On a generalization of the Fox formula for twisted Alexander invariants

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Abstract. We present a generalization of the Fox formula for twisted Alexander invariants associated to representations of knot groups over rings of S-integers of F, where S is a finite set of finite primes of a number field F. As an application, we give the asymptotic growth of twisted homology groups.

This report is based on [Tan].

1 Introduction

The Fox formula is one of the important results in knot theory which expresses the order of the first integral homology group of the n-fold cyclic cover branched over a knot in terms of the Alexander polynomial. Let K be a knot in the 3-sphere S^3, M_n the n-fold cyclic cover branched over K, and $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ the Alexander polynomial of K. If $\Delta_K(t)$ and $t^n - 1$ have no common roots in $\mathbb{C}$, then the Fox formula is given by

$$\# H_1(M_n; \mathbb{Z}) = \left| \prod_{i=1}^{n} \Delta_K(\zeta_n^i) \right|,$$

where $\# G$ denotes the order of a group G and ζ_n is a primitive n-th root of unity ([Fox56]). As an application, it follows immediately from the Fox formula that the asymptotic growth of integral homology groups holds:

$$\lim_{n \to \infty} \frac{1}{n} \log(\# H_1(M_n, \mathbb{Z})) = \log \mathbb{M}(\Delta_K(t)),$$

where $\mathbb{M}(\Delta_K(t))$ is the Mahler measure of $\Delta_K(t)$ ([Mah62]). We remark that this asymptotic growth may be seen as an analogue of the Iwasawa asymptotic formula for p-ideal class groups in a $\mathbb{Z}_p$-extension, p being a prime number ([Iwa59]). The analogies with number theory are the motivation of our study ([Mor12]).

Recently, a generalization of the Alexander polynomial, called a *twisted Alexander invariant*, which was introduced by Lin ([Lin01]) and Wada ([Wad94]), is playing an important role in knot theory ([FV11]). It is known ([KL99], [SW09]) that the twisted Alexander invariant relates to the twisted homology

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group of a knot complement associated to a $\mathrm{GL}_m(R)$-representation of a knot group G_K, where R is a Noetherian UFD.

The purpose of this report is to consider a generalization of the Fox formula for twisted Alexander invariants. As an application, we give the asymptotic growth formulas of twisted homology groups. We remark that these asymptotic growth formulas may be seen as analogues of the asymptotic formula for the Tate–Shafarevich groups or the Selmer groups of p-adic Galois representations in a $\mathbb{Z}_p$-extension, which was firstly studied by Mazur ([Maz72]).

Number theory	**Knot theory**
Iwasawa asymptotic formula for p-ideal class groups	asymptotic growth formula for knot modules
asymptotic formula for Tate–Shafarevich/Selmer groups	asymptotic growth formula for twisted knot modules

Notation. For an integral domain A, we denote by $Q(A)$ the field of fractions of A. For a, b in a commutative ring A, $a \doteq b$ means $a = bu$ for some unit $u \in A^{\times}$. For a field F, we denote $\overline{F}$ the algebraic closure of F. For positive integers m, n, and for a finite set of finite primes $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of a number field F, $m =_S n$ means $m = np_1^{e_1} \cdots p_r^{e_r}$ for some integers $e_1, \ldots, e_r \in \mathbb{Z}$, where $(p_i) = \mathfrak{p}_i \cap \mathbb{Z}$. Note that $m =_S n$ if and only if $|m|_p = |n|_p$ for all $(p) \notin \{(p_1), \ldots, (p_r)\}$, where $|\cdot|_p$ is the p-adic absolute value normalized by $|p|_p = p^{-1}$.

2 Twisted Alexander invariants

The twisted Alexander invariant $\Delta_{K,\rho}(t)$ is defined as follows ([GKM06], [Wad94]).

Let K be a knot in the 3-sphere S^3 and let $X := S^3 \setminus K$ denote the knot complement of K. Let $G_K := \pi_1(X)$ denote the knot group of K. Note that the knot group G_K has a Wirtinger presentation

$$\langle g_1, \ \ldots, g_q \mid r_1, \ \ldots, r_{q-1} \rangle.$$

Let R be a Noetherian UFD. Let F_0 be the free group on the words $g_1, \ldots, g_q$ and $\pi : R[F_0] \to R[G_K]$ denote the natural surjective homomorphism of group rings. We write the same g_i for the image of g_i in G_K. We denote by the same α for the R-algebra homomorphism $R[G_K] \to R[t^{\pm 1}]$, which is induced by the abelianization homomorphism. $\alpha : G_K \to G_K^{\mathrm{ab}} \simeq \mathbb{Z} = \langle t \rangle$.

Let us denote by the same ρ for the R-algebra homomorphism $R[G_K] \to \mathrm{M}_m(R)$, which is induced by a representation $\rho : G_K \to \mathrm{GL}_m(R)$. Then we have the tensor product representation

$$\rho \otimes \alpha : R[G_K] \longrightarrow \mathrm{M}_m(R[t^{\pm 1}]),$$

and the R-algebra homomorphism

$$\Phi := (\rho \otimes \alpha) \circ \pi : R[F_0] \longrightarrow \mathrm{M}_m(R[t^{\pm 1}]).$$

Let us consider the (big) $(q-1)\times q$ matrix P, whose (i,j) component is defined by the $m\times m$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial g_j}\right),$$

where $\frac{\partial}{\partial g_j}: R[F_0]\to R[F_0]$ denotes the Fox derivative ([Fox53]) over R extended from $\mathbb{Z}$. For $1\le j\le q$, let P_j denote the matrix obtained by deleting the j-th column from P and we regard P_j as an $(q-1)m\times(q-1)m$ matrix over $R[t^{\pm1}]$. It is known ([Wad94]) that there is k $(1\le k\le q)$ such that $\det(\Phi(g_k-1))\neq 0$ and that the ratio

$$\Delta_{K,\rho}(t) := \frac{\det(P_k)}{\det\Phi(g_k-1)} \in Q(R)(t)$$

is independent of such k's. We call $\Delta_{K,\rho}(t)$ the *twisted Alexander invariant* of K associated to ρ.

Similarly to the classical case, there is a relation between the twisted Alexander invariant and the order ideals of ρ-twisted Alexander module (cf. [KL99], [SW09]). Let us recall the definition of the order ideal. Let M be a finitely generated R-module. Let us take a finite presentation of M over R:

$$R^m \xrightarrow{\partial} R^n \longrightarrow M \longrightarrow 0,$$

where ∂ is an $m\times n$ matrix over R. We define the *order ideal* $E_0(M)$ of M to be the ideal generated by n-minors of ∂. The order ideal depends only on M and independent of the choice of a presentation. Let $\Delta_0(M)$ be the greatest common divisor of generators of $E_0(M)$, which is well-defined up to multiplication by a unit of R.

Proposition 1 ([KL99], [SW09]) *Let $X_\infty\to X$ be the infinite cyclic cover of X. For any representation $\rho: G_K\to \mathrm{GL}_m(R)$, we have*

$$\Delta_{K,\rho}(t) = \frac{\Delta_0(H_1(X_\infty;V_\rho))}{\Delta_0(H_0(X_\infty;V_\rho))}.$$

In particular, when $\rho: G_K\to\mathrm{GL}_m(R)$ is irreducible over a commutative UFD R, we have the following Corollary. We say that $\rho: G_K\to\mathrm{GL}_m(R)$ is *irreducible* over a commutative UFD R if the composite of ρ with the natural map $\mathrm{GL}_m(R)\to\mathrm{GL}_m(\mathbb{F}(\mathfrak{p}))$ is irreducible over the residue field $\mathbb{F}(\mathfrak{p})=R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of R, where $R_\mathfrak{p}$ is the localization of R at $\mathfrak{p}$.

Corollary 2 *Let $\rho: G_K\to\mathrm{GL}_m(A)$ be an irreducible representation over a PID A. Then we have*

$$\Delta_0(H_0(X_\infty;V_\rho)) \doteq 1 \in A[t^{\pm1}],$$

and hence

$$\Delta_{K,\rho}(t) \doteq \Delta_0(H_1(X_\infty;V_\rho)).$$

In particular, $\Delta_{K,\rho}(t)$ is a Laurent polynomial over A.

In order to prove Corollary 2, we use the following Lemma.

Lemma 3 ([DFJ12, Lemma 2.5]) *Let $\rho_{\mathbb{F}} : G_K \to \mathrm{GL}_m(\mathbb{F})$ be an irreducible representation over a field $\mathbb{F}$. Then we have*

$$\Delta_0(H_0(X_\infty; V_{\rho_{\mathbb{F}}})) \doteq 1 \in \mathbb{F}[t^{\pm 1}].$$

3 Twisted Wang sequences

In this Section, we formulate an analogue of Wang sequence for twisted homology groups. We keep the same notations as before. Recall that R is a Noetherian UFD.

Let $X_\infty \to X$ be the infinite cyclic cover of X. Note that the covering transformation $t : X_\infty \to X_\infty$ induces the action of $\langle t \rangle \simeq \mathbb{Z}$ on $C_*(X_\infty; V_\rho)$ defined by the following:

$$\begin{array}{rcl} t_\# : C_*(X_\infty; V_\rho) & \to & C_*(X_\infty; V_\rho); \\ (r(t) \otimes \boldsymbol{v}) \otimes z & \mapsto & (r(t) \cdot t \otimes \boldsymbol{v}) \otimes z, \end{array}$$

Moreover, note that for the n-fold cyclic cover $X_n \to X$, the covering $p_n : X_\infty \to X_n$ induces the following map:

$$\begin{array}{rcl} p_{n\#} : C_*(X_\infty; V_\rho) & \to & C_*(X_n; V_\rho); \\ (r(t) \otimes \boldsymbol{v}) \otimes z & \mapsto & (r(t_n) \otimes \boldsymbol{v}) \otimes z, \end{array}$$

where $\langle t_n \rangle = \mathbb{Z}/n\mathbb{Z}$.

Lemma 4 *Let $\rho : G_K \to \mathrm{GL}_m(R)$ be a representation. Then we have an exact sequence*

$$0 \to C_*(X_\infty; V_\rho) \stackrel{t_\#^n - 1}{\to} C_*(X_\infty; V_\rho) \stackrel{p_{n\#}}{\to} C_*(X_n; V_\rho) \to 0.$$

Note that Lemma 4 induces the following long exact sequence, which is called the *twisted Wang sequence*:

$$\cdots \to H_1(X_\infty; V_\rho) \stackrel{t_*^n - 1}{\to} H_1(X_\infty; V_\rho) \stackrel{p_{n*}}{\to} H_1(X_n; V_\rho) \stackrel{\partial_{1*}}{\to} H_0(X_\infty; V_\rho) \to \cdots.$$

Hence, when $H_0(X_\infty; V_\rho) = 0$, we have the following relation between $H_1(X_n; V_\rho)$ and $H_1(X_\infty; V_\rho)$:

Proposition 5 *Let $\rho : G_K \to \mathrm{GL}_m(R)$ be a representation. Assume that $H_0(X_\infty; V_\rho) = 0$. Then we have*

$$H_1(X_n; V_\rho) \simeq H_1(X_\infty; V_\rho)/(t^n - 1)H_1(X_\infty; V_\rho).$$

As we discussed in Section 2, when $\rho : G_K \to \mathrm{GL}_m(A)$ is an irreducible representation over a PID A, we have $\Delta_0(H_0(X_\infty; V_\rho)) \doteq 1$, and so by [Hil12, Theorem 3.1], we have $H_0(X_\infty; V_\rho) = 0$. Therefore, we have the following Corollary.

Corollary 6 *Let $\rho : G_K \to \mathrm{GL}_m(A)$ be an irreducible representation over a PID A. Then we have*

$$H_1(X_n; V_\rho) \simeq H_1(X_\infty; V_\rho)/(t^n - 1)H_1(X_\infty; V_\rho).$$

4 Resultants

In this Section, we recall the definition of the resultant and state the relation between the resultant and the order ideal.

Let A be an integral domain. Consider the following two non-zero polynomials in $A[t]$ factor in $\overline{Q(A)}$:

$$f = f(t) = a\prod_{i=1}^{m}(t-\xi_i),\ \ g = g(t) = b\prod_{j=1}^{n}(t-\zeta_j).$$

Then we define the *resultant* $\mathrm{Res}(f,g)$ for polynomials f and g by

$$\mathrm{Res}(f,g) := a^m b^n \prod_{i,j}(\xi_i-\zeta_j) = a^m \prod_i g(\xi_i).$$

For polynomials f, $g \in A[t]$, it is easy to see that $\mathrm{Res}(f,g)=0$ if and only if f and g have a common root in $\overline{Q(A)}$. In addition, the resultant is symmetric up to the sign and is multiplicative ([Lan02, IV.8, IX.3]):

$$\mathrm{Res}(f,g) = (-1)^{\deg(f\cdot g)}\mathrm{Res}(g,f),$$

$$\mathrm{Res}(f,g\cdot h) = \mathrm{Res}(f,g)\cdot\mathrm{Res}(f,h),$$

where f, g, $h \in A[t^{\pm 1}]$. The resultant can be generalized for Laurent polynomials since it is insensitive to units ut^i with $u \in A^\times$ and $i \in \mathbb{Z}$.

The following Lemmas claim that when R is a Noetherian UFD, the greatest common divisor of generators of the order ideal of finitely generated torsion $R[t^{\pm 1}]$-module is computable by using the resultant. Note that we say the Laurent polynomial in $R[t^{\pm 1}]$ is *doubly monic* if the highest and lowest coefficients are units in R.

Lemma 7 ([Hil12, Theorem 3.13]) *Let R be a Noetherian UFD and N a finitely generated torsion $R[t^{\pm 1}]$-module. Let $f(t) \in R[t^{\pm 1}]$ be a doubly monic polynomial. Then $N/f(t)N$ is a torsion R-module if and only if $\Delta_0(N)|_{t=\zeta} \neq 0$ for all non-zero roots ζ of $f(t)$ in $\overline{Q(R)}$.*

Lemma 8 ([Hil12, Corollary 3.13.1]) *Let R be a Noetherian UFD and let $f(t)$, $g(t) \in R[t^{\pm 1}]$ having no common roots in $\overline{Q(R)}$. If $f(t)$ or $g(t)$ is doubly monic, then*

$$\Delta_0(R[t^{\pm 1}]/(f(t),g(t))) \doteq \mathrm{Res}(f(t),g(t)).$$

5 Number theoretic lemmas

Let us recall some facts and notions in number theory, which we shall use in the following. We refer to [MR03, 6.1] and [Ono90, 2.8-2.10] for these materials.

Let F be a number field. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be a finite set of finite primes of F and $\mathcal{O}_{F,S}$ the ring of S-integers, namely

$$\mathcal{O}_{F,S} := \{a \in F \mid v_{\mathfrak{p}}(a) \geq 0 \text{ for all } \mathfrak{p} \in S_F \setminus S\},$$

where $v_{\mathfrak{p}}$ is an additive valuation of F at $\mathfrak{p}$, and S_F is the set of all finite primes of F. It is known that $\mathcal{O}_{F,S}$ is always Noetherian and if we take a sufficiently large finite set T of finite primes of F containing S, then $\mathcal{O}_{F,T}$ turns out to be a PID. Therefore, we may always take S so that $\mathcal{O}_{F,S}$ is a PID. For $a \in F$, we define the *norm* $\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$ *of* a by

$$\mathrm{N}_{F/\mathbb{Q}}(a) := \prod_{\sigma} \sigma(a),$$

where σ runs over all embeddings of F in $\mathbb{C}$. For an integral ideal I of $\mathcal{O}_F$, we define the *norm* $\mathrm{N}I$ by $\#\mathcal{O}_F/I$. It is extended multiplicatively for a fractional ideal of $\mathcal{O}_F$. For $a \in F^{\times}$, we have $|\mathrm{N}_{F/\mathbb{Q}}(a)| = \mathrm{N}(a)$, where $(a) = a\mathcal{O}_F$ is the principal ideal generated by a. We say that a fractional ideal J of $\mathcal{O}_F$ is *prime to* S if any prime factor of J is not in S.

Lemma 9 *For $a \in \mathcal{O}_{F,S} \setminus \{0\}$, we have*

$$\#\mathcal{O}_{F,S}/a\mathcal{O}_{F,S} =_S |\mathrm{N}_{F/\mathbb{Q}}(a)|.$$

Using these norms and Lemma 9, we have the following Lemmas. The proof of Lemma 11 is a generalization of [Web79].

Lemma 10 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let $f(t),\ g(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$ and assume that either $f(t)$ or $g(t)$ is doubly monic. Then $\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t))$ is a torsion $\mathcal{O}_{F,S}$-module if and only if $f(t)$ and $g(t)$ have no common roots in $\overline{F}$. When $f(t)$ and $g(t)$ have no common roots in $\overline{F}$, we have*

$$\#\mathcal{O}_{F,S}[t^{\pm 1}]/(f(t), g(t)) =_S |\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(f(t), g(t)))|.$$

Lemma 11 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let N be a finitely generated $\mathcal{O}_{F,S}[t^{\pm 1}]$-module having no submodule of finite length with $\mathrm{rank}_F(N \otimes_{\mathcal{O}_{F,S}} F) < \infty$. Then $N/(t^n - 1)N$ is a torsion $\mathcal{O}_{F,S}$-module if and only if $\Delta_0(N)$ and $t^n - 1$ have no common roots in $\overline{F}$. When $\Delta_0(N)$ and $t^n - 1$ have no common roots in $\overline{F}$, we have*

$$\#N/(t^n - 1)N =_S |\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(t^n - 1,\ \Delta_0(N)))|.$$

6 Fox formulas for twisted Alexander invariants

In this Section, we formulate an analogue of the Fox formula for twisted Alexander invariants associated to $\mathrm{GL}_m(\mathcal{O}_{F,S})$-representations of knot groups under the assumption $H_0(X_\infty; V_\rho) = 0$. The proof is a generalization of [Cro63].

Theorem 12 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let $\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$ be a representation. Assume that $H_0(X_\infty; V_\rho) = 0$ and let $\Delta_{K,\rho}(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$ be the twisted Alexander invariant of K associated to ρ. If $\Delta_{K,\rho}(t) \neq 0$, and $\Delta_{K,\rho}(t)$ and $t^n - 1$ have no common roots in $\overline{F}$, then we have*

$$\# H_1(X_n; V_\rho) =_S \left| \mathrm{N}_{F/\mathbb{Q}} \left(\prod_{i=1}^{n} \Delta_{K,\rho}(\zeta_n^i) \right) \right|,$$

where ζ_n is a primitive n-th root of unity.

In particular, when $\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$ is irreducible, we have the following Corollary.

Corollary 13 *Let F be a number field. Let S be a finite set of finite primes of F so that the ring of S-integers $\mathcal{O}_{F,S}$ is a PID. Let $\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$ be an irreducible representation, and let $\Delta_{K,\rho}(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$ be the twisted Alexander invariant of K associated to ρ. If $\Delta_{K,\rho}(t)$ and $t^n - 1$ have no common roots in $\overline{F}$, then we have*

$$\# H_1(X_n; V_\rho) =_S \left| \mathrm{N}_{F/\mathbb{Q}} \left(\prod_{i=1}^{n} \Delta_{K,\rho}(\zeta_n^i) \right) \right|,$$

where ζ_n is a primitive n-th root of unity.

7 Asymptotic growth of twisted homology groups

We keep the notation as in Section 6. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Assume that $H_0(X_\infty; V_\rho) = 0$, and $\Delta_{K,\rho}(t)$ and $t^n - 1$ have no common roots in $\overline{F}$ for all positive integers n. Set $\overline{\Delta_{K,\rho}}(t) := \mathrm{N}_{F/\mathbb{Q}}(\Delta_{K,\rho}(t)) \in \mathbb{Z}_{S_0}[t^{\pm 1}]$, where $S_0 = \{\mathfrak{p}_1 \cap \mathbb{Z}, \ldots, \mathfrak{p}_r \cap \mathbb{Z}\}$, and $\mathbb{Z}_{S_0}$ is the ring of S_0-integers of $\mathbb{Q}$. Then by Theorem 12, we have

$$\# H_1(X_n, V_\rho) =_S \prod_{i=1}^{n} \left| \overline{\Delta_{K,\rho}}(\zeta_n^i) \right|. \tag{7.1}$$

As we remarked in Notation, when $(p) \notin S_0$, (7.1) is equivalent to

$$|\# H_1(X_n, V_\rho)|_p = \prod_{i=1}^{n} \left| \overline{\Delta_{K,\rho}}(\zeta_n^i) \right|_p.$$

Here, $|\cdot|_p$ is the p-adic absolute value on $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$, where $\mathbb{C}_p$ is the p-adic completion of an algebraic closure of the p-adic number field.

For $f(t) \in \mathbb{Z}_{S_0}[t^{\pm 1}]$, we define the *Mahler measure* $\mathbb{M}(f(t))$ of $f(t)$ ([Mah62]) by

$$\mathbb{M}(f(t)) := \exp \left(\int_0^1 \log |f(e^{2\pi\sqrt{-1}x})| dx \right).$$

If $f(t)$ factors as $f(t) = at^e \prod_{j=1}^{d}(t-\xi_j)$ in $\mathbb{C}$, then by Jensen's formula, we have $\mathbb{M}(f(t)) = a\prod_{j=1}^{d}\max(|\xi_j|, 1)$. For $f(t) \in \mathbb{C}_p[t^{\pm 1}] \setminus \{0\}$ with no root on roots of unity, we define the *Ueki p-adic Mahler measure* $\mathbb{M}_p(f(t))$ of $f(t)$ ([Uek17]) by

$$\mathbb{M}_p(f(t)) := \exp\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log|f(e^{\frac{2\pi\sqrt{-1}}{n}i})|_p\right).$$

Now we are ready to state our Theorem.

Theorem 14 *When* $(p) \notin S_0$*, we have*

$$\lim_{n\to\infty}\frac{1}{n}\log|\#H_1(X_n, V_\rho)|_p = \log\mathbb{M}_p(\overline{\Delta_{K,\rho}}(t)).$$

In particular, when S *is the empty set, we have*

$$\lim_{n\to\infty}\frac{1}{n}\log(\#H_1(X_n, V_\rho)) = \log\mathbb{M}(\overline{\Delta_{K,\rho}}(t)).$$

Remark 15 Theorem 14 is a generalization of the result by González-Acuña and Short ([GAnS91]), and by Noguchi ([Nog07]), where the case ρ is a trivial representation over $\mathbb{Z}$ was studied.

8 Example

Let K be the figure-eight knot, whose knot group is given by

$$G_K = \langle g_1, g_2 \mid g_1g_2^{-1}g_1^{-1}g_2g_1 = g_2g_1g_2^{-1}g_1^{-1}g_2\rangle.$$

Consider the following representation:

$$\rho : G_K \to \mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right); \quad \rho(g_1) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ \rho(g_2) = \begin{pmatrix}1 & 0\\ \frac{1+\sqrt{-3}}{2} & 1\end{pmatrix},$$

where $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is the ring of integers of $\mathbb{Q}(\sqrt{-3})$. Then we have $\Delta_{K,\rho}(t) = \frac{1}{t^2}(t^2-4t+1) \doteq t^2-4t+1$ and hence by applying Corollary 13, we have the following:

$$\begin{aligned}
\#H_1(X; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-2) = 4,\\
\#H_1(X_2; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-12) = 144,\\
\#H_1(X_3; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-50) = 2500,\\
\#H_1(X_4; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-192) = 36864.
\end{aligned}$$

Since $\overline{\Delta_{K,\rho}}(t) = (t^2-4t+1)^2$, by Theorem 14, we have

$$\lim_{n\to\infty}\frac{1}{n}\log(\#H_1(X_n, V_\rho)) = 2\log(2+\sqrt{3}).$$

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