Two graph homologies and the space of long embeddings

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Abstract

Graph homologies are powerful tools to compute the rational homotopy group of the space of long embeddings. Two graph homologies have been invented from two approaches to study the space of long embeddings: the hairy graph homology from a homotopy theoretical approach, and BCR graph homology from a geometric approach.

The first goal of this article is to construct cycles of the space of long embeddings associated with 2-loop hairy graphs. The second goal is to construct an injective map from the top hairy graph homology to the top BCR graph homology, though the latter graph homology is quite modified. The injectivity plays an essential role in showing that the geometric approach detects the cycles constructed.

Contents

Introduction

A *long embedding* is a smooth embedding of $\mathbb{R}^j$ to $\mathbb{R}^n$ which is standard outside a fixed ball in $\mathbb{R}^j$. We write $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ for the space of long embeddings equipped with the C^∞ topology. Since embeddings are immersions, there is a natural map $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n) \to \mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)$ to the *space of long immersions*. As the space of long immersions is well-studied, we often consider the difference between the two spaces

$$\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n) = \mathrm{hofib}(\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n) \to \mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)),$$

called the *space of long embeddings modulo immersions.*

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The space $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ is nothing but the space of long knots. Haefliger [Hae] started to study $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ for general n and j in the late 1950s. In this article, we consider the following problem.

Problem 0.1. Compute $\pi_*(\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n))$. In particular, compare the cases $n-j=2$ and $n-j \geq 3$.

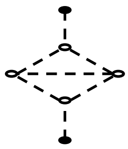
In the 2010s, Fresse, Turchin and Willwacher, following Arone and Turchin, established a significant result on the above problem, by developing a homotopy theoretical approach called *embedding calculus* [GW, GKW, Wei][1]. They showed that if $n-j \geq 3$, the rational homotopy group of $\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)$ depends only on parities of n, j up to degree shifts.

Behind this property, there is a combinatorial complex $HGC_{n,j}$ called the *hairy graph complex*, which is generated by graphs and depends on parities of n and j only.

Theorem 0.2. [AT 1, AT 2, FTW] For $n-j \geq 3$, there is an isomorphism

$$\pi_*(\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)) \otimes \mathbb{Q} \cong H_*(HGC_{n,j}).^2$$

In this article, we focus on the *top hairy graph homology* $\mathcal{B}_{n,j} = H_{\mathrm{top}}(HGC_{n,j})$.

Example 0.3. Let n and j be odd and $n-j \geq 3$. Since  $\neq 0 \in \mathcal{B}_{n,j}$, we have

$$\pi_{3(n-j-2)+(j-1)}(\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)) \otimes \mathbb{Q} \neq 0.$$

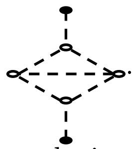
Our main result extends Example 0.3 to the case $n-j=2$.

Theorem 0.4. [Yos 1]If j is odd and satisfies $j \geq 3$,

$$\pi_{j-1}\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^{j+2})_u \otimes \mathbb{Q} \neq 0,$$

where u is the trivial family of the trivial immersion.

Instead of the homotopy theoretical approach, we take a geometric approach called *configuration space integrals*. The first but physical formulation for the integrals was introduced by Witten[Wit] and their mathematical formulations were developed by Bar-Natan[Bar], Bott, Taubes[BT] and so on in the 1990s. Configuration space integrals for higher (co)dimensional embeddings were introduced by Bott [Bot] and developed by Cattaneo, Rossi [CR], Sakai and Watanabe [Sak, SW, Wat 1]. The integrals take value in another graph homology, for which we write $\mathcal{A}_{n,j}$. Unfortunately, the approach has been successful only for 0 and 1-loop graphs (g-loop means the first Betti number is g).

In [Yos 1], we developed configuration space integrals associated with 2-loop graphs to give a geometric cocycle $Z : H_{j-1}(\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^{j+2})) \to \mathcal{A}_{n,j} \otimes \mathbb{R}$. What we do in this article is to give a construction of a geometric cycle $d \in H_{j-1}(\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^{j+2}))$ such that $Z(d) = \pm$ .

Although there is a natural map $\chi : \mathcal{B}_{n,j} \to \mathcal{A}_{n,j}$ between the two graph homologies, the kernel and cokernel of χ are not known. As we can observe in the previous paragraph, the injectivity of χ plays an essential role in showing the non-triviality of cycles.

[1]Embedding calculus gives a tower of approximation $\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n) \to T_k\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)$ which is, if $n-j \geq 3$, higher and higher connected when k increases.

[2]Since $\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)(n-j \geq 3)$ is a group-like homotopy associative H-space, all path-components of $\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)(n-j \geq 3)$ are homotopy equivalent.

Question 0.5. Show that χ is injective. Redefine $\mathcal{A}_{n,j}$ and the geometric approach if necessary so that χ is injective. [3]

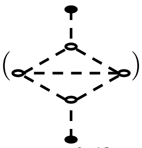
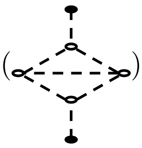
The following is a partial result of the above question.

Theorem 0.6. [Yos 2] The image of any non-trivial element in $\mathcal{B}_{n,j}$ by the map χ does not vanish by STU, IHX and orientation relations of $\mathcal{A}_{n,j}$.

This article is organized as follows. In Section 1, we state our main result more precisely. In Section 2, we construct the geometric cycle d after reviewing Sakai and Watanabe's construction for the 1-loop case. In Section 3, we define the graph homologies $\mathcal{A}_{n,j}, \mathcal{B}_{n,j}$. In Section 4, we introduce configuration space integrals associated with 2-loop graphs. In Section 5, we state a key lemma for cocycle-cycle pairings.

1 Main Result

We write $\overline{\mathcal{K}}_{n,j}$ (resp. $\mathcal{K}_{n,j}$) for $\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^{j+2})$ (resp. $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^{j+2})$).

Notation 1.1. $k(\Gamma) = \#V(\Gamma)/2$. $g(\Gamma) = b_1(\Gamma)$. For example, $k($$) = 3$ and $g($$) = 2$.

Note that $k(\Gamma)$ is an integer for any graph representing an element of $\mathcal{B}_{n,j}$.

Theorem 1.2 (Y.)**.** Let $g = 2$ and let $n, j =$ odd such that $n - j \geq 2$ and $j \geq 3$. If the natural map $\chi : \mathcal{B}_{n,j}(k,g) \to \mathcal{A}_{n,j}(k,g)$ is injective, we have

$$\dim H_{k(n-j-2)+(g-1)(j-1)}(\overline{\mathcal{K}}_{n,j}, \mathbb{Q}) \geq \dim \mathcal{B}_{n,j}(k,g).$$

Observe that if $n - j - 2 = 0$, the degree of the homology does not depend on k. Moreover, we can take sphere cycles in the unknot component as generators of the nontrivial elements:

Corollary 1.3 (Y.)**.** Under the above assumptions, $\pi_{(g-1)(j-1)}(\overline{\mathcal{K}}_{j+2,j})_u \otimes \mathbb{Q}$ is infinite dimensional.

Remark 1.4. If $\pi_{j-1}(\overline{\mathcal{K}}_{j+2,j})_u \otimes \mathbb{Q}$ is infinite-dimensional, so is $\pi_{j-1}(\mathcal{K}_{j+2,j})_u \otimes \mathbb{Q}$.

On the other hand, for $(n,j) = (3,1)$, we have

Remark 1.5. [Hat] $\pi_*(\mathcal{K}_{3,1})_u$ is trivial.

Note that Budney-Gabai [BG] and Watanabe [Wat 2] already showed that $\pi_{j-1}(\mathcal{K}_{j+2,j})_u \otimes \mathbb{Q}$ is infinite-dimensional for $j \geq 2$. The main point of our result is that we construct geometric (co)cycles of the space of long embeddings even for the case $n - j = 2$, associated with 2-loop hairy graphs that produce non-trivial elements for $n - j \geq 3$ in Theorem 0.2.

2 Cycles: ribbon presentations

2.1 Sakai and Watanebe's construction ($g = 1$)

In [Sak, SW, Wat 1], Sakai and Watanabe constructed $k(n-j-2)$-cycles

$$c_k : (S^{n-j-2})^k \to \mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$$

for $k \geq 2$, from specific oriented immersed disks in $\mathbb{R}^3$ called *wheel-like ribbon presentations*. These cycles are detected by configuration space integrals associated with 1-loop graphs.

[3] On the other hand, $\mathcal{B}_{n,j}$ is a "good" graph homology in the sense that $\mathcal{B}_{n,j}$ has an operadic description, and $\mathcal{B}_{n,j}$ is related to (anti-)symmetric polynomials.

Definition 2.1 ([HKS, HS])**.** A *ribbon presentation* $P = \mathcal{D} \cup \mathcal{B}$ is an oriented immersed 2-disk in $\mathbb{R}^3$ which satisfies the following.

- $\mathcal{D} = D_0 \cup D_1 \cdots \cup D_k$ are disjoint $(k+1)$ disks ($D_i \approx D^2$). There is a basepoint $*$ on the boundary of D_0.
- $\mathcal{B} = B_1 \cup \cdots \cup B_k$ are disjoint k bands ($B_i \approx I \times I$).
- Each band connects two disks so that $\{0\} \times I \subset \partial D_i$ and $\{1\} \times I \subset \partial D_j$ for some i and j. Each band can intersect transversally with the interiors of disks except for D_0, along curves parallel to $\{0\} \times I$ in the band. These intersections (with their neighborhoods) are called *crossings* of the ribbon presentation.

We always assume that ribbon presentations are simple: all leaves intersect with exactly one band. A *wheel-like ribbon presentation* is a simple ribbon presentation that satisfies

- B_{i+1} connects D_0 and D_i (B_1 is considered as B_{k+1}).
- B_i intersects with D_i and does not intersect with other disks. (See Figure 1.)

A disk is a *node* if it intersects with no bands. A disk is a *leaf* if it intersects with at least one band and is connected by exactly one band. Note that all disks except D_0 of wheel-like ribbon presentations are leaves.

Remark 2.2. Locally, a crossing of a ribbon presentation is described as follows.

$$B = \{(x_1, x_2, 0) \in \mathbb{R}^3 \,|\, |x_1| \leq 1/2, |x_2| < 3\},$$
$$D = \{(x_1, 0, x_3) \in \mathbb{R}^3 \,|\, |x_1|^2 + |x_3|^2 \leq 1\}.$$

Definition 2.3. For a ribbon presentation P with k crossings, we define the *thickening* V_P of P by

$$V_P = \mathcal{B} \times [-1/4, 1/4]^{j-1} \bigcup \mathcal{D} \times [-1/2, 1/2]^{j-1}.$$

Observe that the thickness of bands and disks are different. After the smoothing of corners of V_P, we take the boundary of V_P and construct the connected sum

$$\varphi_k = \partial V_P \# \iota(\mathbb{R}^j).$$

Then we obtain a long embedding $\varphi_k : \mathbb{R}^j \to \mathbb{R}^n$ after a suitable parametrization. A part corresponding to a crossing of the ribbon presentation is the union of a punctured sphere $\hat{D}_i \approx S^j$ and a tube $\hat{B}_i \approx I \times S^{j-1}$. We call $\hat{D}_i \cup \hat{B}_i$ (with its neighborhood) a *crossing* of φ_k. See Figure 2.

Notation 2.4. We see $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^{n-j-2} \times \mathbb{R}^{j-1}$. Ribbon presentations are constructed in $\mathbb{R}^3$. Thickening is performed using coordinates of $\mathbb{R}^{j-1}$. We see the parameter space S^{n-j-2} as

$$S^{n-j-2} = \{(x_3, \ldots, x_{n-j+1}) \in \mathbb{R}^{n-j-1} \,|\, (x_3 - 1)^2 + x_4^2 + \cdots + x_{n-j+1}^2 = 1\}.$$

Definition 2.5. [Wat 1, SW] The *perturbation* of a crossing to the direction $v \in S^{n-j-2}$ is an operation to replace the band B with the perturbed band $B(v)$. Locally, $B(v)$ is described as

$$B(v) = \{(x_1, x_2, \gamma(x_2)v \in \mathbb{R}^2 \times \mathbb{R}^{n-j-1} \,|\, |x_1| \leq 1/2, |x_2| < 3\},$$

using a test function γ whose support is in $[-3, 3]$. Compare with Lemma 2.2.

Let P be a ribbon presentation with k crossings. For $\mathbf{v} = (v_1, \ldots, v_k) \in (S^{n-j-2})^k$, we can construct the perturbed presentation $P_{\mathbf{v}} = \mathcal{D} \cup \mathcal{B}(\mathbf{v}) = \bigcup D_i \cup \bigcup B_j(v_j)$ and the thickening $V_{P_{\mathbf{v}}}$.

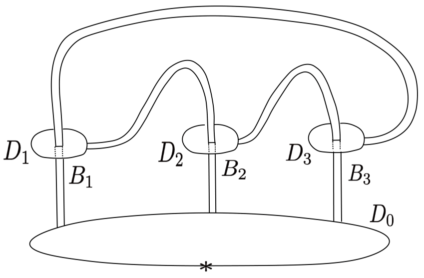


Figure 1: The wheel-like ribbon presentation ($k = 3$)

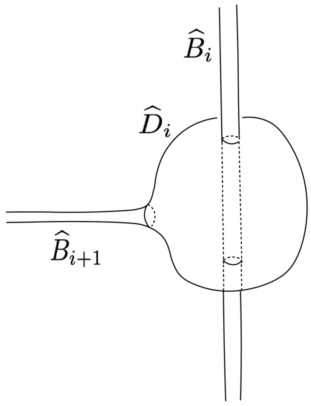


Figure 2: The i-th crossing

Definition 2.6. Let P be the wheel-like ribbon presentation with k crossings. The *wheel-like cycle* $c_k : (S^{n-j-2})^k \to \mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ is defined by

$$\mathbf{v} \longmapsto \varphi_k^{\mathbf{v}} = \partial V_{P_{\mathbf{v}}} \# \iota(\mathbb{R}^j),$$

after smoothing of corners and giving parametrizations form $\mathbb{R}^j$.

Remark 2.7. Each crossing gives a Hopf link $S^{n-j-1} \cup S^j \hookrightarrow \mathbb{R}^n$ because

- $\bigcup_{v_i \in S^{n-j-2}}$ (core of $\hat{B}_i(v_i)) \approx \Sigma S^{n-j-2} \approx S^{n-j-1}$
- $\hat{D}_i \approx$ (punctured) S^j

2.2 Our construction ($g = 2$)

We construct $(k(n-j-2)+(j-1))$-cycles

$$d_{\Theta(p,q,r)} : (S^{n-j-2})^k \times S^{j-1} \to \mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n),$$

($k = p+q+r+1$) detected by configuration space integrals associated with 2-loop graphs. Again, we construct these cycles by perturbating ribbon presentations. Unlike wheel-like ribbon presentations, the ribbon presentations of our cycles have at least one node (drawn in grey.)

Definition 2.8. Let $p, r \geq 1$ and $q \geq 0$. The Θ*-like ribbon presentation of type* (p, q, r) is the ribbon presentation that satisfies the following.

- $\mathcal{D}$ is the union of D_0, $D_{11}, \dots, D_{1p}, D_{1(p+1)}, D_{1(p+2)}$, $D_{21}, \dots, D_{2q}$, $D_{31}, \dots, D_{3r}$.
- $\mathcal{B}$ is the union of $B_{11}, \dots, B_{1p}, B_{1(p+1)}, B_{1(p+2)}$, $B_{21}, \dots, B_{2q}$, $B_{31}, \dots, B_{3r}$.
- B_{*i} connects D_0 and D_{*i} if $(*, i) \neq (1, p+1), (1, p+2)$.
- $B_{1(p+1)}$ connects D_{1p} and $D_{1(p+1)}$. $B_{1(p+2)}$ connects D_{1p} and $D_{1(p+2)}$.
- D_{*i} intersects with $B_{*(i+1)}$ if $(*, i) \neq (1, p), (1, p+1), (1, p+2), (2, q), (3, r)$.
- D_{2q} ($D_{1(p+1)}$ if $q = 0$) and $D_{3(r-1)}$ intersect with B_{3r} so that $D_{3(r-1)}$ is closer to D_0 than D_{2q} ($D_{1(p+1)}$ if $q = 0$) is. D_{33} intersects with B_{11}.
- D_{1p} is a node. $D_{1(p+2)}$ intersects with B_{31}. $D_{1(p+1)}$ intersects with B_{21} (with B_{3r} if $q = 0$). The orientations of these two crossings must be opposite.

The node D_{1p} is connected to two leaves ($D_{1(p+1)}$ and $D_{1(p+2)}$) intersecting with (possibly the same) bands in opposite orientations. In this article, we will focus on the simplest example: the Θ-like ribbon presentation of type $(1,0,1)$. See Figure 3.

In the same way as wheel-like ribbon presentations, we can give a $(S^{n-j-2})^k$ family from k crossings. Moreover, we can give another S^{j-1} family by moving one tube $\hat{B_{12}}$ around another tube $\hat{B_{13}}$. See Figure 4. Note that the cores of $\hat{B_{12}}$ and $\hat{B_{13}}$ are not linked during this process since $n \geq 4$.

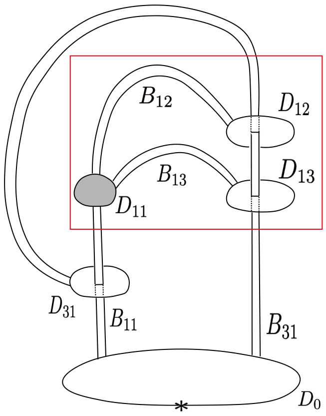


Figure 3: The Θ-like ribbon presentation of type $(1,0,1)$

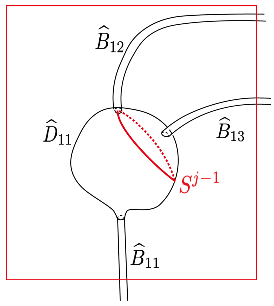


Figure 4: The additional S^{j-1} family

2.3 Replacing the parameter space with sphere

Here, we replace the parameter space of $d_{\Theta(1,0,1)}$ with sphere when $n-j=2$. General case $d_{\Theta(p,q,r)}$ are similar to $d_{\Theta(1,0,1)}$. Recall that $d_{\Theta(1,0,1)} : S^{j-1} \times S^0 \times S^0 \times S^0 \to \mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^{j+2})$. Here, each S^0 is written as

$$S^0 = \{\text{insert the band } (In), \text{remove the band } (Out)\}.$$

Then $d_{\Theta(1,0,1)}$ is the sum of 2^3 components:

$$d_{\Theta(1,0,1)} = \sum_{\varepsilon_i \in S^0} d_{\Theta(1,0,1)}(\varepsilon_1, \varepsilon_2, \varepsilon_3).$$

Lemma 2.9. If the band of some crossing is Out, that is, if some ε_i is Out, the component is null-homotopic, and hence a boundary.

Proof. If the crossing with D_{12} or that with D_{13} is Out, the corresponding S^{j-1} cycle is null-homotopic. See Figure 5. If the crossing with D_{31} is Out, the crossing with D_{12} is also resolved. Then again the corresponding S^{j-1} cycle is null-homotopic. □

By Lemma 2.9, we have $[d_{\Theta(1,0,1)}] = [d_{\Theta(1,0,1)}(\mathrm{In}, \mathrm{In}, \mathrm{In})] \in H_{j-1}\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^{j+2})$.

2.4 Taking the cycle from the unknot component

We show that $d_{\Theta(1,0,1)}(\mathrm{In}, \mathrm{In}, \mathrm{In})$ is in the unknot component when $n-j=2$. For general cases $d_{\Theta(p,q,r)}(\mathrm{In}, \mathrm{In}, \dots, \mathrm{In})$, see Remark 2.11. Note that if $n-j \geq 3$, we can easily show $d_{\Theta(p,q,r)}$ is in the unknot component, because each crossing is not linked if the parameter moving on S^{n-j-2} is fixed.

Recall that the ribbon presentation for $d_{\Theta(1,0,1)}(\mathrm{In}, \mathrm{In}, \mathrm{In})$ has a specific part $\boxed{\quad}$ as in Figure 3.

Lemma 2.10. The part $\boxed{\quad}$ is resolved if the parameter $\theta \in S^{j-1}$ is fixed.

Proof. We use $S4$ move [HKS, HS] of ribbon presentations as in Figure 7. □

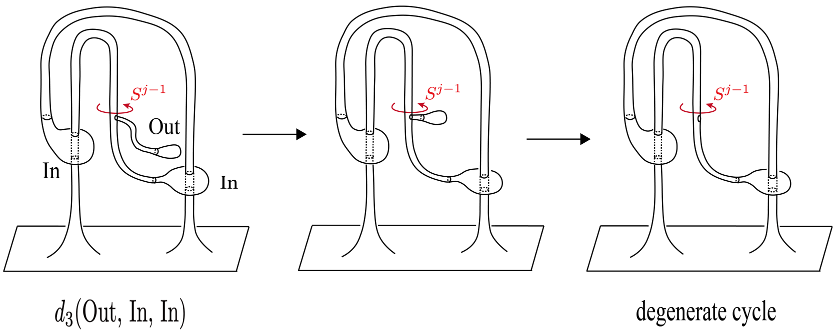


Figure 5: A component with a band Out

So we have $[d_{\Theta(1,0,1)}] = [d_{\Theta(1,0,1)}(\text{In, In, In})] \in \pi_{j-1}\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^{j+2})_u$

Remark 2.11. Recall that the node D_{1p} is connected to two leaves $D_{1(p+1)}$ and $D_{1(p+2)}$. If $q \geq 1$, the two leaves $D_{1(p+1)}$, $D_{1(p+2)}$ intersect with different bands. However, by using the move in Figure 6 repeatedly, we can replace the ribbon presentation so that $D_{1(p+1)}$ and $D_{1(p+2)}$ intersect with the same band. Moreover, we can show that the value of the cocycle-cycle pairing in Section 5 does not change before and after this move. [4]

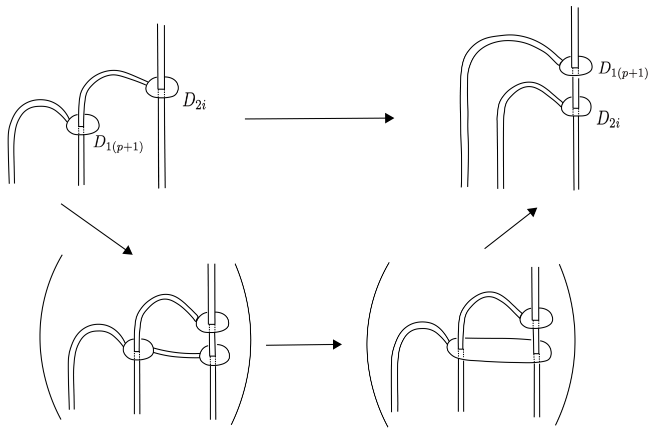


Figure 6: The move which does not change the value of the pairing

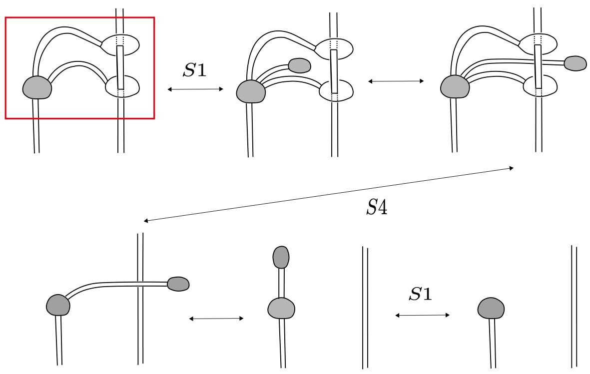


Figure 7: $d_{\Theta(1,0,1)} \subset$ (unknot component)

3 Graph homologies

In knot theory, The following graph homologies $\mathcal{A}(S^1)$ and $\mathcal{B}$ are well known.

$$\mathcal{A}(S^1) = \mathbb{Q}\{\text{Jacobi graphs}\}/\text{STU, IHX, orientation(AS)},$$
$$\mathcal{B} = \mathbb{Q}\{\text{hairy graphs}\}/\text{IHX, orientation(AS)}$$

(The dual of) The quotient $\mathcal{A}(S^1)/1T$ by the 1T relation is known to be isomorphic to the space of $\mathbb{Q}$-valued Vassiliev invariants.

Theorem 3.1. [Bar] There exists an isomorphism $\chi : \mathcal{B} \to \mathcal{A}(S^1)$, called Poincaré-Birkhoff-Witt isomorphism.

[4]This argument is analogous to the fact that the values of Vassiliev invariants of order $\leq k$ are 0, for singular knots with more than k singularities.

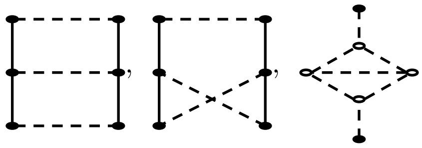
Our graph homology $\mathcal{A}_{n,j}$ is an analog of $\mathcal{A}(S^1)/1T$. $\mathcal{B}_{odd,odd}$ is isomorphic to the connected part of $\mathcal{B}$. We will introduce another graph homology $\mathring{\mathcal{A}}_{n,j}$, which is an analogue of $\mathcal{A}(S^1)$.

Graphs generating $\mathcal{A}_{n,j}$ and $\mathcal{B}_{n,j}$ are introduced by Bott, Cattaneo and Rossi, so they are called *BCR graphs*.

Definition 3.2. A (non-degenerate) *BCR graph* is a connected graph that has

- two types of vertices: black vertices $\bullet$ of degree $(-j)$ and white vertices $\circ$ of degree $(-n)$,
- two types of edges: solid edges ——— of degree $(j-1)$ and dashed edges - - - - - - - of degree $(n-1)$

such that each white (resp. black) vertex has exactly three (resp. one) dashed edges. A *hairy graph* is a BCR graph without solid edges.

Example 3.3.  are BCR graphs. The last one is a hairy graph.

Definition 3.4. We define the three graph homologies $\mathring{\mathcal{A}}_{n,j}$, $\mathcal{A}_{n,j}$ and $\mathcal{B}_{n,j}$ as follows.

$$\begin{aligned}
\mathring{\mathcal{A}}_{n,j} &= \mathbb{Q}\{\text{BCR graphs}\}/\text{“STU”, “IHX”, orientation}\\
\mathcal{A}_{n,j} &= \mathbb{Q}\{\text{BCR graphs}\}/\text{“STU”, “IHX”, chord, orientation}\\
\mathcal{B}_{n,j} &= \mathbb{Q}\{\text{hairy graphs}\}/\text{“IHX”, orientation}
\end{aligned}$$

Here, the “IHX” relation is similar to the IHX relation of Jacobi diagrams. The “STU” and Chord relations are defined as in Fig 8, 9.

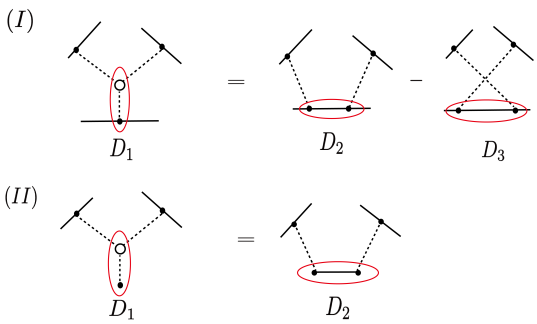


Figure 8: STU relation

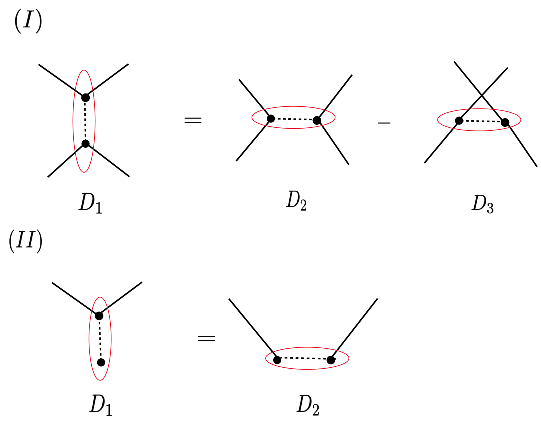


Figure 9: Chord relation

The following is an analogous result to Theorem 3.1.

Theorem 3.5. [Yos 2] The natural map $\chi : \mathcal{B}_{n,j} \to \mathring{\mathcal{A}}_{n,j}$ is a monomorphism.

Proof. The proof is analogous to the proof of Theorem 3.1. We construct a left inverse $\sigma : \mathring{\mathcal{A}}_{n,j} \to \mathcal{B}_{n,j}$ (i.e. $\sigma \circ \chi = \mathrm{id}$) inductively on the number of black vertices. □

4 Cocycles: configuration space integrals

In this section, we quickly review configuration space integrals associated with graphs. Configuration space integrals give cocycles of the space of long embeddings $\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)$.

Definition 4.1 (Configuration space)**.** For a long embedding $\psi \in \mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$, we define the *configuration space* $C_{s,t}(\psi)$ by

$$C_{s,t}(\psi) = \{(x_1, \ldots, x_s, y_{s+1}, \ldots, y_{s+t}) \,|\, x_i \in \mathbb{R}^j, y_j \in \mathbb{R}^n\} \setminus \Delta_\psi,$$

where Δ_ψ is the fat diagonal

$$\Delta_\psi = \bigcup_{1\le i<j\le s} \{x_i = x_j\} \cup \bigcup_{s+1\le i<j\le s+t} \{y_i = y_j\} \cup \bigcup_{\substack{1\le i\le s \\ s+1\le j\le s+t}} \{\psi(x_i) = y_j\}.$$

Intuitively, the first s vertices are on the image of ψ while the last t vertices are in $\mathbb{R}^n$. These $(s+t)$ vertices do not collide each other.

Definition 4.2 (Configuration space bundle)**.** Let $E_{s,t}(\mathbb{R}^j, \mathbb{R}^n)$ be the bundle over $\overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n)$ whose fiber at $\{\widetilde{\psi}(u)\}_{u\in[0,1]}$ is $C_{s,t}(\psi = \widetilde{\psi}(1))$.

Next, we define configuration space integrals associated with graphs. If a graph Γ has s black vertices ● and t white vertices ○, we use the configuration space bundle $E_{s,t}$.

Definition 4.3. Let e be a dashed (resp. solid) edge of Γ. We define the *direction map*

$$P_e : E_{s,t} \to S^{n-1} \quad (\text{resp. } P_e : E_{s,t} \to S^{j-1}),$$

by assigning the direction from the start point to the end point of e. Then a (labeled) graph Γ gives the map $P(\Gamma) : E_{s,t} \longrightarrow \prod S^{j-1} \times \prod S^{n-1}$.

Definition 4.4. Let Γ be a graph such that $k(\Gamma) = k$ and $g(\Gamma) = g$. Let $r_{k,g} = k(n-j-2) + (g-1)(j-1)$. We define the $r_{k,g}$-cochain $I(\Gamma)$ by

$$I(\Gamma)(c) = \int_X c^*\pi_*\Omega(\Gamma) \otimes [\Gamma] \in \mathbb{R} \otimes \mathcal{A}_{n,j}(k,g) \quad (c \text{ is a } d_{k,g}\text{-chain}),$$

where $\Omega(\Gamma) = P(\Gamma)^*(\bigwedge \omega_{S^{j-1}} \wedge \bigwedge \omega_{S^{n-1}})$. [5]

$$\begin{tikzcd}
c^*E_{s,t} \arrow[r] \arrow[d] & E_{s,t} \arrow[r, "P(\Gamma)"] \arrow[d, "\pi"] & \prod S^{j-1} \times \prod S^{n-1} \\
X \arrow[r, "c:\ r_{k,g}\text{-chain}"] & \overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n) &
\end{tikzcd}$$

Theorem 4.5. [SW]$(g=1)$, [Yos 1]$(g=2)$. Define $r_{k,g}$-cochain $Z_{k,g}$ by

$$Z_{k,g}(c) = \sum_{\substack{\Gamma \\ k(\Gamma)=k,\, g(\Gamma)=g}} \frac{I(\Gamma)}{\#\mathrm{Aut}(\Gamma)}(c) \in \mathbb{R} \otimes \mathcal{A}_{n,j}(k,g).$$

Then $Z_{k,g}$ is a cocycle (at least) when n, j: odd and $g = 2$.

[5] Although configuration spaces are open manifolds, there exist canonical compactifications of them, called Fulton Macpherson compactification [Sin]. So the above integral $I(\Gamma)$ is convergent.

5 Cocycle-cycle pairings

Here, we state the key theorem for the parings of the cocycles in Section 4 with the cycles in Section 2.

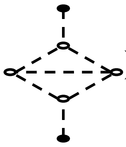
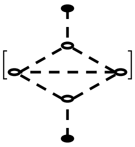
Notation 5.1. The 2-loop hairy graph of type (p, q, r), or the graph $\Theta(p, q, r)$, is the 2-loop planar graph shaped like the character Θ, whose top, middle and bottom edges have p, q and r hairs.

Notation 5.2. Let the graph $\Theta(p, q, r)$ satisfy $k(\Theta(p, q, r))) = k$, or equivalently, $k = p + q + r + 1$. Then we define the $d_{k,2}$-cycle $d(\Theta(p, q, r))$ as $d_{\Theta(p,q,r)}$, which is defined in Section 2.

Theorem 5.3 (Pairing formula)**.** Let $\Theta(p, q, r)$ be as above. Then we have

$$Z_{k,g}(d(\Theta(p, q, r))) = \pm[\Theta(p, q, r)] \in \mathcal{A}_{n,j}(k, 2).$$

Example 5.4.

$Z_{3,2}(d($  $)) = Z_{3,2}(d_{\Theta(1,0,1)}) = \pm[$  $].$

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