THE BREADTH OF THE JONES POLYNOMIAL AND GENERALIZED TWISTS OF A LINK

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ABSTRACT. We investigate the breadth of the Jones polynomial in generalized twists of a link. In particular, we consider how the breadth changes by adding generalized twists. Further, we study the crossing numbers of links obtained by generalized twists.

1. INTRODUCTION

An *m-component link* L is a subset of $\mathbb{R}^3$ that consists of a disjoint union of m-simple closed curves. In particular, a one-component link is a knot.

Let L be a link and D be a diagram of L. The crossing number $c(L)$ of L is defined as follows:

$$c(L) := \min\{c(D) | D \text{ is a diagram of } L\},$$

where $c(D)$ is the crossing number of D. We calculated link invariants by using the software [2].

We list basic facts about linear skein theory. See [3] for details.

Definition 1.1. *The Kauffman bracket is a map from unoriented link diagrams in $\mathbb{R}^2$ to $\mathbb{Z}[A, A^{-1}]$. It is characterized by the following recursive formulas:*

- $\langle \times \rangle = A \langle)\,(\rangle + A^{-1} \langle \asymp \rangle$,
- $\langle \bigcirc \coprod D \rangle = (-A^2 - A^{-2})\langle D \rangle$,
- $\langle \bigcirc \rangle = -A^2 - A^{-2}$.

Definition 1.2. *Let D be an oriented diagram. We define the writhe $w(D)$ of D as the number of positive crossings of D minus the number of negative crossings of D, where a positive crossing and a negative crossing are described in the diagram (See Figure1).*

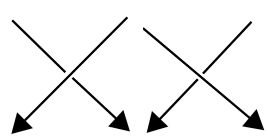


FIGURE 1. The left and right are positive and negative crossings respectively

Definition 1.3. *The Jones polynomial $V_L(t)$ of an oriented link L is a Laurent polynomial in $t^{\frac{1}{2}}$. It is defined by*

$$V_L(t) = \frac{(-A^3)^{-w(D)}}{-A^2 - A^{-2}} \langle D \rangle \mid_{A^2 = t^{-1/2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}],$$

where D is any oriented diagram for L and $w(D)$ is its writhe.

Definition 1.4. *Let L be a link. The breadth of the Jones polynomial $B(V_L(t))$ is defined by the maximum degree of $V_L(t)$ minus the minimum degree of $V_L(t)$. Here, the miximum degree and the minimum degrees are the degrees of $V_L(t)$ with respect to t.*

The following proposition is a basic property of the breadth of the Jones polynomial. Kauffman, Murasugi and Thistlethwaite found this proposition.

Proposition 1.5 ([1], [5], [6], [7])**.** *Let L be a connected link with n-crossing diagram D.*

(1) $B(V_L(t)) \leq n$,
(2) *If D be irreducible and alternating, then $B(V_L(t)) = n$ holds.*

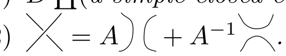
Corollary 1.6. *We have*

$$c(L) \geq B(V_L(t)),$$

where the equality holds when L is alternating.

Definition 1.7. *Let F be an oriented surface with a finite collection of points specified in its boundary ∂F and A be a fixed complex number.*

The linear skein $\mathcal{S}(F)$ of F is the vector space consisting of formal linear sums over $\mathbb{C}$ of link diagrams in F quotiented by the following relations:

(1) $D \coprod$*(a simple closed curve)* $= (-A^{-2} - A^2)D$,
(2) 

Definition 1.8. *We call $S(D^2, 2n)$ the n-th Temperley-Lieb algebra $\mathrm{TL_n}$.*

The $\mathrm{TL_n}$ is generated by elements $\{\mathit{1}_n, e_1, e_2, \ldots, e_{n-1}\}$ shown in the Figure 2 and any base element of the Temperley-Lieb algebra as a vector space is a product of these bases. In diagrams, an integer n beside an arc means n-parallel copies of that arc.

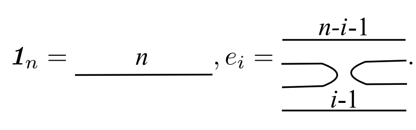
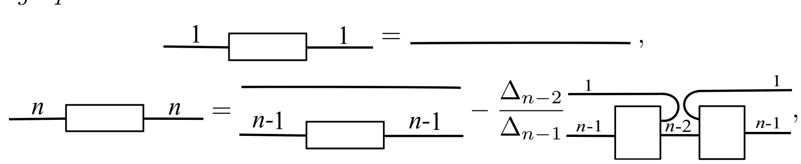


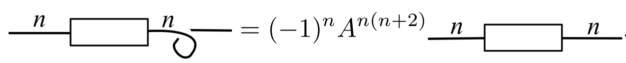
FIGURE 2. The basis of $\mathrm{TL_n}$

Definition 1.9. *We define the Jones-Wenzl idempotent in $\mathrm{TL_n}$ inductively by the following equations:*



where $\Delta_n = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$.

Proposition 1.10. *We use the following formula to calculate the Kauffman bracket:*



Proposition 1.11. *We denote by $f^{(n)}$ the Jones-Wenzl idempotent. The following equalities hold:*

- $f^{(n)} e_i = 0 = e_i f^{(n)} (1 \leq i \leq n-1)$,
- *The closure of $f^{(n)}$ is Δ_n.*

2. GENERALIZED TWISTS OF A LINK

Definition 2.1. *For $(3,3)$-tangles T and S, we denote by TS^n the $(3,3)$-tangle as shown in Figure 3. We also denote by $\widehat{TS^n}$ the link obtained from TS^n by connecting the top and the bottom by three parallel arcs. We call this operation a generalized twist of a link. If S is an ordinary twist, this operation is a twist of the link. This operation is a generalization of twists of links.*

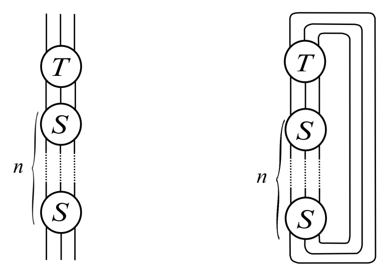
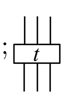
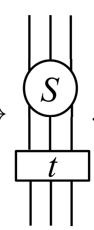
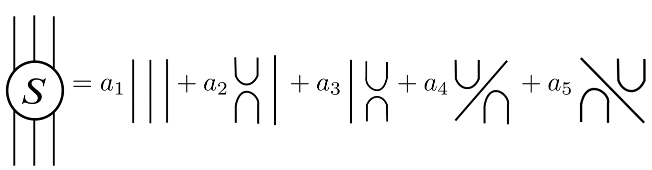


FIGURE 3. The $(3,3)$-tangle TS^n and the link $\widehat{TS^n}$

We consider a linear map S: $\mathrm{TL}_3 \to \mathrm{TL}_3$;  $\mapsto$  . Then, we represent S by five bases of TL_3 as follows:



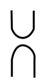
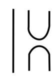
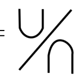
Here, $a_i (i = 1, 2, \ldots, 5)$ are Laurent polynomials of A.

We obtain the following presentation matrix:

Proposition 2.2. *Let S be a $(3,3)$-tangle. Then, the presentation matrix M for the* TL_3 *basis is presented as follows:*

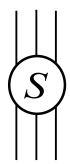
$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_1 + a_2\Delta_1 + a_4 & 0 & 0 & a_2 + a_4\Delta_1 \\ a_3 & 0 & a_1 + a_3\Delta_1 + a_5 & a_3 + a_5\Delta_1 & 0 \\ a_4 & 0 & a_2 + a_4\Delta_1 & a_1 + a_2\Delta_1 + a_4 & 0 \\ a_5 & a_3 + a_5\Delta_1 & 0 & 0 & a_1 + a_3\Delta_1 + a_5 \end{pmatrix}.$$

Proof. For simplicity, v_1, v_2, v_3, v_4 and v_5 are defined as follows:

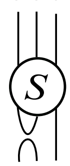
$v_1 = \mathbf{1}_3 =$ , $v_2 = e_1 =$ , $v_3 = e_2 =$ , $v_4 = e_1 e_2 =$ ,

$v_5 = e_2 e_1 =$ .

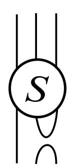
We obtain the following five equalities by applying v_1, v_2, v_3, v_4 and v_5 to the tangle S.



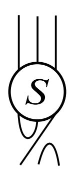
$$= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5,$$



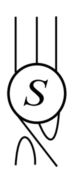
$$= a_1 e_1 + a_2 \Delta_1 e_1 + a_3 e_2 e_1 + a_4 e_1 + a_5 \Delta_1 e_2 e_1$$
$$= (a_1 + a_2\Delta_1 + a_4) v_2 + (a_3 + a_5\Delta_1) v_5,$$



$$= a_1 e_2 + a_2 e_1 e_2 + a_3 \Delta_1 e_2 + a_4 \Delta_1 e_1 e_2 + a_5 e_2$$
$$= (a_1 + a_3\Delta_1 + a_5) v_3 + (a_2 + a_4\Delta_1) v_4,$$



$$= a_1 e_1 e_2 + a_2 \Delta_1 e_1 e_2 + a_3 e_2 + a_4 e_1 e_2 + a_5 \Delta_1 e_2$$
$$= (a_3 + a_5\Delta_1) v_3 + (a_1 + a_2\Delta_1 + a_4) v_4,$$

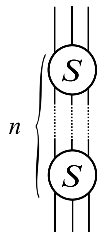


$$= a_1 e_2 e_1 + a_2 e_1 + a_3 \Delta_1 e_2 e_1 + a_4 \Delta_1 e_1 + a_5 e_2 e_1$$
$$= (a_2 + a_4\Delta_1) v_2 + (a_1 + a_3\Delta_1 + a_5) v_5.$$

We obtain the desired result by displaying in a matrix. □

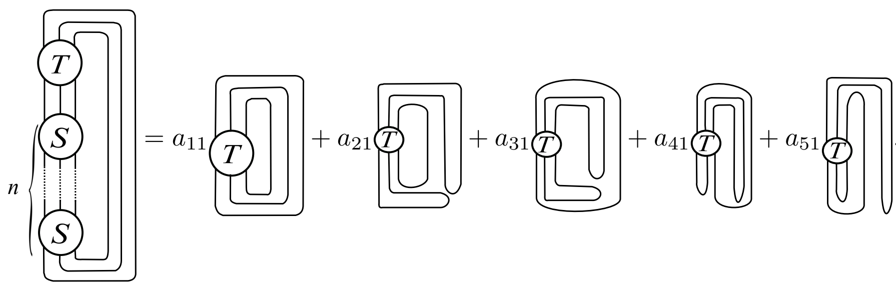
We calculate the n-th power of the presentation matrix. We obtain the eigenvalues and eigenvectors of the presentation matrix, and diagonalize the matrix. Let P be a regular matrix such that $N = P^{-1}MP$ is a diagonal matrix. Note that $M^n = PN^nP^{-1}$. Denote M^n by $(a_{i,j})_{i,j=1,2,\ldots,5}$.

So we can write



$$= a_{11}v_1 + a_{21}v_2 + a_{31}v_3 + a_{41}v_4 + a_{51}v_5.$$

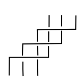
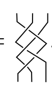
Therefore, we have



3. Main results

We study what the Jones polynomial looks like when the same copies of tangle is composed.

Definition 3.1. *Let T_1, T_o and W be the following tangle diagrams:*

$T_1 :=$  $, T_o :=$  $, W :=$  .

We call T_1 and T_o the ordinary twist.

Remark 3.2. *As elements in TL_3, the following equality holds:*

$$T_o = (-A^9)T_1.$$

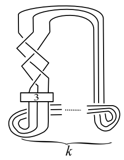
Proposition 3.3. *The breadth of the Jones polynomial and the crossing number of $\widehat{WT_1^k}$ are as follows:*

$$B(V_{\widehat{WT_1^k}}(t)) = \begin{cases} -3k+3 & (k \le -1), \\ 3k+4 & (k \ge 1), \\ 5 & (k = 0), \end{cases} \quad c(\widehat{WT_1^k}) = \begin{cases} -6k-1 & (k \le -2), \\ 6k+1 & (k \ge 1), \\ 6 & (k = -1), \\ 5 & (k = 0). \end{cases}$$

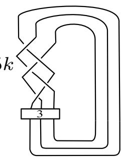
Proof. The Kauffman bracket of $\widehat{WT_o^k}$ is as follows:

$$\begin{aligned}\langle \widehat{WT_o^k} \rangle &= (-1)^k A^{15k+1} \Delta_1 (A^4 + A^{-4}) \\ &\quad - (-1)^k A^{3k-1} \Delta_1 (-A^{10} + 2A^6 - 2A^2 + 2A^{-2} - 2A^{-6} + A^{-10}).\end{aligned}$$

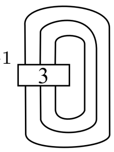
We calculate the Kauffman bracket of $\widehat{WT_o^k}$ in two ways. We have



$= (-1)^k A^{15k}$

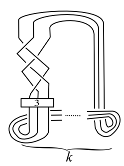


$= (-1)^k A^{15k+1}$



$$= (-1)^k A^{15k+1} \Delta_3$$
$$= (-1)^k A^{15k+1}(-A^6 - A^2 - A^{-2} - A^{-6}).$$

Next, we use the Jones-Wenzl idempotent as follows:



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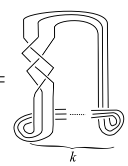
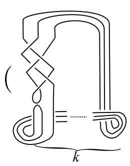


$- \dfrac{\Delta_1}{\Delta_2}($



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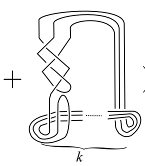

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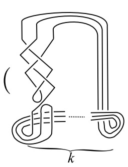
$+ \dfrac{1}{\Delta_2}($



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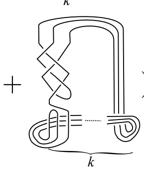


$)$

$$= \langle \widehat{WT_o^k} \rangle + (-1)^k A^{3k-1}(A^{12} - A^8 + A^{-8} - A^{-12}).$$

Therefore, we have

$(-1)^k A^{15k+1}(-A^6 - A^2 - A^{-2} - A^{-6}) = \langle \widehat{WT_o^k} \rangle + (-1)^k A^{3k-1}(A^{12} - A^8 + A^{-8} - A^{-12})$.

The crossing number of $\widehat{WT_1^k}$ can be seen by the linking number.

In the case where $k \leq -2$, we can deform this two-component link as illustrated in Figure 4. This link consists of two components; one component is $(2, -2k-1)$-torus knot $T(2, -2k-1)$, and the other component is the unknot. Further, the linking number of this link is $2k$. Hence, the crossing number between these two components is -2 times the linking number $2k$. Therefore, the crossing number of this link is $-4k + (-2k-1) = -6k - 1$.

Similarly, in the case where $k \geq 1$, this link consists of $(2, 2k+1)$-torus knot and the unknot, and the linking number is $2k$. Therefore, the crossing number is $6k+1$. See Figure 5.

In the case where $k = 0$, this link is the Whitehead link, so the crossing number is 5. Lastly, in the case where $k = -1$, we can reduce the crossings of this diagram significantly. See Figure 6.

Therefore, in the case where $k = -1$, the crossing number is 6. □

Corollary 3.4. *In the case where $k \geq 1$, the following equality holds:*

$$c(\widehat{WT_1^k}) - B(V_{\widehat{WT_1^k}}(t)) = 3k - 3.$$

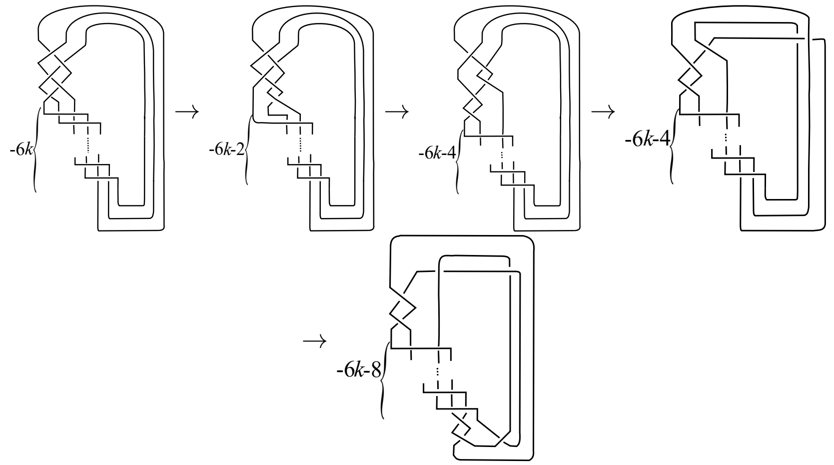


FIGURE 4. In the case where $k \leq -2$

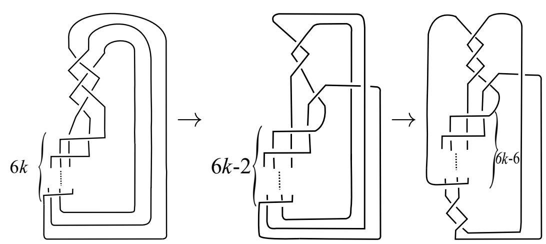


FIGURE 5. In the case where $k \geq 1$

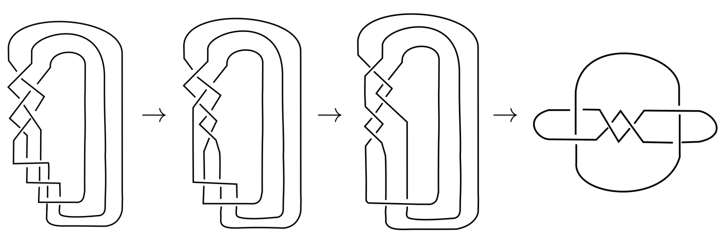
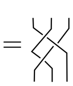


FIGURE 6. The case $k = -1$

In general, the following theorem is obtained:

Theorem 3.5. *Let R be an arbitrary $(3,3)$-tangle. If n is large enough, $B(V_{\widehat{RT_1^n}}(t)) = 3n + a$ holds, where a is an integer.*

Definition 3.6. *Let I be the following diagram:*

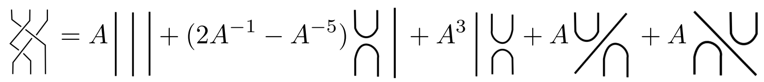
$I :=$  .

Remark 3.7. *When n is odd, The link $\widehat{T_1 I^n}$ is a two-component link. When n is even, the link $\widehat{T_1 I^n}$ is a three-component link.*

Proposition 3.8. *The breadth of the Jones polynomial of* $\widehat{T_1 I^n}$ *is as follows:*

$$B(V_{\widehat{T_1 I^n}}(t)) = \begin{cases} 2n+4 & (0 \le n \le 4), \\ 3n & (n \ge 4), \end{cases} \quad c(\widehat{T_1 I^n}) = \begin{cases} 2n+4 & (1 \le n \le 3), \\ 3n+2 & (n \ge 4). \end{cases}$$

Proof. As an element in TL_3, the following equality holds:



Therefore, we have $a_1 = A, a_2 = 2A^{-1} - A^{-5}, a_3 = A^3, a_4 = A$ and $a_5 = A$.

Hence, we obtain the following presentation matrix $M_2 = (a_{i,j})$.

$a_{11} = A, a_{12} = a_{13} = a_{14} = a_{15} = 0,$
$a_{21} = 2A^{-1} - A^{-5}, a_{22} = A^{-7} - A^{-3}, a_{23} = a_{24} = 0, a_{25} = -A^{-5} + A^{-1} - A^3,$
$a_{31} = A^3, a_{32} = 0, a_{33} = A - A^5, a_{34} = -A^{-1}, a_{35} = 0,$
$a_{41} = A, a_{42} = 0, a_{43} = -A^{-5} + A^{-1} - A^3, a_{44} = A^{-7} - A^{-3}, a_{45} = 0,$
$a_{51} = A, a_{52} = -A^{-1}, a_{53} = a_{54} = 0, a_{55} = A - A^5.$

Next, we calculate M_2^n. Let x_2 be $\sqrt{1 - 2A^4 + 3A^8 + 3A^{16} - 2A^{20} + A^{24}}$. Then, the eigenvalues of M_2 are as follows:

$$A, \frac{1 - A^4 + A^8 - A^{12} - x_2}{2A^7}, \frac{1 - A^4 + A^8 - A^{12} + x_2}{2A^7}.$$

The multiplicity of A is one, and the multiplicity of $\frac{1-A^4+A^8-A^{12}-x_2}{2A^7}$ and $\frac{1-A^4+A^8-A^{12}+x_2}{2A^7}$ are two. The eigenvector of A is $(\frac{1+A^4+A^8}{A^4}, \frac{1+A^4}{A^2}, \frac{1+A^4}{A^2}, 1, 1)$. The eigenvector of $\frac{1-A^4+A^8-A^{12}-x_2}{2A^7}$ is $(0, -\frac{1-A^4-A^8+A^{12}-x_2}{2A^6}, 0, 0, 1)$ and $(0, 0, -\frac{-1+A^4+A^8-A^{12}-x_2}{2A^2(1-A^4+A^8)}, 1, 0)$. The eigenvector of $\frac{1-A^4+A^8-A^{12}+x_2}{2A^7}$ is $(0, -\frac{1-A^4-A^8+A^{12}+x_2}{2A^6}, 0, 0, 1)$ and $(0, 0, -\frac{-1+A^4+A^8-A^{12}+x_2}{2A^2(1-A^4+A^8)}, 1, 0)$.

The Kauffman bracket of $\widehat{T_1 I^n}$ is as follows:

$$\langle \widehat{T_1 I^n} \rangle = 2^{-n}(2^n A^{8+n} + 2^n A^{16+n} + (\frac{1 - A^4 + A^8 - A^{12} + x_2}{A^7})^n + (\frac{1 - A^4 + A^8 - A^{12} - x_2}{A^7})^n))A^{-6}\Delta_1.$$

From the calculation of Kauffman bracket, the breadth $B(V_{\widehat{T_1 I^n}}(t))$ can be divided into the following two types, that is, the value of $B(V_{\widehat{T_1 I^n}}(t))$ changes depending on the magnitude of $16 + n$ and $5n$.

(1) In the case where $16 + n \ge 5n$, that is, $4 \ge n$.
The maximum degree and minimum degree of the Kauffman bracket are $(16 + n) + (-6) = n + 10$ and $(-7n) + (-6) = -7n - 6$, respectively. Therefore, the breadth of the Kauffman bracket is $B(V_{\widehat{T_1 I^n}}(t)) = 2n + 4$.

(2) In the case where $4 \le n$.
The maximum degree and minimum degree of the Kauffman bracket are $5n + (-6) = 5n - 6$ and $(-7n) + (-6) = -7n - 6$, respectively. Therefore, the breadth of the Kauffman bracket is $B(V_{\widehat{T_1 I^n}}(t)) = 3n$.

It turns out that $\widehat{T_1 I}$ is $T(2,6)$ and $\widehat{T_1 I^2}$ is $T(2,4)\#T(2,4)$. Therefore, $c(\widehat{T_1 I}) = 6$ and $c(\widehat{T_1 I^2}) = 8$ are obtained.

The link $\widehat{T_1 I^3}$ is deformed as in Figure 7.

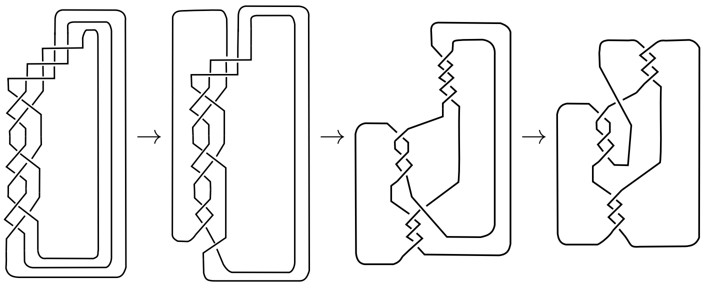


FIGURE 7. The deformation of $\widehat{T_1 I^3}$

In the last figure, the crossing number of the diagram is 8, and this link is alternating. Therefore, we conclude $c(\widehat{T_1 I^3}) = 8$.

Hereafter, we consider the case of $n \geq 4$. We can deform the diagram of $\widehat{T_1 I^n}$ as in Figure 8. The first diagram of $\widehat{T_1 I^n}$ has $3n+6$ crossings, and the last diagram

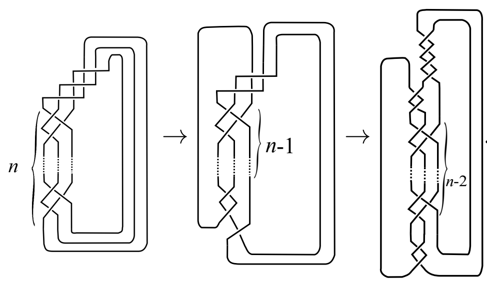


FIGURE 8. The deformation of $\widehat{T_1 I^n}$

has $3n+2$ crossings. Therefore, we have $c(\widehat{T_1 I^n}) \leq 3n+2$.

As noted in the above remark, the link $\widehat{T_1 I^n}$ has different numbers of components depending on the number of twists. Therefore, the proof is divided into the following two cases. See Figure 9.

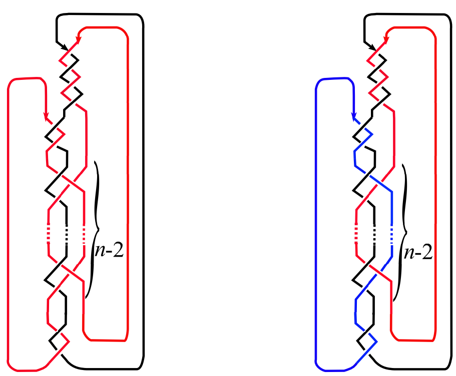


FIGURE 9. In the case of (i)and (ii).

(i) In the case where $n-2$ is odd.

In this case, the number of components is two. In Figure 9, the red component is the $(2, n-2)$-torus knot $T(2, n-2)$, and the black component is the trivial knot. Then, the linking number of $\widehat{T_1 I^n}$ is $n+2$. Therefore, $c(\widehat{T_1 I^n}) \geq (n-2)+2(n+2) = 3n+2$.

(ii) In the case where $n-2$ is even.

In this case, the number of components is three. In Figure 9, each component is the trivial knot, and the red component, the blue component, and the black component are represented by L_1, L_2, L_3, respectively. Then, we have $\mathrm{lk}(L_1, L_2) = \frac{-n+2}{2}$ and $\mathrm{lk}(L_1, L_3) = \mathrm{lk}(L_2, L_3) = \frac{n+2}{2}$.

Therefore, $c(\widehat{T_1 I^n}) \geq |2\,\mathrm{lk}(L_1, L_2)| + 2\,\mathrm{lk}(L_1, L_3) + 2\,\mathrm{lk}(L_2, L_3) = 3n+2$. □

Remark 3.9. *In the case where $1 \leq n \leq 3$, the link $\widehat{T_1 I^n}$ is an alternating link.*

As a corollry, we have

Theorem 3.10. *For $k \geq 4$, the following equality holds:*

$$c(\widehat{T_1 I^n}) - B(V_{\widehat{T_1 I^n}}(t)) = 2.$$

By the above theorem, the following corollary holds:

Corollary 3.11. *There exist infinitely many links with the same $c(L) - B(V_L(t))$, but with distinct the crossing number.*

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