Epimorphisms between genus two handlebody-knot groups

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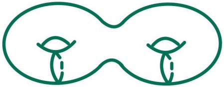
Definition.

- *A genus g handlebody* is an orientable 3-manifold with boundary obtained by attaching g handles to B^3.
- *A genus g handlebody-knot* is a genus g handlebody embedded in the 3-sphere $\mathbb{S}^3$.

a genus 1 handlebody

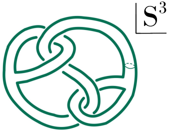
$\longrightarrow$

a genus 1 handlebody-knot



a genus 2 handlebody

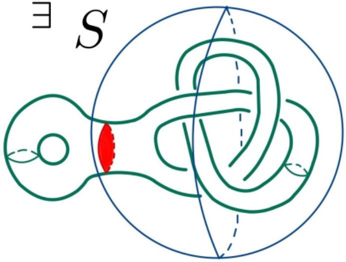
$\longrightarrow$



a genus 2 handlebody-knot

A genus g handlebody-knot H is reducible

$\overset{\text{def.}}{\Longleftrightarrow}$ $^\exists$ $S \subset \mathbb{S}^3$:2-sphere s.t. $H \cap S$ is an essential disk properly embedded in H

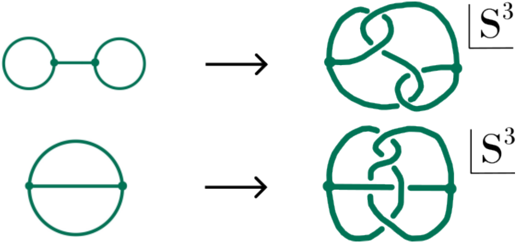


reducible

irreducible

Definition.

A spatial trivalent graph is a finite trivalent graph embedded in $\mathbb{S}^3$.



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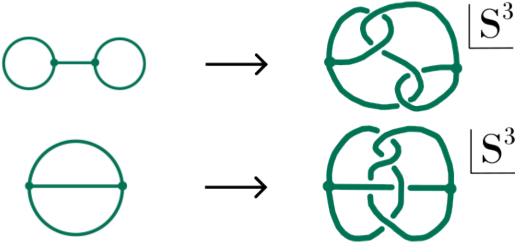


Fact.

Any genus g handlebody-knot is obtained from a regular neighborhood of a spatial trivalent graph.

H : a genus g handlebody-knot in $\mathbb{S}^3$
$G(H) := \pi_1(\mathbb{S}^3 - H)$: the (handlebody-)knot group of H

Definition.

H_1, H_2 : genus g handlebody-knot
$H_1 \geq H_2 \overset{\text{def.}}{\Longleftrightarrow} {}^{\exists}\sigma : G(H_1) \to G(H_2)$: epimorphism

Proposition.

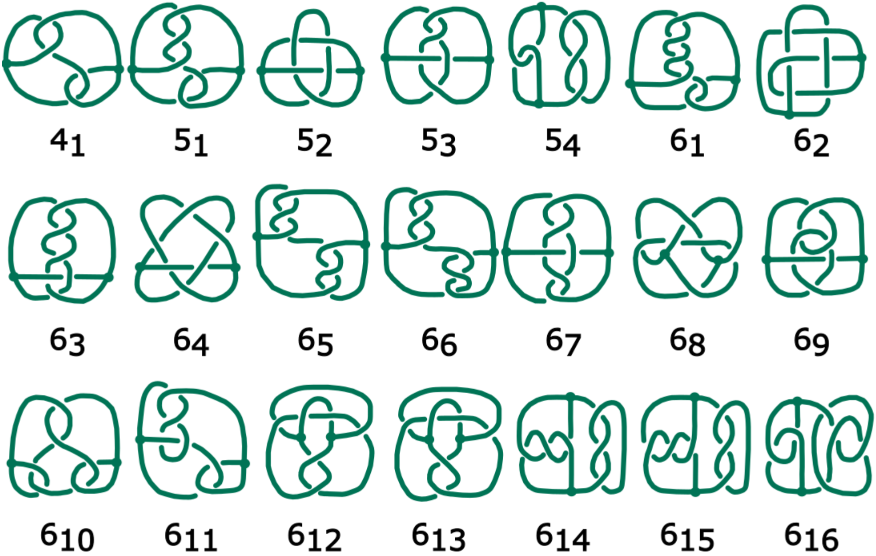
The relation "$\geq$" is a preorder in the set of all equivalence classes of genus 2 handlebody-knots. i.e.

- $H_1 \geq H_1$
- $H_1 \geq H_2$, $H_2 \geq H_3 \Rightarrow H_1 \geq H_3$

Remark.

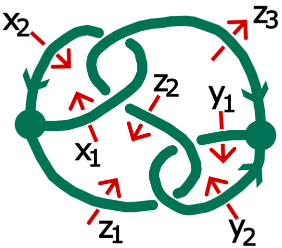
- The relation "$\geq$" is not a partial order.
  $H_1 \geq H_2$, $H_2 \geq H_1 \not\Rightarrow H_1 \cong H_2$
- For prime knots $(g = 1)$:
  The relation "$\geq$" is a partial order.
  This partial order is determined up to 11 crossings.
  [Kitano-Suzuki '05, '11][Horie-Kitano-Matsumoto-Suzuki '11].

The table of irreducible genus 2 handlebody-knots up to 6 crossings [Ishii-Kishimoto-Moriuchi-Suzuki '12]



Ex.

4_1 :



$$G(4_1) \cong \langle x_1, x_2, y_1, y_2, z_1, z_2, z_3 | z_3 x_2 z_3^{-1} x_1^{-1}, x_1 z_3 x_1^{-1} z_2^{-1}, z_2 y_2 z_2^{-1} y_1^{-1}, y_2 z_2 y_2^{-1} z_1^{-1}, z_3^{-1} y_1 y_2^{-1}, z_1 x_1 x_2^{-1} \rangle$$

$$\cong \langle x_1, y_2, z_2 | x_1 z_2 y_2 z_2^{-1} y_2^{-1} x_1^{-1} z_2^{-1} \rangle$$

Definition. (the d-th Alexander ideal of $G(H)$ associated with ψ)

H : a genus 2 handlebody-knot

$G(H) = \langle x_1, x_2, \ldots, x_s \mid r_1, r_2, \ldots, r_{s-2} \rangle$

$\psi : G(H) \longrightarrow \mathbb{Z}$: a homomorphism

$$A\left(G(H), \psi\right) = \left(\ \psi\left(\frac{\partial r_i}{\partial x_j}\right) \ \right) : (s-2) \times s \text{ matrix}$$

: the Alexander matrix of $G(H)$ associated with ψ

$$E_d\left(A\left(G(H), \psi\right)\right) = \begin{cases} (0) & (0 \leq d < 2) \\ \big((s-d)\text{-minors of } \left(A\left(G(H), \psi\right)\right)\big) & (2 \leq d \leq s-1) \\ (1) & (s-1 < d) \end{cases}$$

: the d-th Alexander ideal of $G(H)$ associated with ψ

Fact.

The d-th Alexander ideal of $G(H)$ associated with ψ does not depend on the choice of a presentation of $G(H)$.

Remark.

The d-th Alexander ideal $E_d\left(A\left(G(H),\psi\right)\right)$ depends on $\psi : G(H) \to \mathbb{Z}$.

H : a genus 2 handlebody-knot

$\alpha : G(H) \twoheadrightarrow G(H)/[G(H), G(H)]$: the abelianization

$l \in H_1(H;\mathbb{Z})$

$\psi_l : G(H) \longrightarrow \mathbb{Z} = \langle t \rangle$: a homomorphism

$$x_j \longmapsto t^{lk(\alpha(x_j),l)}$$

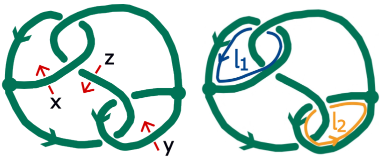
Proposition.

${}^\forall\psi : G(H) \longrightarrow \mathbb{Z} = \langle t \rangle$: homomorphism , ${}^\exists l \in H_1(H)$ s.t. $\psi = \psi_l$

By this proposition, it is sufficient to consider ψ_l for all $l \in H_1(H)$.

Ex. the d-th Alexander ideal of $G(4_1)$ assosiated with ψ_l

4_1 :



$$l = c_1 l_1 + c_2 l_2 \in H_1(4_1) \qquad (c_1, c_2 \in \mathbb{Z})$$

$$\psi_l : G(4_1) \to \mathbb{Z} = \langle t \rangle$$
$$x \longmapsto t^{c_1}$$
$$y \longmapsto t^{c_2}$$
$$z \longmapsto 1$$

$$G(4_1) \cong \langle x, y, z | xzyz^{-1}y^{-1}x^{-1}z^{-1} = r \rangle$$

$$A\left(G(4_1), \psi_l\right) = \begin{pmatrix} \psi_l\left(\frac{\partial r}{\partial x}\right) & \psi_l\left(\frac{\partial r}{\partial y}\right) & \psi_l\left(\frac{\partial r}{\partial z}\right) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & -1 + t^{c1} - t^{c1+c2} \end{pmatrix}$$

$$E_d\left(A\left(G(4_1), \psi_l\right)\right) = \begin{cases} (0) & (d < 2) \\ (1 - t^{c_1} + t^{c_1+c_2}) & (d = 2) \\ (1) & (2 < d) \end{cases}$$

$$E_2\left(A\left(G(4_1), \psi_l\right)\right) = (1 - t^{c_1} + t^{c_1+c_2})$$
$$c_1 = c_2 = 1 : (1 - t + t^2)$$
$$c_1 = 1,\ c_2 = -1 : (2 - t)$$

Proposition.

H_1, H_2 : genus 2 handlebody-knots

If ${}^{\exists}\psi_2 : G(H_2) \to \mathbb{Z}$: a homomorphism, ${}^{\exists}d \geq 0$

s.t. $E_d\left(A\left(G(H_1), \psi_1\right)\right) \not\subset E_d\left(A\left(G(H_2), \psi_2\right)\right)$ for ${}^{\forall}\psi_1 : G(H_1) \to \mathbb{Z}$: homo.

$\Longrightarrow$ ${}^{\nexists}\sigma : G(H_1) \longrightarrow G(H_2)$: an epimorphism

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$\Longrightarrow\ {}^{\nexists}\sigma : G(H_1) \longrightarrow G(H_2)$: an epimorphism

Ex.4_1 and 5_3

$E_2\left(A\left(G(4_1), \psi_l\right)\right) = (1 - t^{c_1} + t^{c_1+c_2})$

$c_1 = 1,\ c_2 = -1 : (2 - t)$

${}^{\forall}\psi_l' : G(5_3) \to \mathbb{Z}$

$E_2\left(A\left(G(5_3), \psi_l'\right)\right) = (2, 1 - t^{c_1} + t^{c_1+c_2})$

5_3:

If $(2, 1 - t^{c_1} + t^{c_1+c_2}) \subset (2 - t) \Longrightarrow 2 = {}^{\exists}f(t)(2 - t) \quad (f(t) \in \mathbb{Z}[t^{\pm 1}])$

In the case of $t = 2$, $2 = 0 \Longrightarrow$ contradiction.

$\therefore 5_3 \not\geq 4_1$

Ex. 6_3 and 6_{10}

6_3

6_{10}

$$^{\forall}\psi : G(6_3) \to \mathbb{Z}$$
$$E_2\left(A\left(G(6_3), \psi\right)\right) = (1)$$

$$^{\forall}\psi' : G(6_{10}) \to \mathbb{Z}$$
$$E_2\left(A\left(G(6_{10}), \psi'\right)\right) = (1)$$

$$\Longrightarrow 6_3 \overset{?}{\geq} 6_{10}, \quad 6_{10} \overset{?}{\geq} 6_3$$

Definition. (the d-th twisted Alexander ideal of $G(H)$ associated with ρ and ψ)

H : a genus 2 handlebody-knot, $\psi : G(H) \longrightarrow \mathbb{Z}$: a homomorphism
$\rho : G(H) \longrightarrow GL(n;\mathbb{Z})$: a representation

$$A\left(G(H), \rho \otimes \psi\right) = \left(\ (\rho \otimes \psi)\left(\frac{\partial r_i}{\partial x_j}\right) \ \right)$$

$$= \begin{pmatrix} (\rho \otimes \psi)\left(\frac{\partial r_1}{\partial x_1}\right) & \cdots & (\rho \otimes \psi)\left(\frac{\partial r_1}{\partial x_s}\right) \\ \vdots & \ddots & \vdots \\ (\rho \otimes \psi)\left(\frac{\partial r_{s-2}}{\partial x_1}\right) & \cdots & (\rho \otimes \psi)\left(\frac{\partial r_{s-2}}{\partial x_s}\right) \end{pmatrix} : n(s-2) \times ns \text{ matrix}$$

: the twisted Alexander matrix of $G(H)$ associated with ρ and ψ

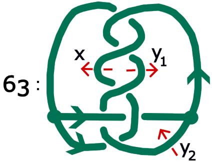
$$E_d\left(A\left(G(H), \rho \otimes \psi\right)\right) = \begin{cases} (0) & (0 \leq d < 2n) \\ \left((ns-d)\text{-minors of}\left(A\left(G(H), \rho \otimes \psi\right)\right)\right) & (2n \leq d \leq ns-1) \\ (1) & (ns-1 < d) \end{cases}$$

: the d-th twisted Alexander ideal of $G(H)$ associated with ρ and ψ

Fact.

The d-th twisted Alexander ideal of $G(H)$ associated with ρ and ψ does not depend on the choice of a presentation of $G(H)$.

Ex. the d-th twisted Alexander ideal of 6_3



$${}^{\forall}\psi_l : G(6_3) \to \mathbb{Z}$$
$$E_2\left(A\left(G(6_3), \psi_l\right)\right) = (1)$$

$$G(6_3) \cong \langle x, y_1, y_2 | y_2 y_1^{-1} x^{-1} y_1 y_2^{-1} x^{-1} y_1 x y_1^{-1} x y_1^{-1} x^{-1} y_1 x^{-1} y_1^{-1} x y_2 y_1^{-1} x y_1 \rangle$$

$\psi : G(6_3) \to \mathbb{Z} = \langle t \rangle,\ \psi(x) = 1,\ \psi(y_i) = t$

$\rho : G(6_3) \to SL(2; \mathbb{Z}/3\mathbb{Z})$

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},\ \rho(y_1) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},\ \rho(y_2) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$A\left(G(6_3), \rho \otimes \psi\right) = \begin{pmatrix} 2+t & 2+2t+2t^2 & 1+2t & 2+t & 1+2t & 2+t \\ 2t+t^2 & 1 & 2+t & 2 & 2+t & 2 \end{pmatrix}$$

$$E_4\left(A\left(G(6_3), \rho \otimes \psi\right)\right) = \left(\begin{vmatrix} 2+t & 2+2t+2t^2 \\ 2t+t^2 & 1 \end{vmatrix}, \begin{vmatrix} 2+2t+2t^2 & 1+2t \\ 1 & 2+t \end{vmatrix}, \cdots \right) = (1-t^2)$$

$$E_5\left(A\left(G(6_3), \rho \otimes \psi\right)\right) = (2+t,\ 2t+t^2,\ 2+2t+2t^2,\ 1, \cdots) = (1)$$

Proposition.

H_1, H_2 : genus 2 handlebody-knots

If ${}^{\exists}\psi_2 : G(H_2) \to \mathbb{Z}$: a homomorphism,

${}^{\exists}\rho_2 : G(H_2) \to GL(n;\mathbb{Z})$: a representation, ${}^{\exists}d \geq 0$

s.t. $E_d\left(A\left(G(H_1), \rho_1 \otimes \psi_1\right)\right) \not\subset E_d\left(A\left(G(H_2), \rho_2 \otimes \psi_2\right)\right)$

for ${}^{\forall}\psi_1 : G(H_1) \to \mathbb{Z}$: homo. and ${}^{\forall}\rho_1 : G(H_1) \to GL(n;\mathbb{Z})$: rep.

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If $^{\exists}\psi_2 : G(H_2) \to \mathbb{Z}$: a homomorphism,
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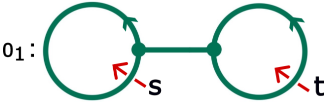
s.t. $E_d\left(A\left(G(H_1), \rho_1 \otimes \psi_1\right)\right) \not\subset E_d\left(A\left(G(H_2), \rho_2 \otimes \psi_2\right)\right)$
for $^{\forall}\psi_1 : G(H_1) \to \mathbb{Z}$: homo. and $^{\forall}\rho_1 : G(H_1) \to GL(n;\mathbb{Z})$: rep.
$\Longrightarrow$ $^{\nexists}\sigma : G(H_1) \longrightarrow G(H_2)$: an epimorphism

Ex. 6_3 and 6_{10}

$^{\exists}\psi_2 : G(6_3) \to \mathbb{Z}$, $^{\exists}\rho_2 : G(6_3) \to SL(2;\mathbb{Z}/3\mathbb{Z})$
$E_4\left(A\left(G(6_3), \rho_2 \otimes \psi_2\right)\right) = (1 - t^2)$

$^{\forall}\psi_1 : G(6_{10}) \to \mathbb{Z}, {}^{\forall}\rho_1 : G(6_{10}) \to SL(2;\mathbb{Z}/3\mathbb{Z})$
$E_4\left(A\left(G(6_{10}), \rho_1 \otimes \psi_1\right)\right) = (1)$

$E_4\left(A\left(G(6_{10}), \rho_1 \otimes \psi_1\right)\right) \not\subset E_4\left(A\left(G(6_3), \rho_2 \otimes \psi_2\right)\right) \quad \therefore 6_{10} \ngeq 6_3$



$G(0_1) = \langle s, t \mid \emptyset \rangle$

Proposition.

$4_1, 5_4, 6_1, 6_7, 6_{10}, 6_{14}, 6_{16} \geq 0_1$

Proposition. [Jaco-McMillan '70]+[S.Suzuki '84]

H : a genus 2 handlebody-knot, $\psi : G(H) \longrightarrow \langle t \rangle$: a homomorphism

$H \geq 0_1 \Longrightarrow E_2\left(A\left(G(H), \psi\right)\right)$ is a principal ideal for every ψ

Corollary.

$5_2, 5_3, 6_2, 6_5, 6_8, 6_9, 6_{12}, 6_{13} \not\geq 4_1, 5_4, 6_1, 6_7, 6_{10}, 6_{14}, 6_{16}$

Proof.

For $5_2, 5_3, 6_2, 6_5, 6_8, 6_9, 6_{12}, 6_{13}$, $E_2\left(A\left(G(H), \psi\right)\right)$ are non-principal ideals.
Hence, $5_2, 5_3, 6_2, 6_5, 6_8, 6_9, 6_{12}, 6_{13} \not\geq 0_1$. □

Proposition.

H_1, H_2 : genus 2 handlebody-knots

If ${}^{\exists}G$: a group

s.t. $\#\mathrm{Hom}(G(H_1), G) < \#\mathrm{Hom}(G(H_2), G)$

$\Longrightarrow H_1 \not\geq H_2$

- $SL(2; \mathbb{Z}/p\mathbb{Z})$ ($p \leq 11$, prime)
- $SL(3; \mathbb{Z}/q\mathbb{Z})$ ($q = 2, 3$)

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H_1, H_2 : genus 2 handlebody-knots

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- $SL(2;\mathbb{Z}/p\mathbb{Z})$ ($p \leq 11$, prime)
- $SL(3;\mathbb{Z}/q\mathbb{Z})$ ($q = 2, 3$)

Ex.

6_5

6_3

$\#\mathrm{Hom}(G(6_5), SL(2;\mathbb{Z}/11\mathbb{Z})) = 3412$, $\quad \#\mathrm{Hom}(G(6_3), SL(2;\mathbb{Z}/11\mathbb{Z})) = 3444$

$\Longrightarrow 6_5 \not\geq 6_3$

Summary

- There are 21 irreducible genus 2 handlebody-knots up to 6 crossings
- $G(5_1) \cong G(6_4), G(5_2) \cong G(6_{13}), G(6_{14}) \cong G(6_{15})$

  $\Rightarrow$We consider $306(= 18 \times 17)$ pairs.

For 251 pairs, we determined the non-existence of epimorphisms.

The unsolved 55 pairs:

$(4_1, 5_1), (4_1, 5_2), (4_1, 5_3), (4_1, 6_2), (4_1, 6_3), (4_1, 6_5), (4_1, 6_6), (4_1, 6_8), (4_1, 6_{11}),$
$(5_2, 5_1), (5_2, 6_3), (5_2, 6_{11}), (5_3, 6_3), (5_3, 6_{11}), (5_4, 5_1), (5_4, 6_3), (5_4, 6_{10}),$
$(5_4, 6_{11}), (6_1, 5_1), (6_1, 5_3), (6_1, 6_2), (6_1, 6_3), (6_1, 6_5), (6_1, 6_6), (6_1, 6_8), (6_1, 6_{11}),$
$(6_2, 6_3), (6_2, 6_{11}), (6_7, 5_1), (6_7, 5_3), (6_7, 6_2), (6_7, 6_3), (6_7, 6_5), (6_7, 6_8), (6_7, 6_{11}),$
$(6_7, 6_{12}), (6_9, 5_1), (6_9, 6_3), (6_9, 6_{11}), (6_{12}, 5_1), (6_{12}, 6_2), (6_{12}, 6_3), (6_{12}, 6_5),$
$(6_{12}, 6_8), (6_{12}, 6_{11}), (6_{14}, 5_1), (6_{14}, 5_2), (6_{14}, 6_3), (6_{14}, 6_9), (6_{14}, 6_{10}),$
$(6_{14}, 6_{11}), (6_{16}, 5_1), (6_{16}, 6_3), (6_{16}, 6_{10}), (6_{16}, 6_{11}).$