an orientation of σ. On the other hand, since M is an oriented 3-manifold, an orientation of σ is induced by that of M. We define ϵ appeared in the symbol of (σ, φ) as follows:

$$\epsilon = \begin{cases} 1 & \text{the orientation of } \sigma \text{ by the local order agrees with the orientation of } M \\ -1 & \text{otherwise.} \end{cases}$$

Theorem 2.1. *Let M be a closed oriented 3-manifold, G a finite group and $\alpha \in Z^3(G, U(1))$. Let K be a triangulation of M with a local order. Let $\sigma_1, \ldots, \sigma_n$ be the tetrahedra of K and a the number of the vertices of K. The Dijkgraaf-Witten invariant $Z(M)$ is defined as follows:*

$$Z(M) = \frac{1}{|G|^a} \sum_{\varphi \in \mathrm{Col}(K)} \prod_{i=1}^{n} W(\sigma_i, \varphi).$$

Then $Z(M)$ is independent of the choice of a triangulation K of M with a local order.

Remark 2.2. In general some triangulation K of M does not admit a local order. Nevertheless, for any closed oriented 3-manifold M, M has a triangulation which admits a local order. For example, a simplicial triangulation K of M admits a local order determined by a total order on the set of the vertices of K.

Remark 2.3. $Z(M)$ only depends on the cohomology class of $\alpha \in Z^3(G, U(1))$. Furthermore, $Z(-M) = \overline{Z(M)}$, where $-M$ is the oriented 3-manifold with the opposite orientation to M.

For a compact oriented 3-manifold M with $\partial M \neq \emptyset$, the DW invariant of M is defined by using a triangulation K of M and a local order of K in the same way as the closed case. However, for $\partial M \neq \emptyset$ case, the DW invariant of M is determined not only by M but also by a triangulation of ∂M and its coloring.

3 Generalization of the Dijkgraaf-Witten invariant

We construct another version of the DW invariant for compact oriented 3-manifolds with non-empty boundary, which we call the generalized DW invariant.

Let M be a compact oriented 3-manifold with boundary. We consider a triangulation of M with ideal vertices such that each boundary component of M converges at an ideal vertex. We call such a triangulation of M with ideal vertices *a generalized ideal triangulation* of M in this paper. In general, a generalized ideal triangulation K of M has both interior vertices and ideal vertices. If $\partial M = \emptyset$, K has no ideal vertices, that is, K is an ordinary triangulation of a closed 3-manifold M. On the other hand, an ideal triangulation is a generalized ideal triangulation without interior vertices.

Let K be a generalized ideal triangulation of M with a local order. Put a coloring φ of K and assign each colored tetrahedron (σ, φ) to the symbol $W(\sigma, \varphi) = \alpha(g, h, k)^{\epsilon}$ by the local order. Then we define the generalized DW invariant in the same way as the original DW invariant for closed 3-manifolds.

Theorem 3.1. *Let M be a compact oriented 3-manifold with boundary, G a finite group and $\alpha \in Z^3(G, U(1))$. Let K be a generalized ideal triangulation of M with a local order. Let $\sigma_1, \ldots, \sigma_n$ be the tetrahedra of K and a the number of the interior vertices of K. The generalized Dijkgraaf-Witten invariant $Z(M)$ is defined as follows:*

$$Z(M) = \frac{1}{|G|^a} \sum_{\varphi \in \mathrm{Col}(K)} \prod_{i=1}^{n} W(\sigma_i, \varphi).$$

Then $Z(M)$ is independent of the choice of a generalized ideal triangulation K of M with a local order.

By using a generalized ideal triangulation K of M, each component of ∂M corresponds to an ideal vertex of K. Hence, the generalized DW invariant of M does not need a triangulation of ∂M nor its coloring. For a closed 3-manifold M, since K has no ideal vertices, the generalized DW invariant of M is no other than the original DW invariant of M defined in Theorem 2.2.

The proof of Theorem 3.1 consists of two parts, the independence of a choice of a local order of a fixed generalized ideal triangulation K of M and the independence of a choice of a generalized ideal triangulation K of M.

The outline of the proof of the indepednece of a choice of a local order is as follows. Let K_1 be a generalized ideal triangulation of M obtained by barycentric subdivision (once) of each tetrahedron of K. We consider the following local order of K_1:

(vertex of K) $<$ (midpoint of an edge of K) $<$ (center of a face of K)
$<$ (center of a tetrahedron of K).

We prove that $Z(M)$ defined by K with an arbitrarily fixed local order coincides with $Z(M)$ defined by K_1 with the above local order, which implies the independence of the choice of a local order of K.

Next we explain the proof of the independence of the choice of a generalized ideal triangulation K of M. In order to show that, we make use of the following theorem by Pachner.

Theorem 3.2 (Pachner)**.** *Any two triangulations of a 3-manifold M can be transformed one to another by a finite sequence of the following two types of transformations shown in Figure 4.*

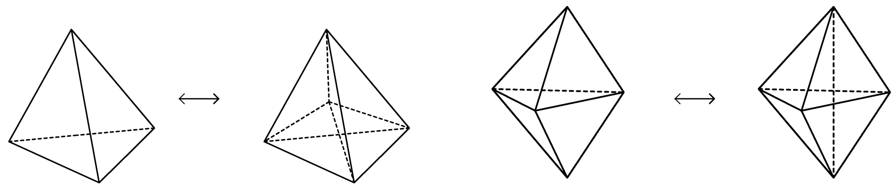


Figure 4: The Pachner moves.

Owing to Theorem 3.2, it suffices to show the invariance of $Z(M)$ under a (1,4)-Pachner move and a (2,3)-Pachner move respectively.

We can show Theorem 2.1 in the same way as Theorem 3.1.

Although we introduce a generalized ideal triangulation in the definition of the generalized DW invariant, in fact it suffices to consider ideal triangulations of M.

Corollary 3.3. *Let M be a compact oriented 3-manifold with $\partial M \neq \emptyset$ or a cusped oriented hyperbolic 3-manifold, G a finite group and $\alpha \in Z^3(G, U(1))$. Let K be an ideal triangulation of M with a local order. Let $\sigma_1, \ldots, \sigma_n$ be the ideal tetrahedra of K. The generalized Dijkgraaf-Witten invariant $Z(M)$ is described by the following form:*

$$Z(M) = \sum_{\varphi \in \mathrm{Col}(K)} \prod_{i=1}^{n} W(\sigma_i, \varphi).$$

Then $Z(M)$ is independent of the choice of an ideal triangulation K of M with a local order.

We present the following theorem proved by Matveev [5; Theorem 1.2.27].

Theorem 3.4 (Matveev)**.** *Any two ideal triangulations of a 3-manifold M can be transformed one to another by a finite sequence of the following two types of transformations shown in Figure 5.*

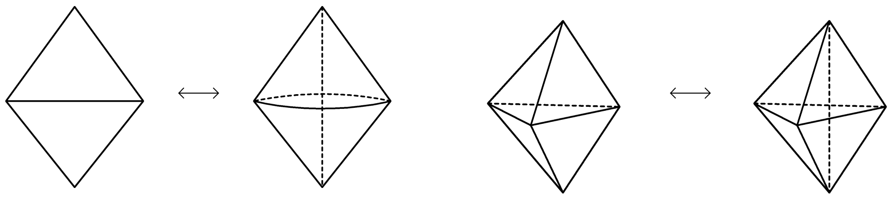


Figure 5: The Pachner moves for ideal triangulations.

We call a (2,3)-Pachner move that increases the number of the ideal tetrahedra *a positive (2,3)-Pachner move* in this paper. In general, a given ideal triangulation of M may not admit a local order. However Benedetti and Petronio proved the existence of an ideal triangulation with a local order [1; Theorem 3.4.9].

Theorem 3.5 (Benedetti-Petronio)**.** *Let M be a compact oriented 3-manifold with boundary and K an ideal triangulation of M. Then there exists a finite sequence of ideal triangulations of M, $K = K_0 \to K_1 \to \cdots \to K_n$, such that K_i is transformed to K_{i+1} by a positive (2,3)-Pachner move and K_n admits a local order.*

4 Examples of cusped hyperbolic 3-manifolds

We calculate the generalized DW invariants of some cusped orientable hyperbolic 3-manifolds. We show that the generalized DW invariants distinguish some pairs of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and with the same Turaev-Viro invariants. We also present an example of a pair of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and with the same homology groups, whereas with distinct generalized DW invariants.

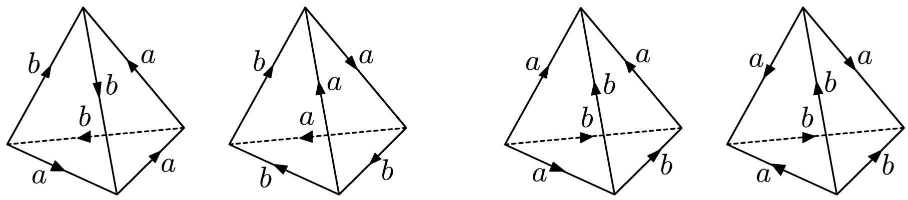


Figure 6: Minimal ideal triangulations of $m003$ and $m004$.

(1) $m003$ and $m004$

According to Regina [2] and SnapPy [3], $m003$ and $m004$ are cusped orientable 3-manifolds with the minimal ideal triangulations shown in Figure 6. The 3-manifold $m004$ is the figure eight knot complement. Their hyperbolic volumes, Turaev-Viro inavariants and homology groups are as follows:

Vol($m003$) = Vol($m004$) ≈ 2.02988,

$$TV(m003) = \sum_{(a,a,b),(a,b,b)\in adm} w_a w_b \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix} \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix} = TV(m004),$$

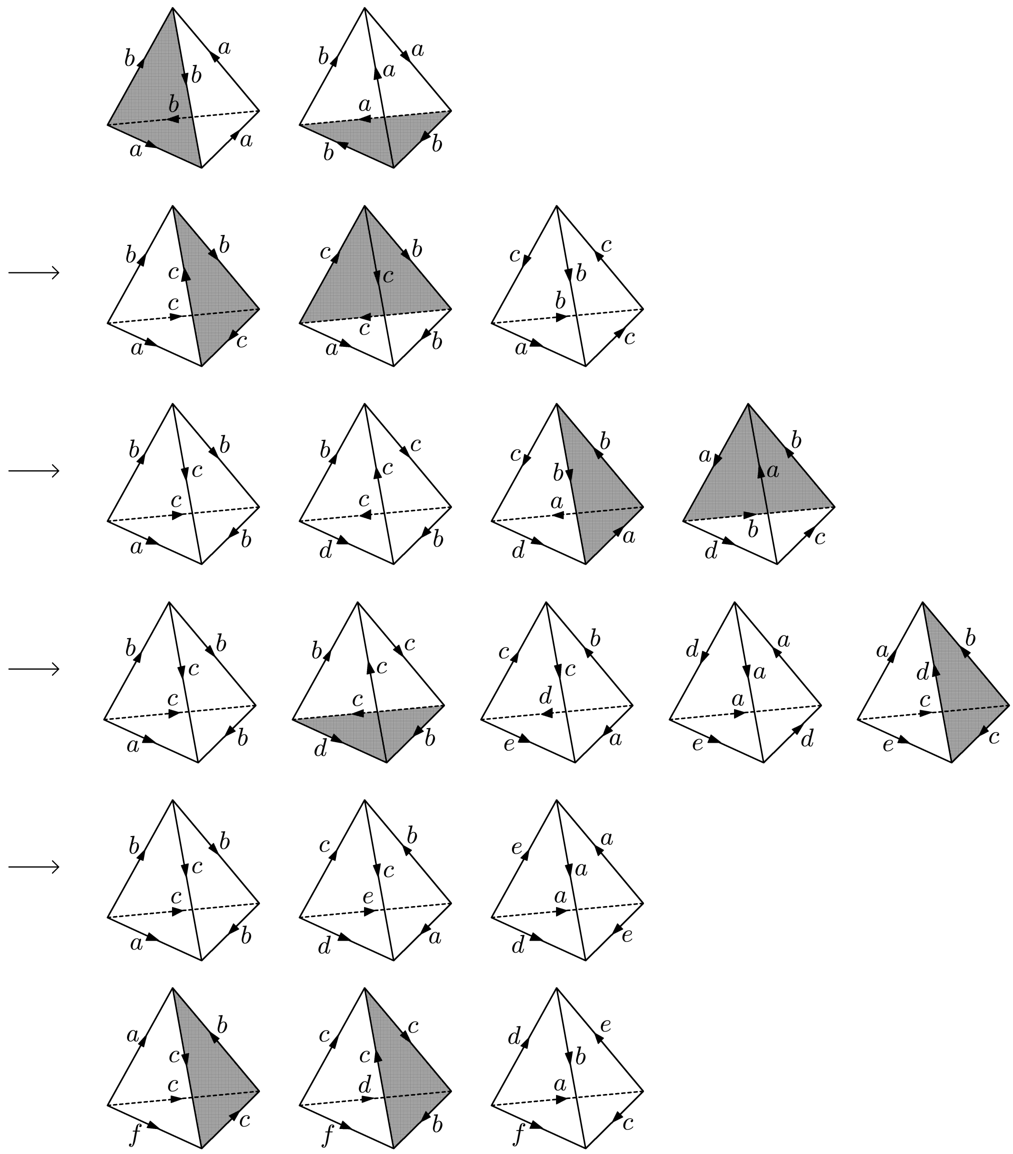
$H_1(m003;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5,\ \ H_1(m004;\mathbb{Z}) = \mathbb{Z}.$

We show that $m003$ and $m004$ have distinct generalized DW invariants.

First we calculate the generalized DW invariant of $m004$. The minimal triangulation of $m004$ admits the local order shown in Figure 6. Identify the labels of edges with the colors of edges. By the left front face of the left tetrahedron of $m004$ shown in Figure 6, $a = ba$. By the right front face of the left tetrahedron of $m004$, $b = ab$. Hence $a = b = 1 \in G$, which implies $m004$ has only a trivial coloring. Therefore, for any finite group G and its any normalized 3-cocycle α,

$$Z(m004) = 1.$$

On the other hand, the minimal ideal triangulation of $m003$ shown in Figure 6 does not admit a local order. In order to assign a local order, transform the ideal triangulation of $m003$ by (2,3)-Pachner moves.



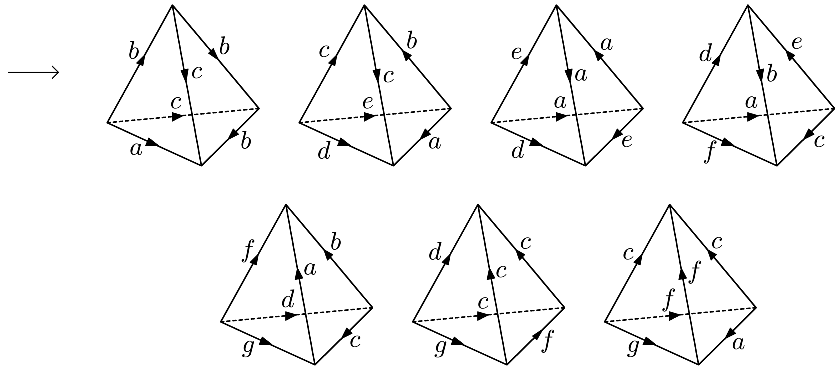


Figure 7: A sequence of (2,3)-Pachner moves for $m003$ to obtain a locally ordered ideal triangulation.

After (2,3)-Pachner moves five times, the ideal triangulation of $m003$ which consists of seven ideal tetrahedra admits the local order shown in Figure 7. The relations between the colors of edges are the following:
$a = b^3$, $c = b^2$, $d = b^4$, $e = b$, $f = 1$, $g = b^2$, $b^5 = 1$.

$$Z(m003) = \sum_{b \in G, b^5=1} \alpha(b,b,b)^{-1}\alpha(b^2,b,b)\alpha(b^3,b^3,b^3)$$
$$\times\alpha(b,b,b^3)\alpha(b,b^2,b^2)\alpha(b^2,b^3,b^2).$$

In order to confirm $Z(m003) \neq Z(m004)$, we calculate $Z(m003)$ for $G = \mathbb{Z}_5$ and a generator α of $H^3(\mathbb{Z}_5, U(1)) \cong \mathbb{Z}_5$.

$$Z(m003) = \frac{1}{2}(5+\sqrt{5}+i\sqrt{10+2\sqrt{5}}).$$

Hence the generalized DW invariants distinguish $m003$ and $m004$.

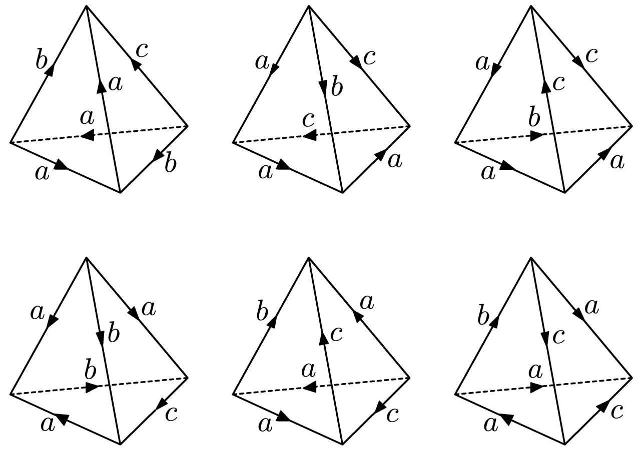


Figure 8: Minimal ideal triangulations of $m006$ and $m007$.

(2) $m006$ and $m007$

According to Regina [2] and SnapPy [3], $m006$ and $m007$ are cusped orientable 3-manifolds with the minimal ideal triangulations shown in Figure 8.

$\mathrm{Vol}(m006) = \mathrm{Vol}(m007) \approx 2.56897$,

$$TV(m006) = \sum w_a w_b w_c \begin{vmatrix} a & b & c \\ a & b & a \end{vmatrix} \begin{vmatrix} a & b & c \\ a & c & a \end{vmatrix} \begin{vmatrix} a & b & c \\ a & c & a \end{vmatrix} = TV(m007),$$

$H_1(m006;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5,\ H_1(m007;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_3.$

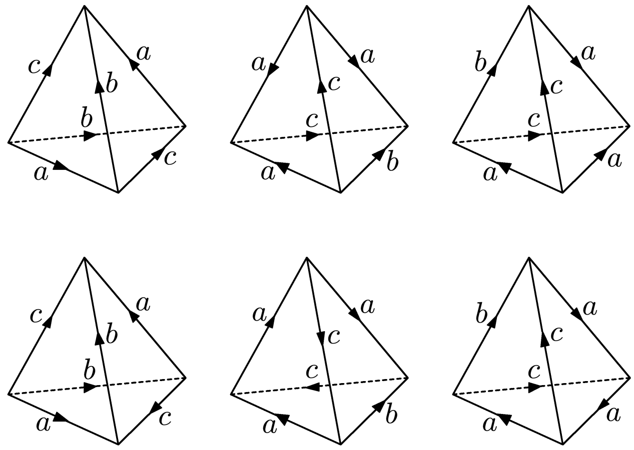
The generalized DW invariants of $m006$ and $m007$ are as follows.

$$Z(m006) = \sum_{a \in G, a^5=1} \alpha(a,a,a)^3 \alpha(a,a^2,a)\alpha(a^3,a^3,a^3).$$

$$Z(m007) = \sum_{a \in G, a^3=1} \alpha(a,a,a)\alpha(a^{-1},a^{-1},a^{-1}).$$

If $G = \mathbb{Z}_5$ and α is a generator of $H^3(\mathbb{Z}_5, U(1)) \cong \mathbb{Z}_5$,

$$\mathrm{Re}(Z(m006)) = -\frac{\sqrt{5}}{2}, \quad Z(m007) = 1.$$



$m009$

$m010$

Figure 9: Minimal ideal triangulations of $m009$ and $m010$.

(3) $m009$ and $m010$

According to Regina [2] and SnapPy [3], $m009$ and $m010$ are cusped orientable 3-manifolds with the minimal ideal triangulations shown in Figure 9.

$\mathrm{Vol}(m009) = \mathrm{Vol}(m010) \approx 2.66674$,

$$TV(m009) = \sum w_a w_b w_c \begin{vmatrix} a & b & c \\ a & b & c \end{vmatrix} \begin{vmatrix} a & b & c \\ a & a & c \end{vmatrix} \begin{vmatrix} a & b & c \\ a & a & c \end{vmatrix} = TV(m010),$$

$H_1(m009;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2,\ H_1(m010;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_6.$

The generalized DW invariants of $m009$ and $m010$ are as follows.

$$Z(m009) = \sum_{a \in G, a^2=1} \alpha(a,a,a).$$

$$Z(m010) = \sum_{b,c \in G, b^3=1, c^2=1, bc=cb} \alpha(b,b,b)^{-1}\alpha(b,b^{-1},b)\alpha(b,c,c)\alpha(cb,c,c)^{-1}.$$

If $G = \mathbb{Z}_3$ and α is a generator of $H^3(\mathbb{Z}_3, U(1)) \cong \mathbb{Z}_3$,

$$Z(m009) = 1, \quad Z(m010) = -\sqrt{3}i.$$

In fact the previous three pairs of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and the same Turaev-Viro invariants are distinguished by their homology groups. The following pair of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and the same homology groups have distinct generalized DW invariants.

$s778$

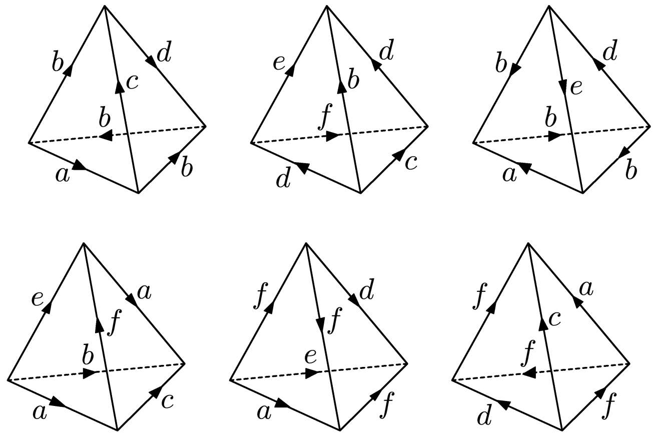


Figure 10: A minimal ideal triangulation of $s778$.

$s788$

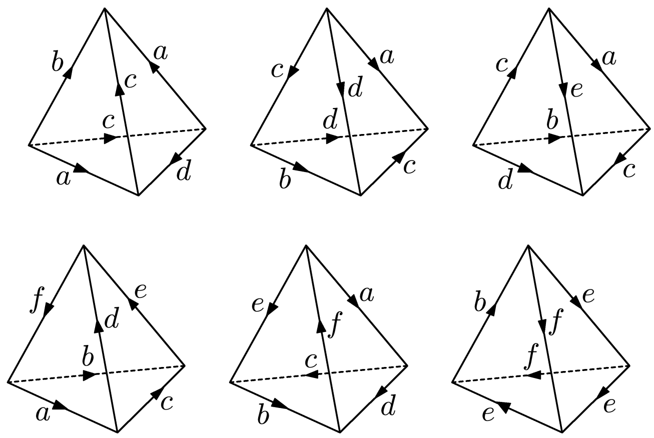


Figure 11: A minimal ideal triangulation of $s788$.

(4) $s778$ and $s788$

According to Regina [2] and SnapPy [3], $s778$ and $s788$ are cusped orientable 3-manifolds with the minimal ideal triangulations shown in Figure 10 and 11 respectively.

$\mathrm{Vol}(s778) = \mathrm{Vol}(s788) \approx 5.33349$,

$H_1(s778;\mathbb{Z}) = H_1(s788;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_{12}$.

The generalized DW invariants of $s778$ and $s788$ are as follows.

$$Z(s778) = \sum_{d \in G, d^{12}=1} \alpha(d,d,d^2)\alpha(d^2,d,d)\alpha(d^2,d,d^2)\alpha(d^3,d^2,d^3)\alpha(d^3,d^{10},d^3) \times\alpha(d^5,d^5,d^{10})\alpha(d^{10},d^5,d^5)\alpha(d^{10},d^5,d^{10}).$$

$$Z(s788) = \sum_{a \in G, a^{12}=1} \alpha(a^5,a,a^2)\alpha(a^6,a^2,a^3)\alpha(a^8,a,a^2)\alpha(a^8,a,a^8)^{-1}\alpha(a^8,a^5,a^8)^{-1} \times\alpha(a^8,a^9,a^8)^{-1}\alpha(a^9,a^5,a^3)^{-1}\alpha(a^9,a^8,a)^{-1}\alpha(a^9,a^9,a^5).$$

If $G = \mathbb{Z}_{12}$ and α is a generator of $H^3(\mathbb{Z}_{12}, U(1)) \cong \mathbb{Z}_{12}$,

$$Z(s778) = -6, \quad Z(s788) = 3 - 2\sqrt{3}.$$

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