Knitting pattern with three components

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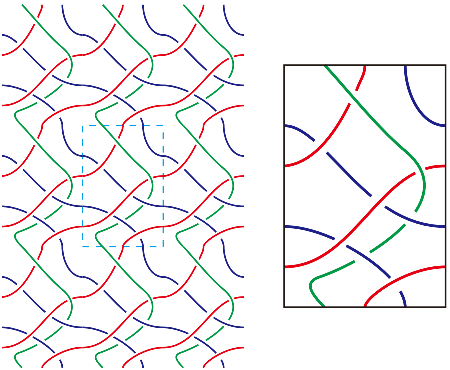
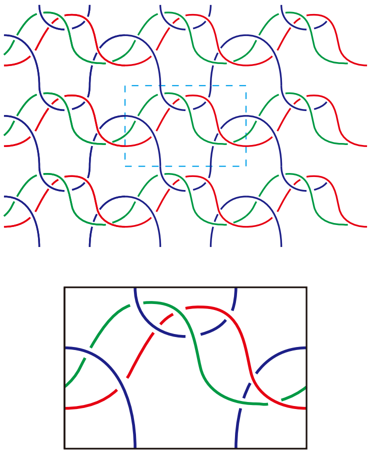
結び目の数理 III

Motivation

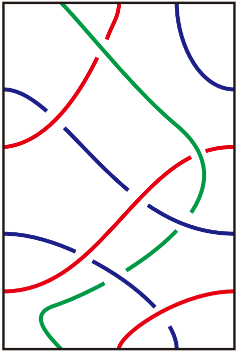
Model 1



Model 2



Today's talk



- Definitions of knitting and its pattern
- Equivalence of them
- Mathematical model of knitting patterns obtained from model 1
- Classification of how green string is entangled in mathematical model

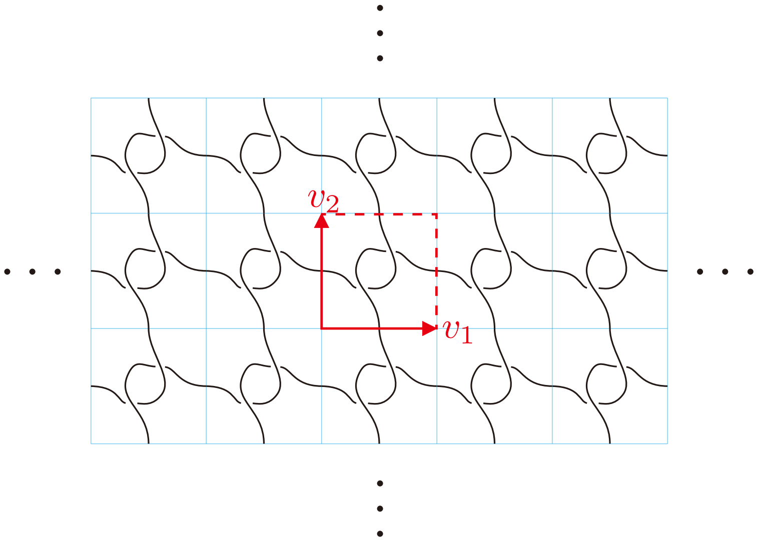
Knitting

$\mathcal{R} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$: basis of $\mathbb{R}^2$
${}^\forall u = (p, q) \in \mathbb{Z}^2$, ${}^\forall \boldsymbol{x} \in \mathbb{R}^2$, $\Phi^{\mathcal{R}}(u, \boldsymbol{x}) = \varphi_u(\boldsymbol{x}) = \boldsymbol{x} + p\boldsymbol{v_1} + q\boldsymbol{v_2}$
$\Rightarrow \Phi^{\mathcal{R}}$: $\mathbb{Z}^2$-*action*.

Definition (Knitting [Kawauchi])

$\widetilde{K}$: quadrivalent graph which are embedded in $\mathbb{R}^2$ and has height information at each of the vertices(crossing points).
$\widetilde{K} \subset \mathbb{R}^2$: knitting $\overset{\text{def}}{\Longleftrightarrow}$ $\widetilde{K}$ is preserved by $\mathbb{Z}^2$-action $\Phi^{\mathcal{R}}$
(i.e., ${}^\exists \mathcal{R} = \langle \boldsymbol{v_1}, \boldsymbol{v_2} \rangle$: basis of $\mathbb{R}^2$ s.t. ${}^\forall u \in \mathbb{Z}^2$, $\varphi_u(\widetilde{K}) = \widetilde{K}$ and φ_u preserves height information at each of the crossing points).

Knitting



Equivalence of knitting

Definition

$\mathcal{R}, \mathcal{R}'$: basis of $\mathbb{R}^2$, $\Phi^{\mathcal{R}}, \Phi^{\mathcal{R}'}$: $\mathbb{Z}^2$-actions

Diffeomorphism $\widetilde{g}$: $(\Phi^{\mathcal{R}}, \Phi^{\mathcal{R}'})$-equivalent

$\overset{\text{def}}{\Longleftrightarrow}$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\widetilde{g}} & \mathbb{R}^2 \\ {\scriptstyle \varphi_u}\downarrow & \circlearrowright & \downarrow{\scriptstyle \varphi'_u} \\ \mathbb{R}^2 & \xrightarrow[\widetilde{g}]{} & \mathbb{R}^2 \end{array}$$

Definition (Equivalence of knittings)

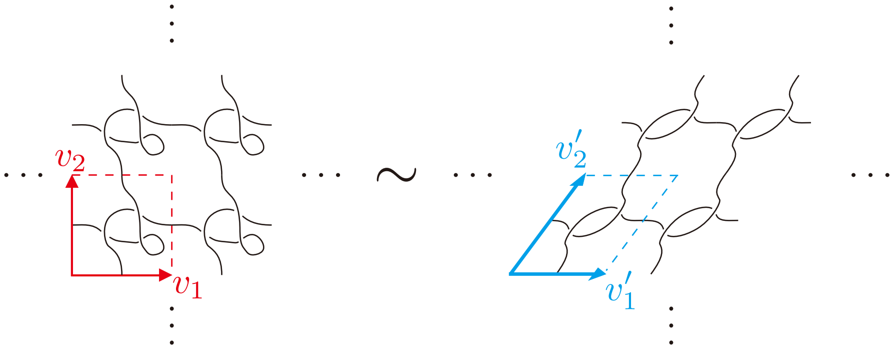
$\widetilde{K}$, $\widetilde{K}'$: knitting

$\widetilde{K}$ and $\widetilde{K'}$ are equivalent ($\widetilde{K} \sim \widetilde{K}'$)

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}\widetilde{g}$: $(\Phi^{\mathcal{R}}, \Phi^{\mathcal{R}'})$-equivalent diffeo.

s.t. $\widetilde{g}(\widetilde{K}) \rightsquigarrow \widetilde{K}'$: $\mathbb{Z}^2$-equivalent Reidemeister moves.

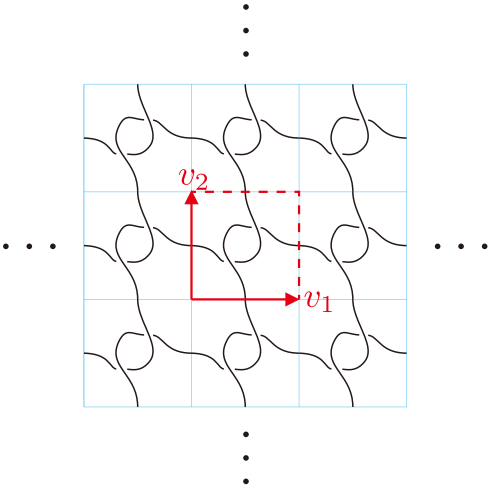
Example



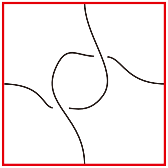
$\mathcal{R} = \langle v_1, v_2 \rangle$: basis of $\mathbb{R}^2$

$\mathcal{R}' = \langle v'_1, v'_2 \rangle$: basis of $\mathbb{R}^2$

Knitting pattern



Knitting



Knitting pattern

Knitting pattern

Definition (Knitting pattern)

$Q \subset \mathbb{R}^2$: closed set, $\mathrm{int}Q$: connected, $\overline{(\mathrm{int}Q)} = Q$

Q : fundamental region of $\widetilde{K}$

$\overset{\mathrm{def}}{\Longleftrightarrow}$ The following conditions are satisfied.

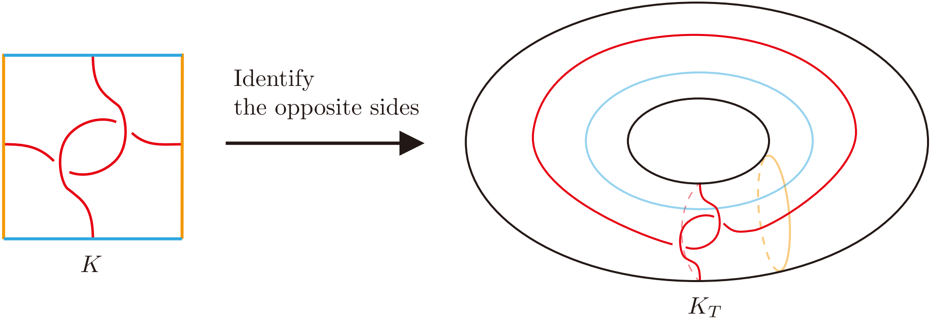
1. $$\bigcup_{u=(p,q)\in\mathbb{Z}^2} \varphi_u(Q) = \mathbb{R}^2$$
2. $\varphi_u(\mathrm{int}Q) \cap \varphi_{u'}(\mathrm{int}Q) = \emptyset \ (u \neq u')$

We assume $Q \cong I \times I$.

Then we call the tangle diagram $K = \widetilde{K} \cap Q$ a knitting pattern.

Link diagram on T^2

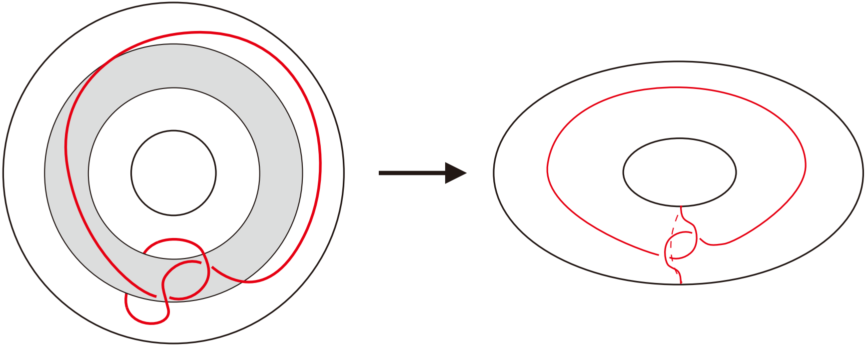
We can obtain a link diagram on T^2 from a knitting pattern.



Link diagram on T^2

$\widehat{T} = T^2 \times I$

pr : $\widehat{T} \to T^2 \times \{0\}$



Link in $\hat{T}$

Link diagram on T^2

Equivalence of knitting pattern

Definition (TR-equivalent)

K_T, K'_T : link diagrams on T^2 obtained from knitting patterns K, K'.
K_T and K'_T are TR-equivalent ($K_T \sim_{TR} K'_T$)
$\overset{\text{def}}{\Longleftrightarrow}$ ${}^\exists g$: orientation preserving diffeomorphism
s.t. $g(K_T) \rightsquigarrow K'_T$: Reidemeister moves on T^2.

Definition

Knitting patterns K and K' are equivalent ($K \sim K'$).
$\overset{\text{def}}{\Longleftrightarrow}$ $K_T \sim_{TR} K'_T$.

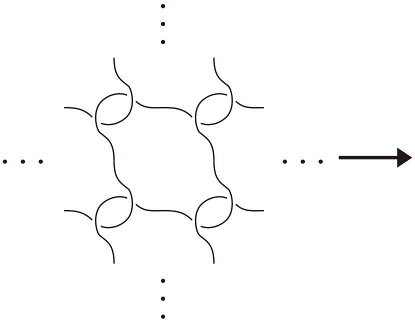
Primitive knitting pattern

Definition

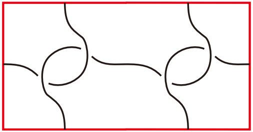
$\widetilde{K}$: knitting, Q : fundamental region of $\widetilde{K}$

K : knitting pattern obtained from $\widetilde{K}$

K : primitive $\overset{\text{def}}{\Longleftrightarrow}$ The area of Q is minimum among all of knitting patterns of $\widetilde{K}$.



primitive knitting pattern



non-primitive knitting pattern

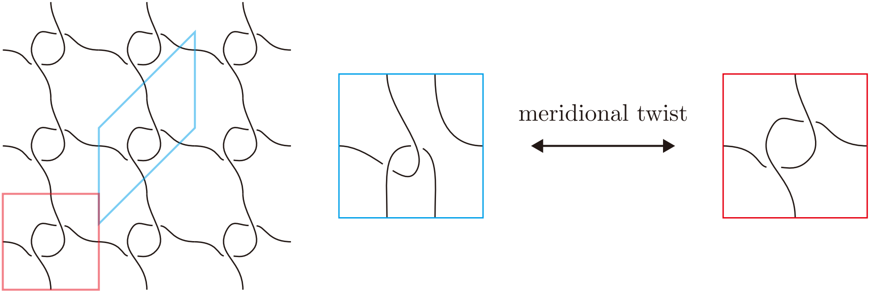
Theorem 1

Theorem 1

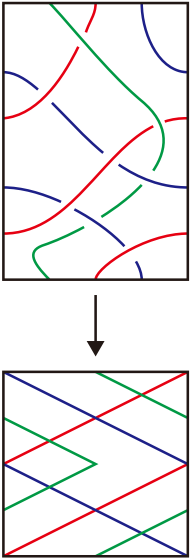
$\widetilde{K}$, $\widetilde{K}'$: knittings

K, K' : primitive knitting patterns obtained from $\widetilde{K}$ and $\widetilde{K}'$.

Then, $\widetilde{K} \sim \widetilde{K}' \Leftrightarrow K \sim K'$



From now, we consider model 1.



Properties of model 1

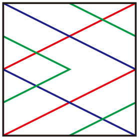
- Consisting of three components and these are essential simple loops on T^2.
- A red string passes over a blue string.
- A green string passes through the complement areas of other strings once.

Step 1 Consider the "slope" for an essential simple component of knitting pattern (Loops in model 1 have slope $\frac{1}{2}$, $-\frac{1}{2}$ and ∞).

Step 2 Classify link projections on T^2 by "word".

Step 3 We get link diagrams on T^2 from link projections obtained in step 2.

In the end, we give the classification of the case $n = 2$. (Theorem 3)



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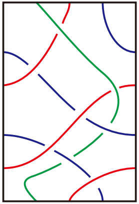

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Slope for a component of a knitting pattern

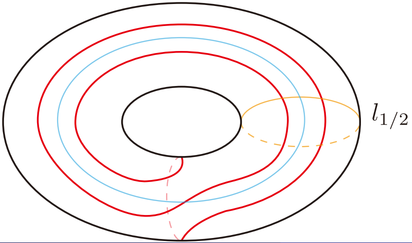
We consider knitting $\widetilde{K}$ consisting only of components which are homeomorphic to $\mathbb{R}$ and these components induce essential simple loops on T^2.

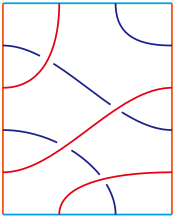
Definition (Slope for a knitting pattern)

l : essential simple loop on T^2

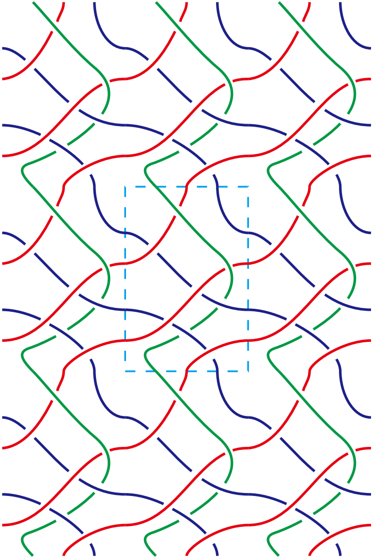
l has slope $\frac{q}{p}$ $(p \in \mathbb{Z}_{\geq 0},\ q \in \mathbb{Z},\ \gcd(p,q) = 1)$

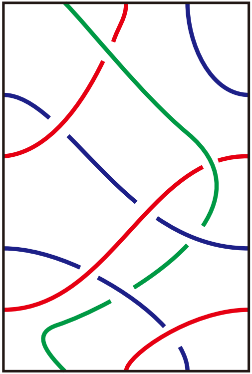
Then, a component l in knitting pattern has slope $\frac{q}{p}$ $(l = l_{q/p})$.





Knitting pattern consisting of $l_{1/2}, l_{-1/2}$

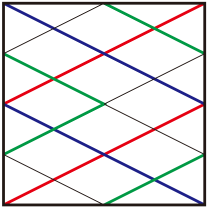


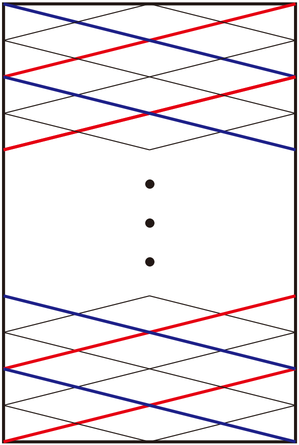


Mathematical model

Properties of model 1

- Consisting of essential simple loops $l_{1/2}$, $l_{-1/2}$ and l_∞ on T^2.
- l_∞ passes through the complement areas of $l_{1/2} \cup l_{-1/2}$ once.





G : graph on T^2 dual to $l_{1/n} \cup l_{-1/n}$.

$l_{q/p}$: essential simple loop on G.

Assumption

$n \in \mathbb{N}$, $p \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{Z}$, gcd(p, q)=1

We consider knitting pattern with three components $l_{1/n}$, $l_{-1/n}$ and $l_{q/p}$ on G.

Definition

$p_1, p_2, q_1, q_2 \in \mathbb{Z}$, gcd$(p_1, q_1)$=gcd$(p_2, q_2)$=1

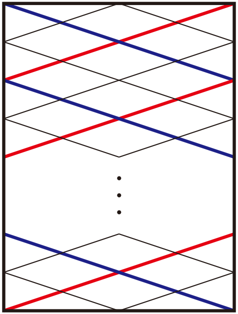
$$\Delta(q_1/p_1, q_2/p_2) = |p_2 q_1 - p_1 q_2|$$

denotes the minimum crossing number between l_{q_1/p_1} and l_{q_2/p_2}.

Observation

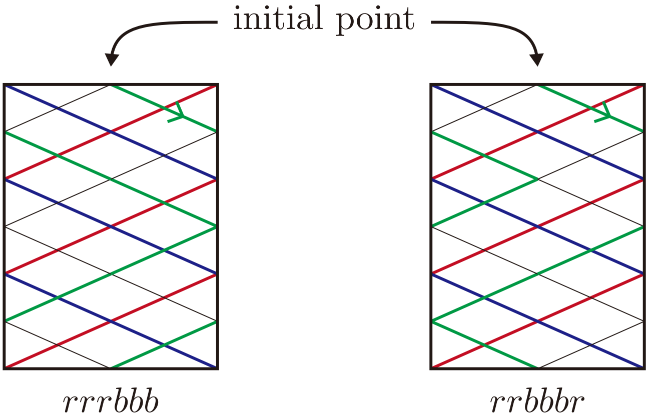
$c(l_{1/n}, l_{q/p}) = \Delta(1/n, q/p)$

(resp. $c(l_{-1/n}, l_{q/p}) = \Delta(-1/n, q/p)$)



Link projection on T^2

Label the crossing point between $l_{q/p}$ and $l_{1/n}$ as r, and $l_{-1/n}$ as b. Then, we get a "word" generated by $\{r, b\}$.



Link projection on T^2

Proposition

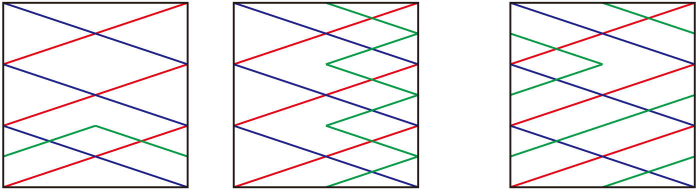
$l_{1/n}, l_{-1/n}$: *essential s.c.c. on* T^2 *with slope* $\frac{1}{n}, -\frac{1}{n}$

$l_{q/p}$: *essential s.c.c. on* T^2 *on* G

$\#r = c(l_{1/n}, l_{q/p}), \#b = c(l_{-1/n}, l_{q/p})$

Then,

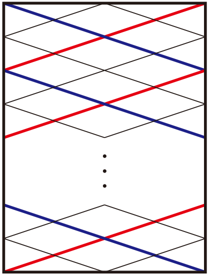
- $\frac{q}{p} = 0 \Rightarrow c = \#r + \#b = 1 + 1 = 2$
- $\frac{q}{p} \neq 0 \Rightarrow c = \#r + \#b = 2n$



$c = 1 + 1 = 2$ $\qquad$ $c = 3 + 3 = 6$ $\qquad$ $c = 2 + 4 = 6$

Link projection on T^2

From now, we consider the case $\frac{q}{p} \neq 0$, and we can assume $|q| = 1$.



Remark

$w_{q/p}$: a word induced from $l_{q/p}$

$\#r = \#\{r \in w_{q/p}\}$, $\#b = \#\{b \in w_{q/p}\}$

Proposition

$l_{1/n}, l_{-1/n}$: *essential s.c.c. on T^2 with slope $\frac{1}{n}, -\frac{1}{n}$*

$l_{q/p}$: *essential s.c.c. on T^2 on G*

$\#r = c(l_{1/n}, l_{q/p})$, $\#b = c(l_{-1/n}, l_{q/p})$

$$^{\forall}t \in \{0, 1, \ldots, 2n\},\ (\#r, \#b) = (t, 2n - t) \Leftrightarrow \frac{q}{p} = \frac{1}{n-t}$$

Link projection on T^2

In this slide, we consider link projections on T^2

Definition (Equivalence of word)

Two words with length $2n$ generated by $\{r, b\}$ are equivalent
$\overset{\text{def}}{\Longleftrightarrow}$ Two words are same up to circular permutation.

Definition (Equivalence of link projection on T^2)

Two link projections are equivalent
$\overset{\text{def}}{\Longleftrightarrow}$ $^{\exists}$orientation preserving diffeomorphism of T^2 taking one projection to the other.

Link projection on T^2

Proposition

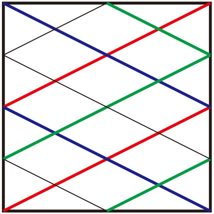
$w = x_1 x_2 \cdots x_{2n}$ $(x_i \in \{r, b\})$: *word*

$w' = x'_1 x'_2 \cdots x'_{2n}$ $(x'_i \in \{r, b\})$: *word*

K_T, K'_T : *link projections on* T^2 *induced by* w *and* w'.

Then,

w *and* w' *are equivalent* $\Leftrightarrow$ K_T *and* K'_T *are equivalent*



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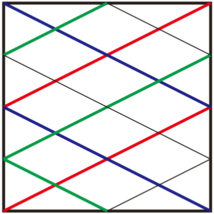

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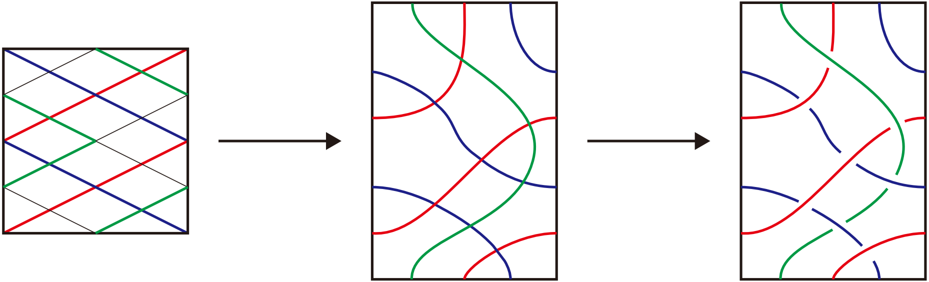
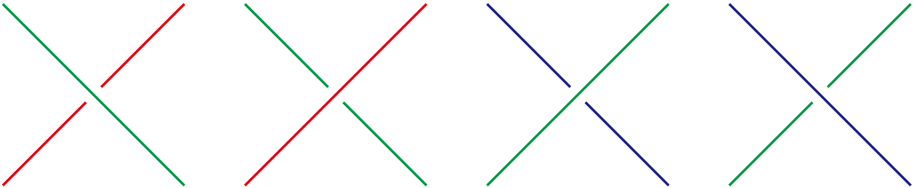


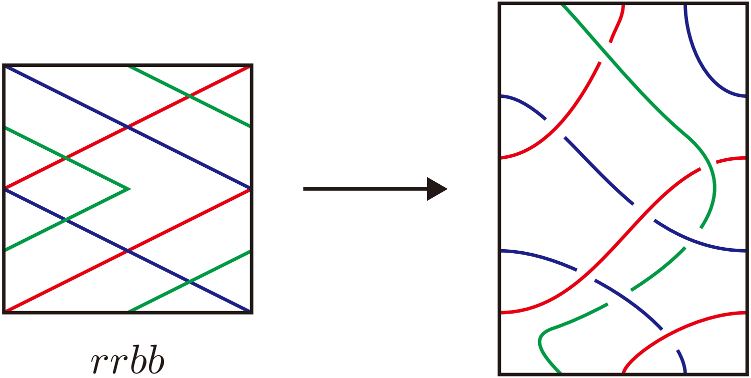
Link diagram on T^2

We consider link diagrams on T^2 with three components $l_{1/n}$, $l_{-1/n}$ and $l_{q/p}$, and $l_{1/n} \geq l_{-1/n}$(i.e. $l_{1/n}$ passes over $l_{-1/n}$ at all of the crossing points.) holds.
Label the crossing points between $l_{q/p}$ and $l_{1/n} \cup l_{-1/n}$ in the link diagram on T^2 either $\overline{r}$, $\underline{r}$, $\overline{b}$, or $\underline{b}$.



$\overline{r}$ $\underline{r}$ $\overline{b}$ $\underline{b}$

Example



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Split link diagram

Definition

D : link diagram on T^2 with n components $l_1, l_2, \ldots, l_n$ $(n \in \mathbb{N})$

$l_i \geq l_j$ (resp. $l_i \leq l_j$) $(i, j \in \{1, 2, \ldots, n\}, i \neq j)$

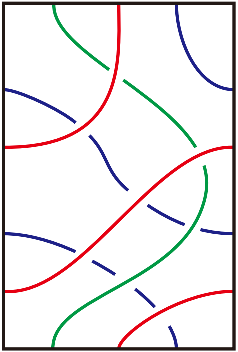
$\overset{\text{def}}{\Longleftrightarrow}$ l_i passes over (resp. under) l_j at all of the crossing points of l_i and l_j or $l_i \cap l_j = \emptyset$ in D.

D : split

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}k \in \{1, 2, \ldots, n\}$ s.t. ${}^{\forall}m \in \{1, \ldots, k-1, k+1, \ldots, n\}$,

$$\begin{cases} l_k \geq l_m \\ l_k \leq l_m \end{cases}$$

Example



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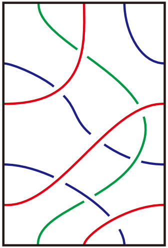
Non split link diagram on T^2

Proposition

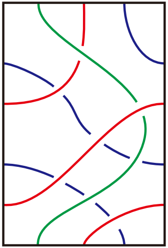
$w = x_1 x_2 \cdots x_{2n}$ $(x_i \in \{\overline{r}, \underline{r}, \overline{b}, \underline{b}\})$: *word*

K_T : *link diagram on* T^2 *induced by* w *with* $l_{1/n} \geq l_{-1/n}$

$$\begin{cases} x_i \in \{\underline{r}, \overline{b}, \underline{b}\} \\ x_i \in \{\overline{r}, \underline{r}, \overline{b}\} \end{cases} \Leftrightarrow K_T : \textit{split}$$



$\underline{r}\underline{r}\overline{b}\underline{b}$



$\overline{r}\underline{r}\overline{b}\overline{b}$

RIII-equivalent

Definition (RIII-equivalent)

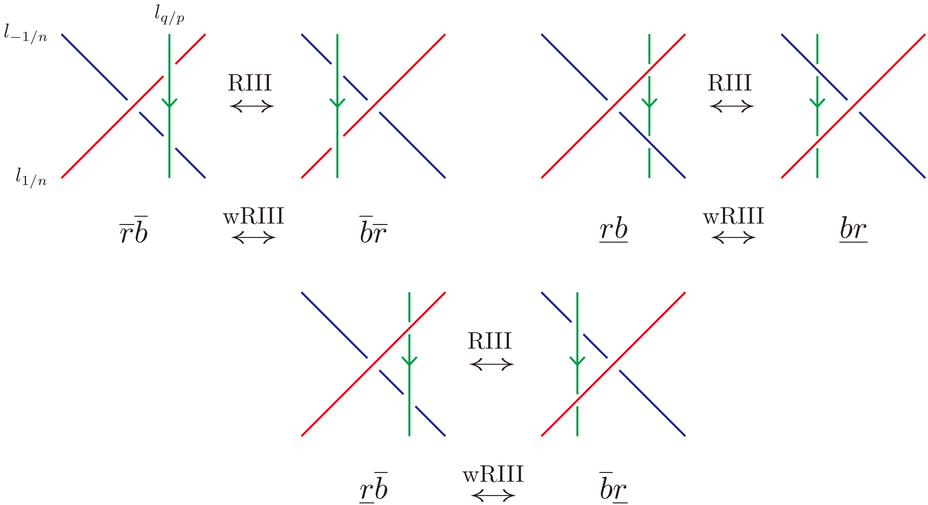
K_T, K'_T : link diagrams on T^2 obtained from knitting patterns K, K'.

K_T and K'_T are RIII-equivalent

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}g$: orientation preserving diffeomorphism

s.t. $g(K_T) \rightsquigarrow K'_T$: Reidemeister moves III on T^2.

Reidemeister move III



Link diagram on T^2

Definition (Equivalence of word)

Two words with length $2n$ generated by $\{\overline{r}, \underline{r}, \overline{b}, \underline{b}\}$ are equivalent $\overset{\text{def}}{\Longleftrightarrow}$ Two words are same words up to circular permutation after finite sequence of wRIIIs.

Theorem 2

$w = x_1 x_2 \cdots x_{2n}$ $(x_i \in \{\overline{r}, \underline{r}, \overline{b}, \underline{b}\})$: word
$w' = x'_1 x'_2 \cdots x'_{2n}$ $(x'_i \in \{\overline{r}, \underline{r}, \overline{b}, \underline{b}\})$: word
K_T, K'_T : link diagrams on T^2 induced by w and w' with $l_{1/n} \geq l_{-1/n}$

w and w' are equivalent $\Leftrightarrow$ K_T and K'_T are RIII-equivalent

Result

Algorithm

$n \in \mathbb{N}$, $t \in \{0, 1, \ldots, 2n\}$

How to get knitting pattern consisting of $l_{1/n}$, $l_{-1/n}$ and $l_{1/(n-t)}$.

1. Consider words generated by $\{r, b\}$ with $(\#r, \#b) = (t, 2n - t)$ up to cyclic permutation.
2. Give hight information to words and exclude word consisting of $\{\underline{r}, \overline{b}, \underline{b}\}$ or $\{\overline{r}, \underline{r}, \overline{b}\}$.(Exclude split knitting patterns)
3. Consider words up to cyclic permutation after finite sequences of wRIIIs.

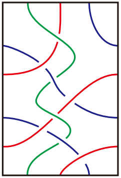
Theorem 3(Classification of the case $n = 2$)

$p \in \{0, 1, 2\}$, $|q| = 1$, gcd(p, q)=1

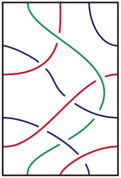
$l_{1/2}$, $l_{-1/2}$, $l_{q/p}$: essential simple loops on T^2 with slope $\frac{1}{2}$, $-\frac{1}{2}$ and $\frac{q}{p}$

There are nine non split knitting patterns consisting of $l_{1/2}$, $l_{-1/2}$ and $l_{q/p}$.

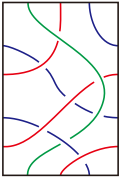
$\frac{q}{p} = \infty$



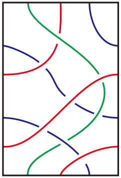
$(\frac{1}{2}, -\frac{1}{2}, \infty)[\overline{r}\underline{b}\overline{r}\underline{b}]$



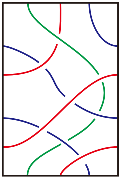
$(\frac{1}{2}, -\frac{1}{2}, \infty)[\overline{rr}\underline{bb}]$



$(\frac{1}{2}, -\frac{1}{2}, \infty)[\overline{rr}\bar{b}\underline{b}]$

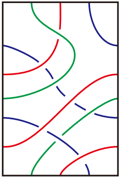


$(\frac{1}{2}, -\frac{1}{2}, \infty)[\overline{r}\underline{r}\bar{b}\underline{b}]$

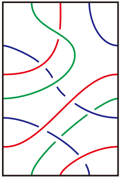


$(\frac{1}{2}, -\frac{1}{2}, \infty)[\overline{r}\underline{rbb}]$

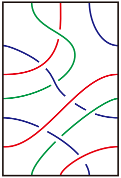
$\frac{|q|}{p} = 1$



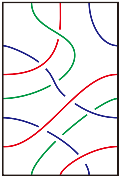
$(\frac{1}{2}, -\frac{1}{2}, 1)[\overline{r}\overline{bb}\underline{b}]$



$(\frac{1}{2}, -\frac{1}{2}, 1)[\overline{r}\bar{b}\underline{bb}]$



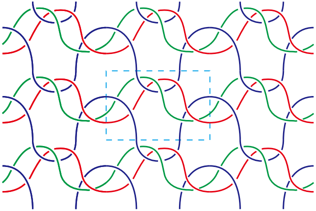
$(\frac{1}{2}, -\frac{1}{2}, 1)[\overline{r}\underline{b}\bar{b}\underline{b}]$

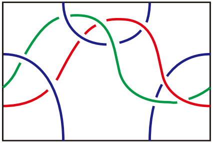


$(\frac{1}{2}, -\frac{1}{2}, 1)[\overline{r}\underline{bbb}]$

Future works

- The relation between equivalence of words and equivalence of link diagrams on T^2 up to Reidemeister moves I, II and III.
- Knitting patterns consisting of $l_{n/m}$, $l_{-n/m}$ and $l_{q/p}$.
- Considering model 2.





Reference

- A. Kawauchi, Complexities of a knitting pattern, Reactive and Functional Polymers, 131, (2018) 230 − 236.
- A. A. Akimova, S. V. Matveev and V. V. Tarkaev, Classification of Links of Small Complexity in the Thickened Torus, Proc.SteklovInst.Math. 303, (Suppl 1) (2018) 12 − 24.
- A. A. Akimova, S. V. Matveev and V. V. Tarkaev, Classification of prime links of in the thickened torus having crossing number 5, Journal of Knot Theory and Its Ramifications 29(03), (2020) 2050012.
- A. A. Akimova, Classification of knots in a thickened torus whose minimal octahedral diagrams do not lie in an annulus, Vestn. Yuzhno-Ural. Gos. Univ., Ser. Mat. Mekh. Fiz. 7(1), (2015) 5 − 10.

Reference

- Sergey V. Matveev, Prime decompositions of knots in $T^2 \times I$, Topology and its Applications, 159, (2012) 1820 − 1824.
- S. Grishanov, V. Meshkov and A. Omelchenko, A Topological Study of Textile Structures. Part I: Topological Invariants in Application to Textile Structures, Textile Research Journal, 79(8), (2009) 702 − 713.
- S. Grishanov, V. Meshkov and A. Omelchenko, A Topological Study of Textile Structures. Part II: Topological Invariants in Application to Textile Structures, Textile Research Journal, 79(9), (2009) 822 − 836.