COMPLEXES INDUCED FROM SPHERICAL CURVES AND DISTANCES DERIVED FROM THEM

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ABSTRACT. In this paper, we introduce a distance $\tilde{d}_{w3}$ on the equivalence classes of spherical curves under deformations of type RI and ambient isotopies. We obtain an inequality that estimate its lower bound (Theorem 1). In Theorem 2, we show that if for a pair of spherical curves P and P', $\tilde{d}_{w3}([P],[P'])$ $= 1$ and P and P' satisfy a certain technical condition, then P' is obtained from P by a single weak RIII only. In Theorem 3, we show that if P and P' satisfy other conditions, then P' is ambient isotopic to a spherical curve that is obtained from P by a sequence of a particular local deformations, which realizes $\tilde{d}_{w3}([P],[P'])$.

1. INTRODUCTION

A spherical curve is the image of a generic immersion of a circle into a 2-sphere. Any two spherical curves can be transformed each other by a finite sequence of deformations, each of which is either one of type RI, type RII, or type RIII that is a replacement of a part of the curve contained in a disk as in Figure 1, and ambient isotopies. These deformations are obtained from Reidemeister moves of type Ω_1,

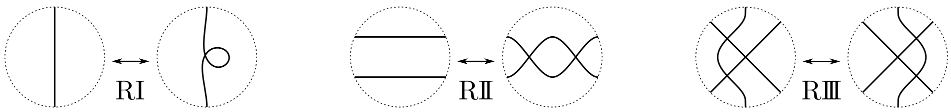


FIGURE 1. Reidemeister moves: type RI, type RII, and type RIII.

type Ω_2, and type Ω_3 on knot diagrams by ignoring over/under information.

Viro [8] suggested to decompose RIII into the following two types: suppose that P_1 is transformed into P_2 by a deformation of type RIII. Note that in RIII of Figure 1, a triangle is observed in each of the disks. We say that P_1 and P_2 are *related by a strong RIII* if the orientations on the edges of the triangles induced by an orientation of the spherical curves are coherent. If the orientations are not coherent, then we say that P_1 and P_2 are *related by a weak RIII*. See Figure 2.

Let $\mathcal{C}$ be the set of ambient isotopy classes of spherical curves. We say that two elements v and v' of $\mathcal{C}$ are *RI-equivalent*, denoted by $v \sim_{RI} v'$ if there are representatives P, P' of v, v' respectively such that P' is obtained from P by a sequence of deformations of type RI and ambient isotopies. We note that $\sim_{RI}$ is an equivalence relation on $\mathcal{C}$ (see the proof of Proposition 1). Then $\tilde{\mathcal{C}}$ denotes the quotient set $\mathcal{C}/\sim_{RI}$ and for a spherical curve P, $[P]$ denotes the quotient containing P. Then we obtain a 1-complex, denoted by $\tilde{\mathcal{K}}_{w3}$ by:

Key words and phrases. spherical curve; Reidemeister move; chord diagram.

FIGURE 2. Strong RⅢ (left) and weak RⅢ (right) in a disk D. Dotted curves indicate the connection patterns of six points on ∂D.

- the set of vertices of $\tilde{\mathcal{K}}_{w3}$ corresponds to $\tilde{\mathcal{C}}$, and
- two vertices v and v' are joined by an edge if there are representatives P and P' of v and v' respectively such that P' is obtained from P by a sequence consisting of one deformation of type weak RⅢ, and some (possibly, empty) deformation(s) of type RI and ambient isotopies.

Then $\tilde{d}_{w3}$ denotes the path-metric distance of $\tilde{\mathcal{C}}$ induced by $\tilde{\mathcal{K}}_{w3}$. That is, for each pair of vertices v and v', we define $\tilde{d}_{w3}$ by:

$\tilde{d}_{w3}(v, v') = \min\{\text{the number of the edges of } J \mid J : \text{ path in } \tilde{\mathcal{K}}_{w3} \text{ joining } v \text{ and } v'\}$

Let P and P' be spherical curves. We say that P and P' are *equivalent under RI and weak RⅢ* if there are spherical curves $\tilde{P}$ and $\tilde{P}'$ such that $\tilde{P}$ is ambient isotopic to P, $\tilde{P}'$ is ambient isotopic to P' and $\tilde{P}'$ is obtained from $\tilde{P}$ by a sequence of deformations each of which is of type RI or type weak RⅢ. Then, we define $d_{w3}(P, P')$ by: $d_{w3}(P, P')$

$$= \min \left\{ \begin{array}{l|l} \text{the number of weak RⅢ('s)} & \tilde{P} \text{ and } \tilde{P}' \text{ are equivalent under RI and} \\ \text{in the sequence of} & \text{weak RⅢ, } \tilde{P} \text{ is ambient isotopic to } P, \\ \text{deformations from } \tilde{P} \text{ to } \tilde{P}' & \text{and } \tilde{P}' \text{ is ambient isotopic to } P' \end{array} \right\}$$

If P and P' are not equivalent under RI and weak RⅢ, let $d_{w3}(P, P') = \infty$. Maybe the next proposition is well-known to the experts, but it will be worth to give a concrete statement for understanding the proofs of the theorems of this paper.

Proposition 1. *For any pair of spherical curves P, P' which are equivalent under RI and weak RⅢ, we have:*

$$d_{w3}(P, P') = \tilde{d}_{w3}([P], [P'])$$

For a proof of this proposition, see Appendix.

Some equivalence classes under RI and weak RⅢ are studied by [1, 2, 3, 4]. Let $\mathcal{P}_{\leq 7}$ be the set of the ambient isotopy classes of all of the prime spherical curves with at most seven double points. In Figure 45 of [3], there is a diagram consisting of certain elements of $\mathcal{P}_{\leq 7}$, which describes how the elements are related by a weak or strong RⅢ. The elements are named after the notations in Rolfsen's table [7]. However, we should note that the spherical curves of the diagram are treated up to mirror image. (For example, the spherical curve denoted 7_6 is transformed into the mirror image of 6_3 in the diagram, and not transformed into 6_3 itself by a single deformation of type weak RⅢ and some deformations of type RI, nevertheless the diagram in [3] says that 7_6 is transformed into 6_3 by a single weak RⅢ and some

RI's.) For a spherical curve P, we use P^* to denote the mirror image of P. Figure 3 is a similar diagram obtained from all of the elements of $\mathcal{P}_{\leq 7}$. (We note that, by using elementary geometric arguments together with Fact 3 in Section 3, it is easy to show that 6_3 is not ambient isotopic to 6_3^*, that 7_6 is not ambient isotopic to 7_6^*, that 7_B is not ambient isotopic to 7_B^*, and that $\mathcal{P}_{\leq 7}$ consists of the 21 elements in Figure 3. A systematic proof of this fact will be found in [5].) In [1], an idea for detecting spherical curves which are not equivalent under RI and weak RIII by using positive knot diagrams is introduced (for details, see [1, Corollary 3.2]). By using the idea, it is elementary to show that the quotient set of $\mathcal{P}_{\leq 7}$ under RI and weak RIII consists of nine or ten equivalence classes. The ambiguity "nine or ten" had come from the issue that whether 7_5 and 7_C in Figure 3 are equivalent under RI and weak RIII, or not. Later, it was shown that 7_5 and 7_C are equivalent under RI and weak RIII with passing through spherical curves each with eight double points [4, Figure 6], and this shows that the quotient set consists of nine elements, depicted in Figure 3.

In this paper, as a sequel of these reserches, we study the distance $\tilde{d}_{w3}$. We obtain an inequality that estimate its lower bound (Theorem 1). Further, we show that if $\tilde{d}_{w3}([P],[P']) = 1$ and P and P' satisfy a certain technical condition, then P' is obtained from P by a single weak RIII only (Theorem 2). Theorem 2 can be regarded as a kind of "rigidity" of the deformations from P to P'. In Theorem 3, we give a similar result under a different setting. We say that P' is obtained from P by a *deformation of type* α, if P' is obtained by replacing a part of P contained in a disk as in Figure 4. Then we show that if the pair P, P' satisfies other conditions, then P' is obtained from P by applying a sequence of deformations of type α and ambient isotopies, and this sequence realizes $\tilde{d}_{w3}([P],[P'])$ (Theorem 3).

Further by using the results, we study the distance of each pair of elements in the equivalence classes of Figure 3. We note that we can show, by Remark 3 of Section 3, that the spherical curves in Figure 3 are not mutually RI-equivalent.

Since each of the equivalence classes (1), (4), (5), (7), (9) has only one element, this problem is trivial for these equivalence classes. Since each of the equivalence classes (2), (3) consists of two elements and they are related by a single weak RIII and RI, this problem is solved for these equivalence classes. For the equivalence class (8), we will see that $\tilde{d}_{w3}([7_5],[7_C]) > 1$ (Example 2). On the other hand, by [3, Figure 45] we see that $\tilde{d}_{w3}([7_5],[7_C]) \leq 2$. Hence we have $\tilde{d}_{w3}([7_5],[7_C]) = 2$. For the equivalence class (6), we show: For each pair P, P' of elements in the equivalence class (6), $\tilde{d}_{w3}([P],[P'])$ is realized by the minimal number of arrows in the paths joining P and P' in the diagram (Example 3, and Assertion 1).

2. Preliminaries

Definition 1 (Gauss word). Let $\hat{n} = \{1,2,3,\ldots,n\}$. A *word* w of length n is a map from $\hat{n}$ to $\mathbb{N}$. The word is represented by $w(1)w(2)w(3)\cdots w(n)$. Then, we call each element of $w(\hat{n})$ a *letter*. A *Gauss word* of length $2n$ is a word w of length $2n$ satisfying that each letter in $w(\hat{2n})$ appears exactly twice in $w(1)w(2)w(3)\cdots w(2n)$. Let cyc and rev be maps $\hat{2n} \to \hat{2n}$ where $\mathrm{cyc}(p) \equiv p+1 \pmod{2n}$ and $\mathrm{rev}(p) \equiv -p+1 \pmod{2n}$. Two Gauss words, v and w, of length $2n$ are *isomorphic* if there exists a bijection $f : v(\hat{2n}) \to w(\hat{2n})$ satisfying that there exists $t \in \mathbb{Z}$ such that $w \circ (\mathrm{cyc})^t \circ (\mathrm{rev})^\epsilon = f \circ v$ ($\epsilon = 0$ or 1). The isomorphisms give an equivalence relation on the Gauss words. For a Gauss word v of length $2n$, the equivalence

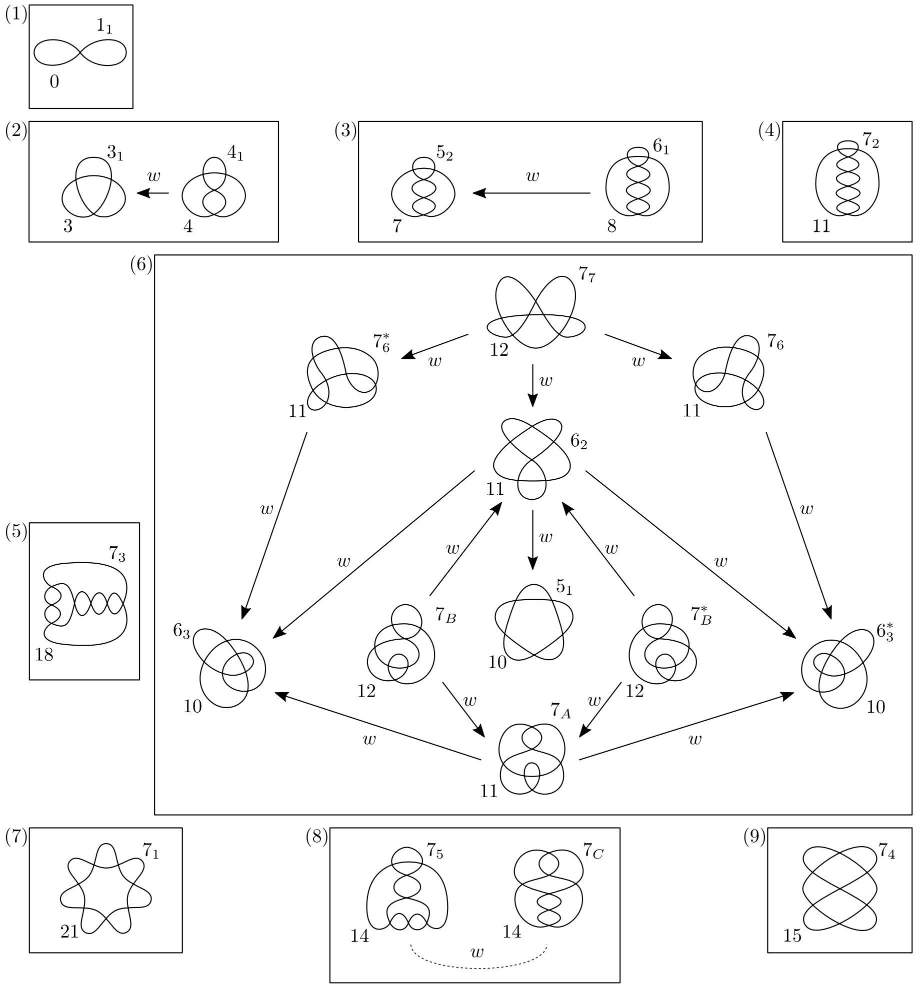


FIGURE 3. The elements of $\mathcal{P}_{\leq 7}$. Fourteen of them are obtained from the knot table of Rolfsen [7] by ignoring over/under informations of the crossings. We assign them the same symbols as Rolfsen's. The spherical curve 7_A (7_B, 7_C resp.) is obtained from 7_6 (7_7, 7_5 resp.) by a single flype. The arrow from a spherical curve P to a spherical curve P' means that P is transformed into P' by a transformation consisting of one deformation of type positive weak RⅢ (see Definition 8) and some deformation(s) of type RI.

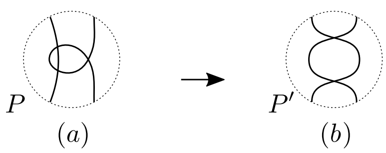


FIGURE 4

class containing v is denoted by $[v]$. A Gauss word v' is called a *sub-Gauss word* of the Gauss word v if v' is obtained from v by ignoring some letters of v. Then, the set of sub-Gauss words of v is denoted by $\mathrm{Sub}(v)$.

Definition 2 (chord diagram)**.** A *chord diagram* is a configuration of n pair(s) of points on a circle up to ambient isotopy and reflection of the circle. Traditionally, two points of each pair are connected by a (straight) arc. This arc is called a *chord*. We say that a chord in a chord diagram is *isolated* if there is no chord transversely intersecting the chord.

We note that the equivalence classes of the Gauss words of length $2n$ have one to one correspondence with the chord diagrams consisting of n chords as in Figure 5. In this paper, we identify these four expressions in Figure 5, and freely use either

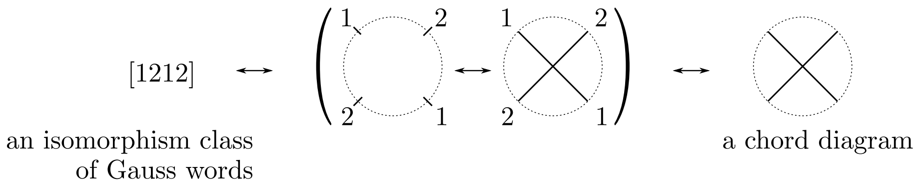


FIGURE 5. Four expressions.

one of them depending on situations.

Definition 3 (a chord diagram CD_P of a spherical curve P)**.** Let P be a spherical curve. Then, there is a generic immersion $g : S^1 \to S^2$ such that $g(S^1) = P$. We define a *chord diagram* of P (e.g., Figure 6) as follows: Let k be the number of the double points of P, and $m_1, m_2, \ldots,$ and m_k mutually distinct positive integers. For

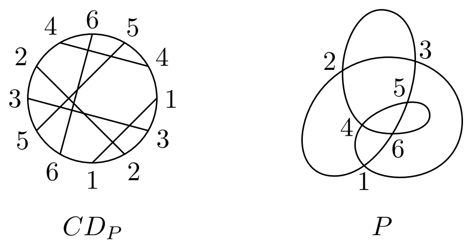


FIGURE 6. A chord diagram CD_P of a spherical curve P.

P, fix a base point, which is not a double point on P, and choose an orientation of P. We start from the base point, and proceed along P according to the orientation of P. Then we assign m_1 to the first double point that we encounter. We assign m_2 to the next double point that we encounter provided it is not the first double point. Suppose that we have already assigned $m_1, m_2, \ldots,$ and m_p. Then, we assign m_{p+1} to the next double point that we encounter if it has not been assigned yet. Following the same procedure, we finally label all the double points of P. We note that g^{-1}(double point assigned m_i) consists of two points on S^1 and we shall assign m_i to them. The chord diagram represented by g^{-1}(double point assigned m_1), g^{-1}(double point assigned m_2), $\ldots$, and g^{-1}(double point assigned m_k) on S^1 is denoted by CD_P, and is called *a chord diagram of the spherical curve* P. Clearly if P' is a spherical curve that is ambient isotopic to P, then $CD_{P'} = CD_P$ as chord diagrams.

We note that $P \to CD_P$ induces a map from $\mathcal{C}$ to chord diagrams. Recall that CD_P is identified with an equivalence class of Gauss words, say $[v_P]$. By these facts, we see that there is a map from $\mathcal{C}$ to the set of the equivalence classes of the Gauss words $P \to [v_P]$.

Definition 4. Let CD be a chord diagram. Then $f_c(CD)$ denotes the number of the connected components of the union of the chords of CD. For a spherical curve P, $f_c(P)$ denotes $f_c(CD_P)$.

Definition 5 (connected sum)**.** Let P_i $(i = 1, 2)$ be a spherical curve. Suppose that the ambient 2-spheres are oriented. Let p_i be a point on P_i where p_i is not a double point $(i = 1, 2)$. Let d_i be a sufficiently small disk with center p_i $(i = 1, 2)$ where $d_i \cap P_i$ consists of an arc properly embedded in d_i. Let $\hat{d}_i = cl(S^2 \setminus d_i)$ and $\hat{P}_i = P_i \cap \hat{d}_i$. Let $h : \partial \hat{d}_1 \to \partial \hat{d}_2$ be an orientation reversing homeomorphism such that $h(\partial \hat{P}_1) = \partial \hat{P}_2$. Then, $\hat{P}_1 \cup_h \hat{P}_2$ is a spherical curve in the oriented 2-sphere $\hat{d}_1 \cup_h \hat{d}_2$. The spherical curve $\hat{P}_1 \cup_h \hat{P}_2$ in the oriented 2-sphere is denoted by $P_1 \sharp_{(p_1,\ p_2),h} P_2$ (or, simply $P_1 \sharp P_2$). This spherical curve $P_1 \sharp_{(p_1,\ p_2),h} P_2$ is called a *connected sum* of the spherical curves P_1 and P_2 at the pair of points p_1 and p_2 (see Figure 7).

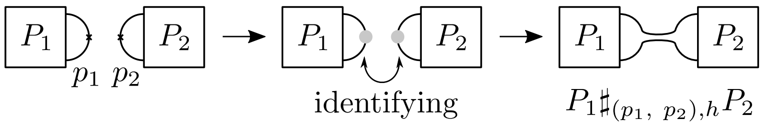


FIGURE 7. Two spherical curves P_1 and P_2 and a connected sum $P_1 \sharp_{(p_1,\ p_2),h} P_2$.

Definition 6 (trivial spherical curve, prime spherical curve)**.** A spherical curve P is *trivial* if P is a simple closed curve. A spherical curve P is *prime* if P is nontrivial and is not a connected sum of two nontrivial spherical curves.

Remark 1. For a spherical curve P, it is easy to see that $f_c(P)$ is the number of prime factors of P.

Recall that $\mathcal{C}$ denotes the ambient isotopy classes of spherical curves and $\tilde{\mathcal{C}}$ the quotient set $\mathcal{C}/\sim_{RI}$, where $[P]$ denotes the element of $\tilde{\mathcal{C}}$ represented by the spherical curve P.

Definition 7 ($x(CD)$)**.** Let x be a chord diagram. For a chord diagram CD, fix a Gauss word G representing CD. Then let $\mathrm{Sub}_x(G) = \{H \mid H \in \mathrm{Sub}(G),\ [H] = x\}$, where $\mathrm{Sub}(G)$ denotes the set of the sub-Gauss words of G, defined in Definition 1. The cardinality of this subset is denoted by $x(G)$, that is, $x(G) = \sharp\,\mathrm{Sub}_x(G)$. By the definition of the isomorphic Gauss words, for another Gauss word G' representing CD, it is easy to see $x(G') = x(G)$. Hence, we shall denote the number by $x(CD)$. If CD is a chord diagram of a spherical curve P, then $x(P)$ denotes $x(CD)$. Clearly if P is ambient isotopic to P', then $x(P) = x(P')$. This show that x induces a map from $\mathcal{C}$ to $\mathbb{Z}_{\geq 0}$, which will be also denoted by $x(\cdot)$.

Because each equivalence class of the Gauss words is identified with a chord diagram, we can calculate the number $x(CD)$ by using geometric observations. We explain this philosophy in Example 1.

Example 1. We consider the chord diagram CD in Figure 8 (Note that $CD = CD_P$ in Figure 6), and we label the chords of CD by α_i $(1 \le i \le 6)$ as in Figure 8. We consider the subset of the power set of $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$, each element of which represents a chord diagram isomorphic to $\otimes$. It is elementary to see that this subset consists of ten elements, those are, $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_1, \alpha_4\}$, $\{\alpha_1, \alpha_5\}$, $\{\alpha_2, \alpha_3\}$, $\{\alpha_2, \alpha_4\}$, $\{\alpha_2, \alpha_5\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_3, \alpha_6\}$, and $\{\alpha_4, \alpha_6\}$, and this fact shows that $\bigotimes(CD) = 10$. Similarly, we can show that $\circledast(CD) = 6$ and $\ominus(CD) = 8$.

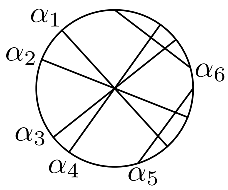


FIGURE 8. CD.

Remark 2. The integer assigned to each spherical curve P in Figure 3 denotes $\bigotimes(P)$.

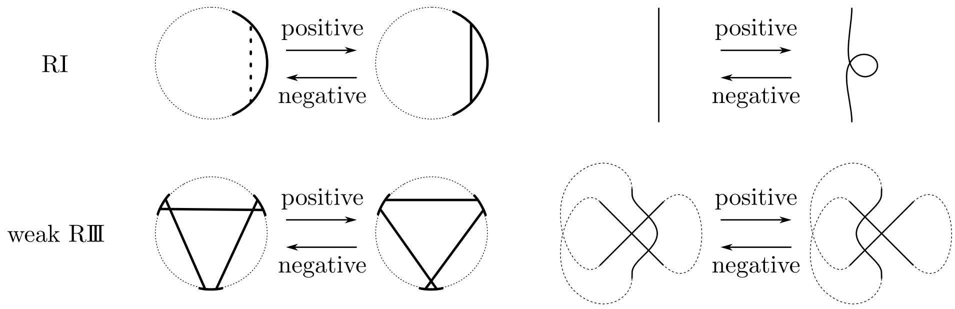


FIGURE 9. Reidemeister moves.

Notation 1. For a chord diagram CD, $n(CD)$ denotes $\ominus(CD)$, that is the number of the chords of CD. For a spherical curve P, $n(CD_P)$ is denoted by $n(P)$, that is the number of the double points of P.

A calculation of the number $x(CD)$ by using geometric observations as in Example 1 together with Figure 9, we have Facts 1 and 2 below.

Fact 1 ([3], Theorem 2(1))**.** *Let $c, c' \in \mathcal{C}$. Suppose that there are representatives P and P' of c and c' respectively such that P' is obtained from P by a deformation of type RI. Then, $\bigotimes(c') = \bigotimes(c)$, $n(c') = n(c) + 1$ or $n(c') = n(c) - 1$.*

Fact 2 ([3], Theorem 2(3))**.** *Let $c, c' \in \mathcal{C}$. Suppose that there are representatives P and P' of c and c' respectively such that P' is obtained from P by a deformation of type weak RIII. Then, $\bigotimes(c') = \bigotimes(c) - 1$ or $\bigotimes(c') = \bigotimes(c) + 1$.*

Definition 8. Let $c, c' \in \mathcal{C}$. Suppose that there are representatives P and P' of c and c' respectively such that P' is obtained from P by a deformation of type RI. If $n(c') = n(c) + 1$ ($n(c') = n(c) - 1$ resp.), then we call such RI a *positive* (*negative* resp.) RI.

Suppose that there are representatives P and P' of c and c' respectively such that P' is obtained from P by a deformation of type weak RIII. If $\bigotimes(P') = \bigotimes(P) - 1$ ($\bigotimes(P') = \bigotimes(P) + 1$ resp.), then we call such weak RIII a *positive* (*negative* resp.) weak RIII.

By Facts 1 and 2, it is easy to show Proposition 2 below.

Proposition 2. *Let P, P' be spherical curves such that $d_{w3}(P, P') < \infty$. Then $\tilde{d}_{w3}([P], [P'])(= d_{w3}(P, P')) \equiv 0 \pmod 2$ if and only if $\bigotimes(P) \equiv \bigotimes(P') \pmod 2$.*

Example 2. Let 7_5, 7_C be the spherical curves in Figure 3. In Figure 45 of [3], it is shown that $\tilde{d}_{w3}([7_5], [7_C]) \leq 2$. On the other hand, since $\bigotimes(7_5) = \bigotimes(7_C) = 14$ (Remark 2), Proposition 2 shows that $\tilde{d}_{w3}([7_5], [7_C]) \equiv 0 \pmod 2$. These show that $\tilde{d}_{w3}([7_5], [7_C]) = 2$.

Example 3. In the diagram of the equivalence class (6) in Figure 3, for any two spherical curves P and P', which are joined by a path consisting of two arrows in the diagram, we can show $\tilde{d}_{w3}([P], [P']) = 2$. For example, take 7_6 and 7_A in Figure 3. Since they are joined by a path consisting of two arrows, we have $\tilde{d}_{w3}([7_6], [7_A]) \leq 2$. On the other hand, since $\bigotimes(7_6) = \bigotimes(7_A) = 11$ (Remark 2), Proposition 2 shows that $\tilde{d}_{w3}([7_6], [7_A]) \equiv 0 \pmod 2$. These shows that $\tilde{d}_{w3}([7_6], [7_A]) = 2$. Similar arguments works for all of the pairs each of which is joined by a path consisting of two arrows, those are $(5_1, 6_3)$, $(5_1, 7_7)$, $(5_1, 7_B)$, $(6_2, 7_6)$, $(6_2, 7_6^*)$, $(6_2, 7_A)$, $(6_3, 6_3^*)$, $(6_3, 7_7)$, $(6_3, 7_B)$, $(6_3, 7_B^*)$, $(6_3^*, 7_7)$, $(6_3^*, 7_B)$, $(6_3^*, 7_B^*)$, $(7_6, 7_6^*)$, $(7_6, 7_A)$, $(7_6^*, 7_A)$, $(7_7, 7_B)$, $(7_7, 7_B^*)$, $(7_B, 7_B^*)$. Details are left to the reader.

3. Main results

For spherical curves P, P', let $n(P)$, $f_c(P)$, $\tilde{d}_{w3}([P], [P'])$ and $d_{w3}(P, P')$ be as in Section 2.

Theorem 1. *Let P, P' be spherical curves. Suppose that $n(P) > n(P')$, and that $f_c(P) = f_c(P')$. Then we have:*

$$\tilde{d}_{w3}([P], [P']) \geq n(P) - n(P').$$

Example 4. For each $N \in \mathbb{N}$, there exist P and P' such that $f_c(P) = f_c(P')$, and that $\tilde{d}_{w3}([P], [P']) = n(P) - n(P') = N$. In fact, let P_N, P'_N be spherical curves as in Figure 10. It is easy to see that $d_{w3}(P_N, P'_N)(= \tilde{d}_{w3}([P_N], [P'_N])) \leq N$. On the other hand, since $n(P_N) - n(P'_N) = 4N - 3N = N$ and $f_c(P_N) = f_c(P'_N) = N$, we have $\tilde{d}_{w3}([P_N], [P'_N]) \geq n(P_N) - n(P'_N) = N$ by Theorem 1.

We say that *a spherical curve P contains a* 1*-gon* if there is an open disk component of $S^2 \setminus P$ that contains exactly one corner. A spherical curve P is called *RI-minimal* if P does not contain a 1-gon. It is clear that any spherical curve is transformed to a minimal one by successively applying deformations of type negative RI. The following fact is known.

Fact 3 ([1] (cf. [6])). *For any spherical curve P, the RI-minimal spherical curves obtained from P are mutually ambient isotopic.*

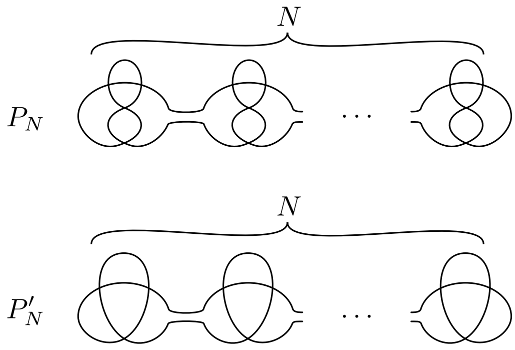


FIGURE 10

Theorem 2. *Let P, P' be RI-minimal spherical curves such that $f_c(P) - n(P) = f_c(P') - n(P')$. Suppose that $\tilde{d}_{w3}([P],[P']) = 1$. Then there is a spherical curve $\tilde{P}'$ such that $\tilde{P}'$ is ambient isotopic to P', and that $\tilde{P}'$ is obtained from P by applying a deformation of type weak RⅢ only, where no RI is required.*

Recall that the transformation of spherical curves depicted in Figure 4 is called a deformation of type α. Here we note that each deformation of type α is represented by successively applying a deformation of type RⅢ, and a deformation of type RI (Figure 11). Then we call it a *deformation of type weak (strong resp.)* α if the type

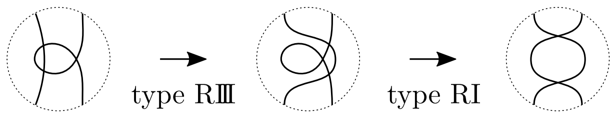


FIGURE 11

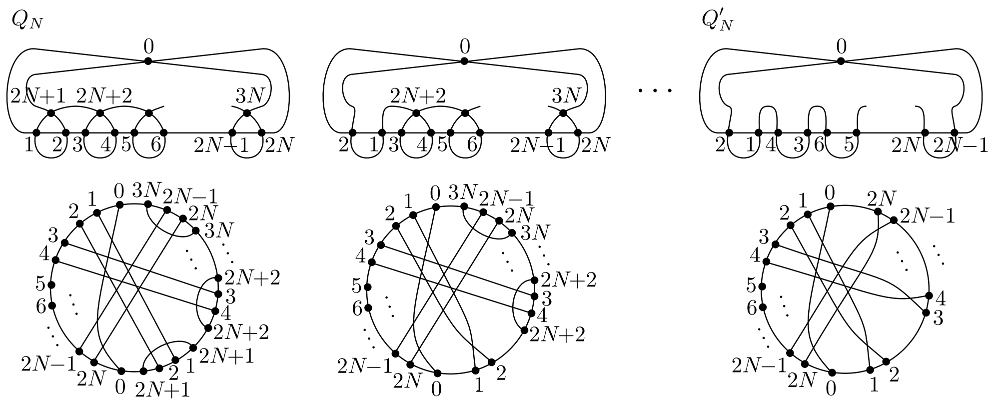
of the RⅢ deformation is weak (strong resp.). By Figure 9 and Figure 11, we have:

Claim 1. *If the deformation of type α is of type weak α, then the deformatin of type RⅢ relevent to it is always positive.*

Theorem 3. *Let P, P' be RI-minimal spherical curves such that $f_c(P) = f_c(P') = 1$, and that $\tilde{d}_{w3}([P],[P']) = q < \infty$. Suppose that $n(P) - n(P') = q$, and that $\bigotimes(P) - \bigotimes(P') = q$. Then there is a spherical curve $\tilde{P}'$ such that $\tilde{P}'$ is ambient isotopic to P', and that $\tilde{P}'$ is obtained from P by applying deformations of type weak α successively q times and this sequence realizes $\tilde{d}_{w3}([P],[P'])$.*

Example 5. For each $N \in \mathbb{N}$, let Q_N, Q'_N be spherical curves as in Figure 12. It is easy to see that $f_c(Q_N) = f_c(Q'_N) = 1$, $n(Q_N) = 3N+1, n(Q'_N) = 2N+1$, hence $n(Q_N) - n(Q'_N) = N$, and $\bigotimes(Q_N) = 2N^2 + 2N, \bigotimes(Q'_N) = 2N^2 + N$, hence $\bigotimes(Q_N) - \bigotimes(Q'_N) = N$. By Theorem 1, we see that $\tilde{d}_{w3}([Q_N],[Q'_N]) \geq N$. On the other hand, it is easy to show that Q'_N is obtained from Q_N by applying deformations of type weak α N times. This shows that $\tilde{d}_{w3}([Q_N],[Q'_N]) \leq N$. Hence $\tilde{d}_{w3}([Q_N],[Q'_N]) = N$. These show that the statement in Theorem 3 is exact.

For the proofs of Theorems 1, 2 and 3, we prepare some notations and lemmas.



$f_c(Q_N) = 1$
$n(Q_N) = 3N + 1$
$\bigotimes(Q_N) = 2N^2 + 2N$

$f_c(Q'_N) = 1$
$n(Q'_N) = 2N + 1$
$\bigotimes(Q'_N) = 2N^2 + N$

FIGURE 12

Notation 2. Let P and P' be two spherical curves that are equivalent under RI and weak RⅢ, i.e., there exists a finite sequence of spherical curves $P = P_0, P_1, \ldots, P_m = P'$, where P_i is obtained from P_{i-1} by a deformation of type RI or type weak RⅢ. Then, Op_i denotes the deformation from P_{i-1} to P_i, and these settings are expressed by using the notation:

$$P = P_0 \overset{Op_1}{\to} P_1 \overset{Op_2}{\to} \cdots \overset{Op_m}{\to} P_m = P'.$$

Since every deformation of type RⅢ does not change the number of double points of P $(= n(P))$, we have the next lemma.

Lemma 1. *$n(P') - n(P) = \sharp\{i \mid Op_i : positive\ RI\} - \sharp\{j \mid Op_j : negative\ RI\}$.*

By Figure 9, we immediately have :

Lemma 2. *If Op_i is of type positive RI, then $f_c(P_i) - f_c(P_{i-1}) = 1$.*

Let P and P' be spherical curves. Suppose that P' is obtained from P by a deformation of type positive weak RⅢ. Then, Figure 13 describes the corresponding

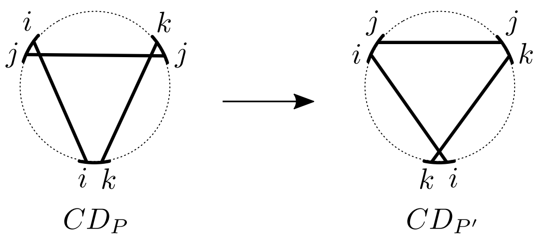


FIGURE 13. A single positive weak RⅢ.

transformation on chord diagrams. The chords i, j, and k are called *the triple relevant to the weak RⅢ*. We call the chord j in the right chord diagram in Figure 13 *the isolated chord in the triple relevant to the weak RⅢ.* Then we define the value, denoted $\mu_{w3}(P, P')$ as follows.

$$\mu_{w3}(P, P') = \sharp\{\text{chord in } CD_{P'} \text{ intersecting the chord } j \text{ transversely}\}.$$

By geometric observations of Figure 14, we have Lemma 3 below.

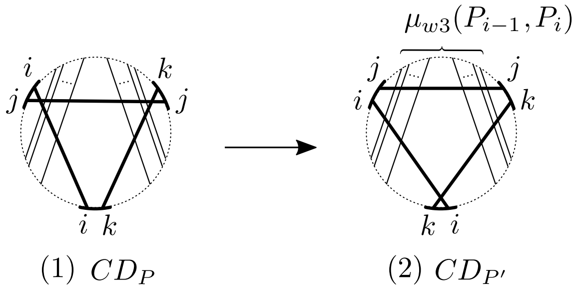


FIGURE 14. CD_P and $CD_{P'}$ under a positive weak RⅢ sending P to P'.

Lemma 3. *If Op_i is of type positive weak RⅢ, then $f_c(P_i) - f_c(P_{i-1}) = 0$ or 1. Further, $f_c(P_{i-1}) - f_c(P_i) = 0$ if and only if $\mu_{w3}(P_{i-1}, P_i) \neq 0$.*

By Lemma 2 and Lemma 3, we have the following equality.

$$\begin{aligned} f_c(P') = f_c(P) &+ \sharp\{i \mid Op_i : \text{positive RI}\} - \sharp\{j \mid Op_j : \text{negative RI}\} \\ &+ \sharp\{k \mid Op_k : \text{positive weak RⅢ}, \mu_{w3}(P_{k-1}, P_k) = 0\} \\ &- \sharp\{l \mid Op_l : \text{negative weak RⅢ}, \mu_{w3}(P_l, P_{l-1}) = 0\}. \end{aligned} \tag{1}$$

Let reduced(P) be an RI-minimal spherical curve obtained from P by successively applying deformations of type negative RI. Note that $CD_{\text{reduced}(P)}$ is obtained from CD_P by successively removing outermost isolated chords (e.g., Figure 15).

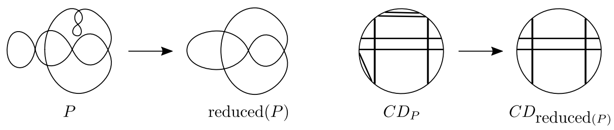


FIGURE 15. (P, reduced(P)) and (CD_P, $CD_{\text{reduced}(P)}$).

Remark 3. For spherical curves P and P', we see, by Fact 3, that $[P] = [P']$ if and only if reduced(P) is ambient isotopic to reduced(P').

Lemma 4. *Suppose that $f_c(\text{reduced}(P_{i-1})) = 1$. If Op_i is a positive weak RⅢ, then $f_c(\text{reduced}(P_i)) = 1$, and $n(\text{reduced}(P_{i-1})) - n(\text{reduced}(P_i)) = 0$ or 1. Further $n(\text{reduced}(P_{i-1})) - n(\text{reduced}(P_i)) = 0$ if and only if $\mu_{w3}(P_{i-1}, P_i) \neq 0$.*

Proof. Since Op_i is of type positive weak RⅢ, the chords relevant to the weak RⅢ Op_i is not isolated in $CD_{P_{i-1}}$. It is observed by Figure 16 that $n(\text{reduced}(P_{i-1})) - n(\text{reduced}(P_i)) = 0$ or 1, and if $\mu_{w3}(P_{i-1}, P_i) \neq 0$, (hence, the isolated chord in the triple relevant to the weak RⅢ is not isolated in CD_{P_i}) then $f_c(\text{reduced}(P_i)) = 1$, and $n(\text{reduced}(P_{i-1})) - n(\text{reduced}(P_i)) = 0$. Suppose that $\mu_{w3}(P_{i-1}, P_i) = 0$. Then

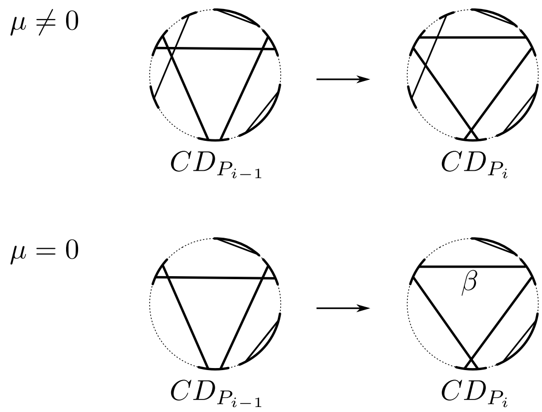


FIGURE 16

we see by Figure 16 that
"the set of the isolated chords of CD_{P_i}" = "the set of the isolated chords of $CD_{P_{i-1}}$" $\cup\{\beta\}$, where β is the isolated chord in the triple relevant to the weak RⅢ Op_i. Since $f_c(\text{reduced}(P_{i-1})) = 1$, the double points of P_{i-1} corresponding to the isolated chords are removed by successively applying deformation of type negative RI. Note that by this transformation all of the isolated chords of $CD_{P_{i-1}}$ is removed. This together with Figure 16 shows that $f_c(\text{reduced}(P_i)) = 1$, and $n(\text{reduced}(P_{i-1})) - n(\text{reduced}(P_i)) = 1$. These complete the proof of the lemma. □

Proof of Theorem 1. Let

$$P = P_0 \stackrel{Op_1}{\to} P_1 \stackrel{Op_2}{\to} \cdots \stackrel{Op_m}{\to} P_m,$$

where P_m is ambient isotopic to P', be a sequence realizing $\tilde{d}_{w3}([P],[P'])$ $(= d_{w3}(P, P'))$.

By (1) above,

$$\begin{aligned} f_c(P') - f_c(P) = \sharp\{i \mid Op_i : \text{positive RI}\} - \sharp\{j \mid Op_j : \text{negative RI}\} \\ + \sharp\{k \mid Op_k : \text{positive weak RⅢ}, \mu_{w3}(P_{k-1}, P_k) = 0\} \\ - \sharp\{l \mid Op_l : \text{negative weak RⅢ}, \mu_{w3}(P_l, P_{l-1}) = 0\}. \end{aligned}$$

This together with Lemma 1 shows

$$\begin{aligned} f_c(P') - f_c(P) = n(P') - n(P) + \sharp\{k \mid Op_k : \text{positive weak RⅢ}, \mu_{w3}(P_{k-1}, P_k) = 0\} \\ - \sharp\{l \mid Op_l : \text{negative weak RⅢ}, \mu_{w3}(P_l, P_{l-1}) = 0\}. \end{aligned}$$

By the assumption of Theorem 1($f_c(P) = f_c(P')$), this implies

$$(2) \qquad n(P) - n(P') = \sharp\{k \mid Op_k : \text{positive weak RIII}, \mu_{w3}(P_{k-1}, P_k) = 0\}$$
$$- \sharp\{l \mid Op_l : \text{negative weak RIII}, \mu_{w3}(P_l, P_{l-1}) = 0\}.$$

By the definition of $d_{w3}(P, P')$, we have

$$d_{w3}(P, P') = \sharp\{k \mid Op_k : \text{positive weak RIII}\} + \sharp\{l \mid Op_l : \text{negative weak RIII}\}$$

Then we note that

$$\begin{aligned} &\sharp\{k \mid Op_k : \text{positive weak RIII}\} + \sharp\{l \mid Op_l : \text{negative weak RIII}\} \\ &\geq \sharp\{k \mid Op_k : \text{positive weak RIII}\} \\ &\geq \sharp\{k \mid Op_k : \text{positive weak RIII}, \mu_{w3}(P_{k-1}, P_k) = 0\} \\ &\geq \sharp\{k \mid Op_k : \text{positive weak RIII}, \mu_{w3}(P_{k-1}, P_k) = 0\} \\ &\quad - \sharp\{l \mid Op_l : \text{negative weak RIII}, \mu_{w3}(P_l, P_{l-1}) = 0\} \\ &= n(P) - n(P') \text{ (by (2))}. \end{aligned}$$

These together with Proposition 1 imply that

$$\tilde{d}_{w3}([P], [P']) = d_{w3}(P, P') \geq n(P) - n(P').$$

□

Proof of Theorem 2. Let

$$P = P_0 \overset{Op_1}{\to} P_1 \overset{Op_2}{\to} \cdots \overset{Op_m}{\to} P_m \text{ , where } P_m \text{ is ambient isotopic to } P'$$

be a sequence realizing $\tilde{d}_{w3}([P], [P']) = 1$ (see Proposition 1), hence there exists unique s $(1 \leq s \leq m)$ such that Op_s is of type weak RIII. By exchanging P and P', if necessary, we may suppose that Op_s is of type positive weak RIII.

Claim 2.

$$\mu_{w3}(P_{s-1}, P_s) \neq 0.$$

Proof. Suppose, for a contradiction, that $\mu_{w3}(P_{s-1}, P_s) = 0$.
By Lemma 1 and (1) above,

$$\begin{aligned} f_c(P') - f_c(P) = n(P') - n(P) &\\ &+ \sharp\{k \mid Op_k : \text{positive weak RIII}, \mu_{w3}(P_{k-1}, P_k) = 0\} \\ &- \sharp\{l \mid Op_l : \text{negative weak RIII}, \mu_{w3}(P_l, P_{l-1}) = 0\}. \end{aligned}$$

This together with the assumption of Theorem 2 ($f_c(P) - n(P) = f_c(P') - n(P')$) shows

$$\begin{aligned} &\sharp\{k \mid Op_k : \text{positive weak RIII}, \mu_{w3}(P_{k-1}, P_k) = 0\} \\ &\quad = \sharp\{l \mid Op_l : \text{negative weak RIII}, \mu_{w3}(P_l, P_{l-1}) = 0\}. \end{aligned}$$

Since $\mu_{w3}(P_{s-1}, P_s) = 0$, this shows that there exists another deformation of type weak RIII in the sequence, contradicting the fact that $P_0 \to P_1 \to \cdots \to P_m$ realizes $\tilde{d}_{w3}([P], [P']) = 1$. □

Recall that $Op_1, Op_2, \ldots, Op_{s-1}$ are of type RI. Since P contains no 1-gon, by Fact 3 we obtain the next claim.

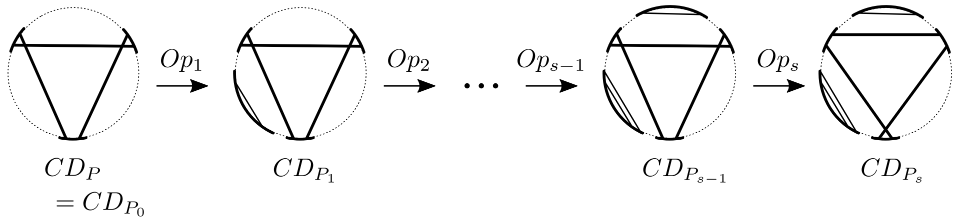


FIGURE 17

Claim 3. *P is ambient isotopic to* reduced(P_{s-1})*, in particular CD_P is obtained from $CD_{P_{s-1}}$ by successively removing outermost isolated chords, say $a_1, a_2, \ldots, a_t$.*

Since each member of the triple in $CD_{P_{s-1}}$ relevant to the weak RⅢ Op_s is not isolated in $CD_{P_{s-1}}$, we see that each a_i $(1 \leq i \leq t)$ is not a member of the triple relevant to the weak RⅢ Op_s. Hence a_1, a_2, $\ldots$, a_t survive in CD_{P_s}. By Claim 2, we see that the isolated chord in the triple relevant to the weak RⅢ Op_s is not isolated in CD_{P_s}. These show that $CD_{\mathrm{reduced}(P_s)}$ is obtained from CD_{P_s} by successively removing the chords $a_1, a_2, \ldots, a_t$ that are outermost at each step. Since $Op_{s+1}, \ldots, Op_m$ are of type RI, and P' is RI-minimal, by Fact 3, this shows:

Claim 4. reduced(P_s) *is ambient isotopic to P', and that $CD_{P'}$ is obtained from CD_{P_s} by removing a_1, a_2, $\ldots$, a_t.*

Claim 3 and Claim 4 show that reduced(P_{s-1}) and reduced(P_s) are related by a single weak RⅢ, and, hence P and P' are related by a single weak RⅢ. This completes the proof of Theorem 2. □

Proof of Theorem 3. Let $P = P_0 \overset{Op_1}{\to} P_1 \overset{Op_2}{\to} \cdots \overset{Op_m}{\to} P_m$, where P_m is ambient isotopic to P', be the sequence realizing $\tilde{d}_{w3}([P],[P'])$ $(= d_{w3}(P,P'))$. Then let $\{i_1, i_2, \ldots, i_q\}$ $(1 \leq i_1 < i_2 < \cdots < i_q \leq m)$ be the set of the numbers such that $Op_{i_1}, Op_{i_2}, \ldots, Op_{i_q}$ are of type RⅢ. By the assumption of Theorem 3 ($\bigotimes(P) - \bigotimes(P') = q$), and Facts 1 and 2, we see that each Op_{i_j} $(j = 1, 2, \ldots, q)$ is of type positive weak RⅢ.

Claim 5. $\mu(P_{i_j-1}, P_{i_j}) = 0$ $(j = 1, 2, \ldots, q)$.

Proof. We first consider the pair P_{i_1-1}, P_{i_1}. Since $Op_1, Op_2, \ldots, Op_{i_1-1}$ are of type RI, we immediately have $n(\mathrm{reduced}(P_0)) = n(\mathrm{reduced}(P_1)) = \cdots = n(\mathrm{reduced}(P_{i_1-1}))$. Then by Lemma 4, we have that $n(\mathrm{reduced}(P_{i_1-1})) - n(\mathrm{reduced}(P_{i_1})) = 0$ or 1, and that $n(\mathrm{reduced}(P_{i_1-1})) - n(\mathrm{reduced}(P_{i_1})) = 1$ if and only if $\mu_{w3}(P_{i_1-1}, P_{i_1}) = 0$. By using the same arguments inductively, we have that for each j, $n(\mathrm{reduced}(P_{i_j})) = n(\mathrm{reduced}(P_{i_j+1})) = \cdots = n(\mathrm{reduced}(P_{i_{j+1}-1}))$, where $P_{i_{q+1}-1} = P_m$, $n(\mathrm{reduced}(P_{i_j-1})) - n(\mathrm{reduced}(P_{i_j})) = 0$ or 1, and that $n(\mathrm{reduced}(P_{i_j-1})) - n(\mathrm{reduced}(P_{i_j})) = 1$ if and only if $\mu_{w3}(P_{i_j-1}, P_{i_j}) = 0$. Since $P_0 = P$, and P_m is ambient isotopic to P', these together with the assumptions of Theorem 3 (RI minimality of P and P', and the equality $n(P) - n(P') = q$) show that $\mu_{w3}(P_{i_j-1}, P_{i_j}) = 0$ $(j = 1, 2, \ldots, q)$. □

Let $\{a_1^{(0)}, \ldots, a_{t_0}^{(0)}\}$ be the set of isolated chords in $CD_{P_{i_1-1}}$. Since $f_c(\mathrm{reduced}(P_{i_1-1})) = f_c(P) = 1$ (by the assumption of Theorem 3), CD_{P_0} is obtained from $CD_{P_{i_1-1}}$ by successively removing the chords $a_1^{(0)}, \ldots, a_{t_0}^{(0)}$ that are outermost at each step.

Since $\mu_{w3}(P_{i_1-1}, P_{i_1}) = 0$ by Claim 5, we see that the isolated chord relevant to the weak RⅢ, say β_1, is isolated in $CD_{P_{i_1}}$. These show that $CD_{\mathrm{reduced}(P_{i_1})}$ is obtained from $CD_{P_{i_1}}$ by removing $a_1^{(0)}, \ldots, a_{t_0}^{(0)}, \beta_1$. Let $P_1^{(1)} = \mathrm{reduced}(P_{i_1})$. Since Op_{i_1} is positive weak RⅢ, each $a_k^{(0)}$ is not a member of the triple relevant to the weak RⅢ Op_{i_1}. On the other hand, we note that reduced(P_{i_1-1}) is ambient isotopic to P_0. Hence we can apply a deformation of type RⅢ, say $Op_1^{(1)'}$, corresponding to Op_{i_1} on P_0 to obtain a spherical curve $P_1^{(1)'}$ such that the chord diagram $CD_{P_1^{(1)'}}$ contains exactly one isolated chord corresponding to β_1. Since $f_c(P_0) = 1$ by the assumption, removing β_1 is realized by a deformation of type negative RI, say $Op_1^{(1)}$. As a conclusion we have obtained a sequence $P = P_0 \stackrel{Op_1^{(1)'}}{\to} P_1^{(1)'} \stackrel{Op_1^{(1)}}{\to} P_1^{(1)}$. Further by Lemma 4, we see $n(P_1^{(1)}) = n(P_0) - 1$, $f_c(P_1^{(1)}) = 1$. As an upshot, we obtain the next claim.

Claim 6. *There is a sequence* $P = P_0 \stackrel{Op_1^{(1)'}}{\to} P_1^{(1)'} \stackrel{Op_1^{(1)}}{\to} P_1^{(1)}$*, where* $Op_1^{(1)'}$ *is of type positive weak RⅢ,* $Op_1^{(1)}$ *is of type negative RI with* $P_1^{(1)}$ *is RI-minimal,* $n(P_1^{(1)}) = n(P_0) - 1$*,* $f_c(P_1^{(1)}) = 1$*.*

Then we can retake the sequence realizing $d_{w3}(P, P')$ as follows.

$$P = P_0 \stackrel{Op_1^{(1)'}}{\to} P_1^{(1)'} \stackrel{Op_1^{(1)}}{\to} P_1^{(1)} \stackrel{Op_1^{(1)''}}{\to} \cdots \stackrel{Op_{t_0}^{(1)''}}{\to} P_{t_0+1}^{(1)} = P_{i_1}$$

(, where $Op_1^{(1)''}, \ldots, Op_{t_0}^{(1)''}$ are of type positive RI that produce isolated chords in $CD_{P_{i_1}}$, which correspond to $a_1^{(0)}, \ldots, a_{t_0}^{(0)}$) followed by the subsequence

$$P_{i_1} \stackrel{Op_{i_1+1}}{\to} P_{i_1+1} \stackrel{Op_{i_1+2}}{\to} \cdots \stackrel{Op_m}{\to} P_m$$

of the original sequence. Then by Claim 6 we can apply the above arguments to the sequence from $P_1^{(1)}$ to $P_m = P$ to obtain a sequence such that

$$(P = P_0 \stackrel{Op_1^{(1)'}}{\to} P_1^{(1)'} \stackrel{Op_1^{(1)}}{\to}) P_1^{(1)} \stackrel{Op_2^{(2)'}}{\to} P_2^{(2)'} \stackrel{Op_2^{(2)}}{\to} P_2^{(2)}$$

(, where $Op_2^{(2)'}$ is positive weak RⅢ, $Op_2^{(2)}$ is negative RI) followed by some positive RI's, from $P^{(2)'}$ to P_{i_2}, then further followed by the subsequence

$$P_{i_2} \to P_{i_2+1} \to \cdots \to P_m = P'$$

of the original sequence, where $P_2^{(2)}$ is RI-minimal, $n(P_2^{(2)}) = q - 2$, $f_c(P_2^{(2)}) = 1$. By repeating the arguments, we obtain a sequence

$$P = P_0 \stackrel{Op_1^{(1)'}}{\to} P_1^{(1)'} \stackrel{Op_1^{(1)}}{\to} P_1^{(1)} \stackrel{Op_2^{(2)'}}{\to} P_2^{(2)'} \stackrel{Op_2^{(2)}}{\to} P_2^{(2)} \to \ldots$$

$$\cdots \to P_j^{(j)} \stackrel{Op_{j+1}^{(j+1)'}}{\to} P_{j+1}^{(j+1)'} \stackrel{Op_{j+1}^{(j+1)}}{\to} P_{j+1}^{(j+1)} \to \cdots \stackrel{Op_q^{(q)}}{\to} P_q^{(q)} \to \cdots \to P_m,$$

where all of the deformations in the subsequence $P_q^{(q)} \to \cdots \to P_m$ are of type RI. Here we note that $P_q^{(q)}$ is RI minimal. Hence by Fact 3 we have P_m is ambient isotopic to $P_q^{(q)}$.

Claim 7. *Each* $P_j^{(j)} \stackrel{Op_{j+1}^{(j+1)'}}{\to} P_{j+1}^{(j+1)'} \stackrel{Op_{j+1}^{(j+1)}}{\to} P_{j+1}^{(j+1)}$ *represents a deformation of type weak* α*.*

Proof. The above arguments imply that $P_j^{(j)}$ is RI-minimal, and that $CD_{P_j^{(j)\prime}}$ contains exactly one isolated chord, say β_j, and $CD_{P_{j+1}^{(j+1)}}$ is obtained from $CD_{P_j^{(j)\prime}}$ by removing β_j. Since $f_c(P_j^{(j)}) = 1$, this deformation is realized by a deformation of type negative RI (see Figure 18). It is directly observed from Figure 18, that these represent a deformation of type weak α. $\square$

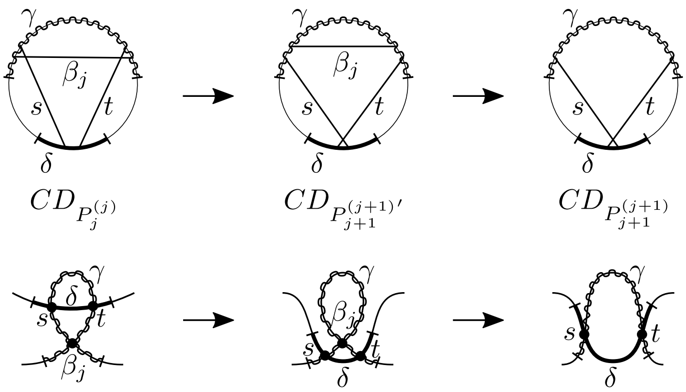


FIGURE 18

Claim 7 shows that by putting $\tilde{P}' = P_q^{(q)}$ we obtain the conclusion of Theorem 3. $\square$

4. Distance between elements of the equivalence class (6).

In this section, we show that for each pair P, P' of elements in the equivalence class (6) in Figure 3, $\tilde{d}_{w3}([P],[P'])$ is realizes by the minimal number of arrows in the paths joining P and P' in the diagram of the equivalence class (6).

By Example 3, for a proof of this, it is enough to prove the next assertion.

Assertion 1. For any pair of elements P and P' in equivalence class (6) in Figure 3 such that the minimal number of arrows in the paths in Figure 3 joining P and P' is 3, we have $\tilde{d}_{w3}([P],[P'])(= d_{w3}(P,P')) = 3$.

Proof. We first consider the pair $(5_1, 7_6)$. By Figure 3 we have $d_{w3}(5_1, 7_6) \leq 3$. Since $\bigotimes(5_1) = 10$, $\bigotimes(7_6) = 11$, we obtain, by Proposition 2 that $d_{w3}(5_1, 7_6) \equiv 1 \pmod 2$. On the other hand, since $f_c(5_1) = f_c(7_6)(= 1)$, we have $d_{w3}(5_1, 7_6) \geq 7 - 5 = 2$ by Theorem 1. These show that $d_{w3}(5_1, 7_6) = 3$. Similar argument works for the pairs $(5_1, 7_6^*)$, $(5_1, 7_A)$. (Details are left to the reader.)

Then we consider the pair $(7_6, 7_B)$. By Figure 3 we have $d_{w3}(7_6, 7_B) \leq 3$. Since $\bigotimes(7_6) = 11$, $\bigotimes(7_B) = 12$, we obtain $d_{w3}(7_6, 7_B) \equiv 1 \pmod 2$. Hence, it is enough to show $d_{w3}(7_6, 7_B) \neq 1$. Suppose that $d_{w3}(7_6, 7_B) = 1$. Since $n(7_6) = n(7_B) = 7$, and $f_c(7_6) = f_c(7_B) = 1$, we see that 7_6 and 7_B are related by a deformation of type weak RⅢ only by Theorem 2. For 7_B, we have four regions where we can apply deformations of type weak RⅢ, that are depicted in Figure 19. It is directly observed that 7_6 is not produced by applying the deformation of type weak RⅢ there. This contradicts Theorem 2. These show that $d_{w3}(7_6, 7_B) = 3$. Similar argument works for the pairs $(7_6, 7_B^*)$, $(7_6^*, 7_B^*)$, $(7_6^*, 7_B)$, $(7_A, 7_7)$.

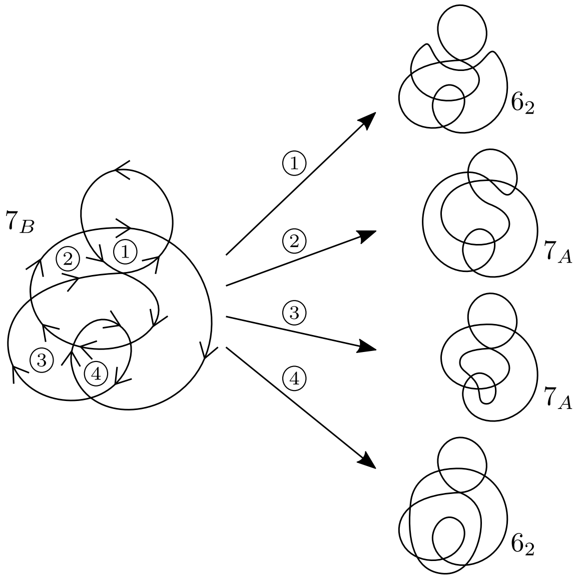


FIGURE 19

Then we consider the pair $(7_6, 6_3)$. In this case, since $\bigotimes(7_6) = 11$, $\bigotimes(6_3) = 10$ it is enough to show $d_{w3}(7_6, 6_3) \neq 1$. Suppose that $d_{w3}(7_6, 6_3) = 1$. Then we can apply Theorem 3 to show that 7_6 is related to 6_3 by exactly one deformation of type weak α. However we can show that this is not the case, by using the analysis as in Figure 20. These show that $d_{w3}(7_6, 6_3) = 3$. Similar argument works for the pair $(7_6^*, 6_3^*)$. These complete the proof of Assertion 1. $\square$

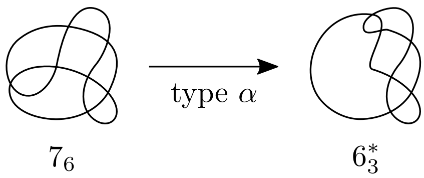


FIGURE 20

ACKNOWLEDGEMENTS

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Appendix A. Proof of Proposition 1

Proof of Proposition 1. By the definition of $\tilde{d}_{w3}$ and d_{w3}, it is easy to see that $\tilde{d}_{w3}([P],[P']) \leq d_{w3}(P,P')$. Hence we show that $d_{w3}(P,P') \leq \tilde{d}_{w3}([P],[P'])$. Let $m = \tilde{d}_{w3}([P],[P'])$. This means that there is a sequence $P = P_0 \to P_1 \to \cdots \to P_k$ ($k \geq m$) and ambient isotopies ϕ_t^i ($i = 0, 1, \ldots, k$) such that there is a subset $\{i_1, i_2, \ldots, i_m\} \subset \{i = 0, 1, \ldots, k\}$ ($i_1 < i_2 < \cdots < i_m$) such that

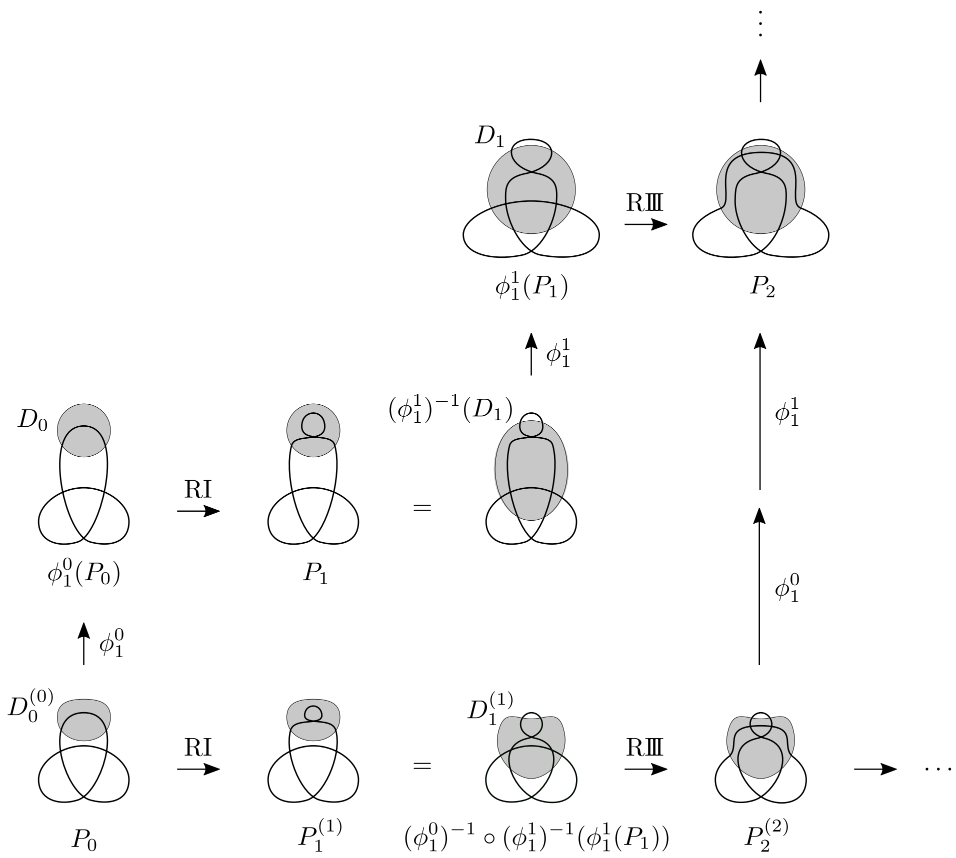
- $\phi_0^i = \mathrm{id}_{S^2}$
- $\phi_1^j(P_j)$ is transformed to P_{j+1} ($j = 0, 1, \ldots, k-1$) by a deformation of type RI (if $j \notin \{i_1, i_2, \ldots, i_m\}$) or by a deformation of type RⅢ (if $j \in \{i_1, i_2, \ldots, i_m\}$)
- $\phi_1^k(P_k) = P'$.

Let D_j be the disk in S^2 such that the deformation $\phi_1^j(P_j) \to P_{j+1}$ of type RI or type RⅢ is performed within D_j. Then let $D_0^{(0)} = (\phi_1^0)^{-1}(D_0)$, and $P_1^{(1)} = (\phi_1^0)^{-1}(P_1)$ (hence $P_1^{(1)}$ is ambient isotopic to P_1). Since $(\phi_1^0)^{-1}(\phi_1^0(P_1)) = P_1$, we see that $P_1^{(1)}$ is obtained from P_0 by a deformation of type RI or type weak RⅢ that is performed within $D_0^{(0)}$. Then let $D_1^{(1)} = (\phi_1^0)^{-1} \circ (\phi_1^1)^{-1}(D_1)$, $P_2^{(2)} = (\phi_1^0)^{-1} \circ (\phi_1^1)^{-1}(P_2)$ (, hence $P_2^{(2)}$ is ambient isotopic to P_2). It is easy to see that $P_2^{(2)}$ is obtained from P_0 by applying a deformation of type RI or type weak RⅢ that is performed within $D_0^{(0)}$, then applying a deformation of type RI or type weak RⅢ that is performed within $D_1^{(1)}$. For each $j \in \{3, 4, \ldots, m-1\}$, let

$D_j^{(j)} = (\phi_1^0)^{-1} \circ (\phi_1^1)^{-1} \circ \cdots \circ (\phi_1^j)^{-1}(D_j)$, and

$P_{j+1}^{(j+1)} = (\phi_1^0)^{-1} \circ (\phi_1^1)^{-1} \circ \cdots \circ (\phi_1^j)^{-1}(P_{j+1})$ (hence $P_{j+1}^{(j+1)}$ is ambient isotopic to P_{j+1}).

Then $P_m^{(m)}$ is obtained from P_0 by successively applying deformation of type RI or type weak RⅢ that is performed within the disk $D_j^{(j)}$. Since $P_m^{(m)}$ is ambient isotopic to P', this shows that $d_{w3}(P,P') \leq \tilde{d}_{w3}([P],[P'])$. □



D_1
RⅢ
$\phi_1^1(P_1)$
P_2
ϕ_1^1
D_0
RI
$=$
$(\phi_1^1)^{-1}(D_1)$
$\phi_1^0(P_0)$
P_1
ϕ_1^0
$D_0^{(0)}$
$D_1^{(1)}$
P_0
$P_1^{(1)}$
$(\phi_1^0)^{-1} \circ (\phi_1^1)^{-1}(\phi_1^1(P_1))$
$P_2^{(2)}$
$\cdots$

FIGURE 21