Quandle coloring quivers for vitrual links

Ryotaro Ueda

Osaka University

研究集会「結び目の数理III」
December 24, 2020

Today's contents

Today's contents

Virtual link diagrams

Definition (Kauffman, 1999)

Virtual link diagram $\Leftrightarrow$ 平面へはめ込まれたいくつかの向き付けられた円周であり, その多重点は有限個の横断的に交わる 2 重点のみで, そこには *classical crossing* または *virtual crossing* の情報が与えられているもの.

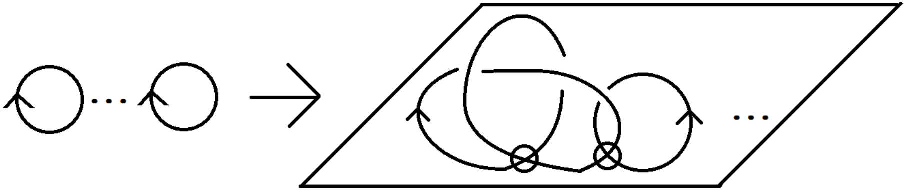


Figure: Virtual link diagram.

Virtual links

Definition (Kauffman, 1999)

D, D' : virtual link diagrams.
$D \sim D' \Leftrightarrow$ 有限回の *generalized Reidemeister moves* で移り合う.
特に有限回の oriented generalized Reidemeister moves で移り合う.
virtual link $\Leftrightarrow$ virtual link diagram の同値類.

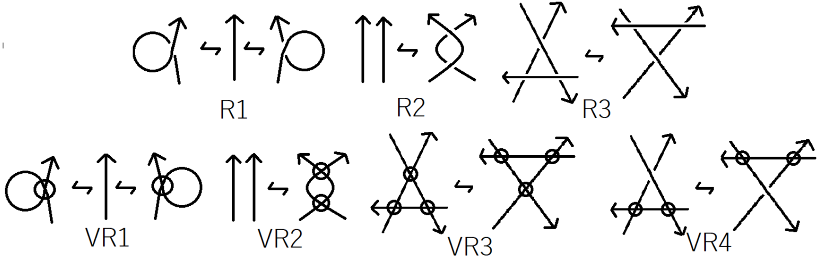


Figure: A generating set of generalized Reidemeister moves.

Quandles

Definition (Joyce, 1982)

次の 3 条件を満たす集合 X と 2 項演算 $*$ の組 $(X,*)$ を *quandle* という.

1. $^\forall x \in X$, $x * x = x$.
2. $^\forall x, y \in X$, $^{\exists!} z \in X$ s.t. $z * y = x$.
3. $^\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

以後 quandle $(X, *)$ を X と書き, 有限なものとする.

Example

X を剰余群 $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, 演算 $*$ を $x * y = 2y - x$ とすれば quandle になる,
この quandle を位数 n の *dihedral quandle* と呼び, R_n と書く.

Quandle automorphisms and endomorphisms

Notation

X : quandle
$\mathrm{End}(X) := \{\ f : X \to X \mid\ f$ は X の endomorphism $\}$.
$\mathrm{Aut}(X) := \{\ f : X \to X \mid\ f$ は X の automorphism $\}$.

Fact

$n \in \mathbb{N}$, ${}^{\forall} f : R_n \to R_n$; endomorphism , ${}^{\exists} a, b \in \mathbb{Z}_n$ s.t. $f(x) = ax + b$.
特に f が automorphism の時は $(a, n) = 1$.

Virtual quandles

Definition (Manturov, 2002)

X : quandle , $f \in \mathrm{Aut}(X)$. このとき 組 (X, f) を *virtual quandle* という.

注意として (X, f) は quandle ではないが, Manturov に従ってこのように呼ぶ.

Definition (Ceniceros-Nelson, 2009)

$(X, f), (X, g)$: virtual quandles.
$(X, f) \sim (X, g) \Leftrightarrow {}^{\exists} \phi \in$ Aut(X) s.t. $f = \phi^{-1} \circ g \circ \phi$.

$$
\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "g"] \\
X \arrow[r, "\phi"'] & X
\end{tikzcd}
$$

Example of virtual quandles

Example

R_3 : 位数 3 の dihedral quandle ,
$f, g \in \mathrm{Aut}(R_3)$, $f(x) = x + 1,\ g(x) = x + 2$.
このとき $(R_3, f) \sim (R_3, g)$ である.
実際, $\phi(x) = 2x$ とすると $f = \phi^{-1} \circ g \circ \phi$ となる.

Fact

$\mathrm{Aut}(R_3) \cong S_3$: symmetric group である.
このとき S_3 の共役類は $(1), (12), (123)$ なので, R_3 上の virtual quandle の同値類は 3 種類ある.

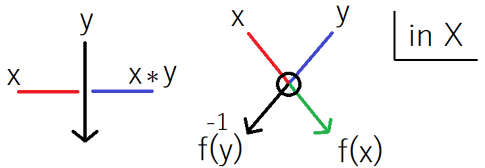
Virtual quandle colorings

Definition

D : virtual link diagram.
D の *virtual arc* $\Leftrightarrow$ D の arc を virtual crossing でさらに分割したもの.
$\mathcal{VA}(D) :=$ { D の virtual arc } .

Definition

(X, f) : virtual quandle, D : virtual link diagram.
$c : \mathcal{VA}(D) \to X$ が (X, f)-*coloring* $\Leftrightarrow$ D の各交点で以下の図の条件を満たす.
$\mathrm{Col}_{(X,f)}(D) :=$ { D の (X, f)-coloring }.
特に f が恒等写像のときは $\mathrm{Col}_X(D)$ とかく.



Example of virtual quandle colorings

D : virtual Hopf link diagram, (R_3, f) : virtual quandle, $f(x) = 2x$.
このとき virtual quandle coloring は次の 3 通りである.

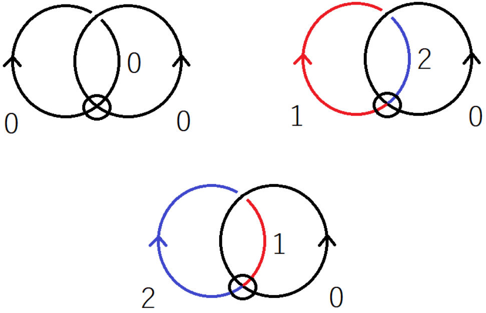


Figure: Virtual Hopf link diagram の virtual quandle coloring.

Property of virtual quandle colorings (1)

Proposition (Manturov, 2002)

(X, f) : virtual quandle.
$D \sim D' \Rightarrow |\mathrm{Col}_{(X,f)}(D)| = |\mathrm{Col}_{(X,f)}(D')|$.
つまり $|\mathrm{Col}_{(X,f)}(D)|$ は virtual link の不変量である.

Proposition (U.)

D : virtual link diagram.
$(X, f) \sim (X, g) \Rightarrow |\mathrm{Col}_{(X,f)}(D)| = |\mathrm{Col}_{(X,g)}(D)|$.
つまり $|\mathrm{Col}_{(X,f)}(D)|$ は virtual quandle の同値類の不変量である.

Property of virtual quandle colorings (2)

Theorem (Ceniceros-Nelson, 2009)

D : virtual link diagram.
$^\exists(X,f),(X,g)$: virtual quandles s.t. $|\mathrm{Col}_{(X,f)}(D)| \neq |\mathrm{Col}_{(X,g)}(D)|$
$\Rightarrow {}^\forall D' \sim D$, D' は virtual crossing を含む.

Example

D : virtual trefoil diagram, $(R_3, f_1), (R_3, f_2)$: virtual quandles, $f_1(x) = 2x+1, f_2(x) = x+1$.
$|\mathrm{Col}_{(R_3,f_1)}(D)| = 3$, $|\mathrm{Col}_{(R_3,f_2)}(D)| = 0$ なので, ${}^\forall D' \sim D$, D' は virtual crossing を含む.

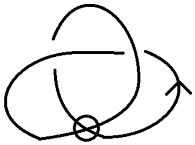


Figure: Virtual trefoil diagram.

Today's contents

1. Virtual links and virtual quandle colorings
2. Definition of virtual quandle coloring quivers
3. Main results

Quandle coloring quivers

Definiton (Cho-Nelson, 2018)

D : classical link diagram , X : quandle.
$S \subset \mathrm{End}(X)$,
$Q_X^S(D)$: *quandle coloring quiver*
$\Leftrightarrow Q_X^S(D) = (V, E)$: an oriented graph.

1. $V = \mathrm{Col}_X(D)$.
2. $E = \{ (v, w, g) \in V \times V \times S \mid w = g \circ v \}$.

ここで辺集合 E には第 1 成分を始点, 第 2 成分を終点として向きを入れる.

Theorem (Cho-Nelson, 2018)

$D \sim D' \Rightarrow {}^{\forall}S \subset \mathrm{End}(X)$, $Q_X^S(D) \cong Q_X^S(D')$.
つまり $Q_X^S(D)$ は classical link の不変量である.

Virtual quandle coloring quivers

Definiton (U.)

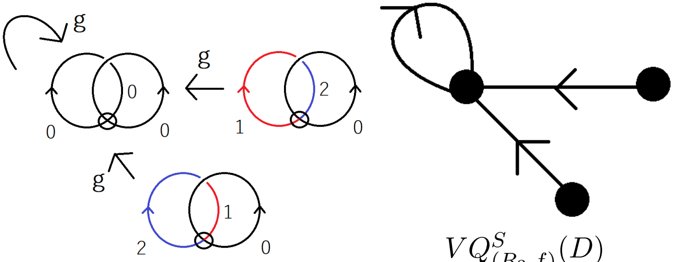
D : virtual link diagram , (X, f) : virtual quandle.
$S \subset \mathrm{Com}_f(X) := \{\ g \in \mathrm{End}(X) \mid f \circ g = g \circ f\ \}$,
$VQ^S_{(X,f)}(D)$: *virtual quandle coloring quiver*
$\Leftrightarrow VQ^S_{(X,f)}(D) = (V, E)$: an oriented graph.

1. $V = \mathrm{Col}_{(X,f)}(D)$.
2. $E = \{\ (v, w, g) \in V \times V \times S \mid w = g \circ v\ \}$.

ここで辺集合 E には第 1 成分を始点, 第 2 成分を終点として向きを入れる.

Example of virtual quandle coloring quivers (1)

D : virtual Hopf link diagram,
(R_3, f) : virtual quandle, $f(x) = 2x$, $g(x) = 0$, $S = \{g\} \subset \mathrm{Com}_f(R_3)$



$VQ^S_{(R_3,f)}(D)$

Today's contents

Main results

Main theorem (1) (U.)

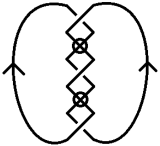
$D \sim D' \Rightarrow {}^{\forall}S \subset \mathrm{Com}_f(X)$, $VQ^S_{(X,f)}(D) \cong VQ^S_{(X,f)}(D')$.
つまり $VQ^S_{(X,f)}(D)$ は virtual link の不変量である.

Remark

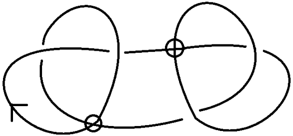
$VQ^S_{(X,f)}(D)$ は $|\mathrm{Col}_{(X,f)}(D)|$ より真に強い virtual link の不変量である.

Example of virtual quandle coloring quivers (2)

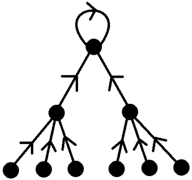
(R_9, f) : virtual quandle, $f(x) = 2x$, $g(x) = 3x$, $S = \{g\} \subset \mathrm{Com}_f(R_9)$,
$|\mathrm{Col}_{(R_9,f)}(D)| = |\mathrm{Col}_{(R_9,f)}(D')| = 9$.



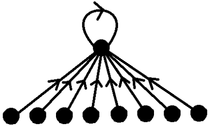
D



D'



$VQ^S_{(R_9,f)}(D)$

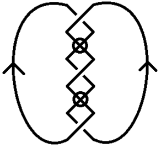


$VQ^S_{(R_9,f)}(D')$

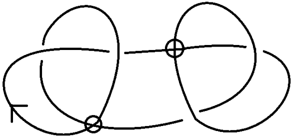
つまり $D \not\sim D'$ $\square$

Example of virtual quandle coloring quivers ($2'$)

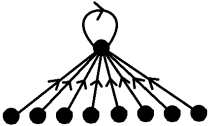
(R_9, f) : virtual quandle, $f(x) = 2x$, $g'(x) = 0$, $S' = \{g'\} \subset \mathrm{Com}_f(R_9)$,
$|\mathsf{Col}_{(R_9,f)}(D)| = |\mathsf{Col}_{(R_9,f)}(D')| = 9$.



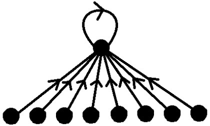
D



D'



$VQ^{S'}_{(R_9,f)}(D)$



$VQ^{S'}_{(R_9,f)}(D')$

つまり S の選び方によって $D \nsim D'$ か分からない場合もある.

Main results

Theorem (Tanigichi, 2020)

p : odd prime, D, D' : classical link diagrams.
$|\mathrm{Col}_{R_p}(D)| = |\mathrm{Col}_{R_p}(D')|$
$\Rightarrow {}^\forall S \subset \mathrm{End}(R_p)$, $Q^S_{R_p}(D) \cong Q^S_{R_p}(D')$.

Main theorem (2) (U.)

p : odd prime, D, D' : virtual link diagrams, $f \in \mathrm{Aut}(R_p)$.
$|\mathrm{Col}_{(R_p,f)}(D)| = |\mathrm{Col}_{(R_p,f)}(D')|$
$\Rightarrow {}^\forall S \subset \mathrm{Com}_f(R_p)$, $VQ^S_{(R_p,f)}(D) \cong VQ^S_{(R_p,f)}(D')$.

Remark

$VQ^S_{(R_p,f)}(D) \cong VQ^S_{(R_p,f)}(D')$
$\Rightarrow \ |\mathrm{Col}_{(R_p,f)}(D)| = |\mathrm{Col}_{(R_p,f)}(D')|$

Main results

Main theorem (3) (U.)

D : virtual link diagram,
$(X,f) \sim (X,g)$, つまり $^{\exists}\phi \in \mathrm{Aut}(X)$ s.t. $f = \phi^{-1} \circ g \circ \phi$.
このとき Φ_ϕ : $\mathrm{Com}_f(X) \to \mathrm{Com}_g(X)$; $h \mapsto \phi h \phi^{-1}$ とすると,
$^{\forall}S \subset \mathrm{Com}_f(X)$, $VQ^{S}_{(X,f)}(D) \cong VQ^{\Phi_\phi(S)}_{(X,g)}(D)$.

Remark

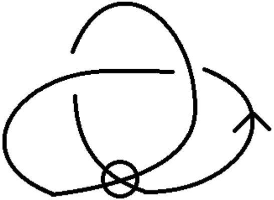
特に Φ_ϕ は全単射なので, $VQ^{\mathrm{Com}_f(X)}_{(X,f)}(D) \cong VQ^{\mathrm{Com}_g(X)}_{(X,g)}(D)$.
つまり $VQ^{\mathrm{Com}_f(X)}_{(X,f)}(D)$ は virtual quandle の同値類の不変量である.

Remark

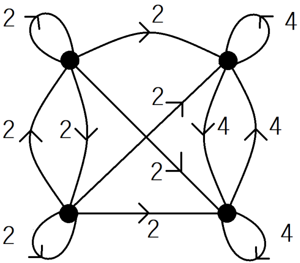
$VQ^{\mathrm{Com}_f(X)}_{(X,f)}(D)$ は $|\mathrm{Col}_{(X,f)}(D)|$ より真に強い virtual quandle の同値類の不変量である.

Example of virtual quandle coloring quivers (3)

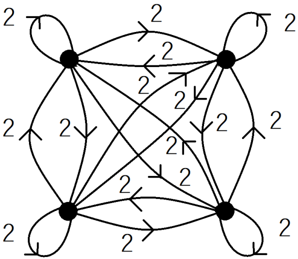
D : virtual trefoil diagram.
$(R_4, f_1), (R_4, f_2)$: virutual quandles, $f_1(x) = 3x + 2, f_2(x) = x + 2,$
$|\mathrm{Col}_{(R_4,f_1)}(D)| = |\mathrm{Col}_{(R_4,f_2)}(D)| = 4.$



D



$VQ^{\mathrm{Com}_{f_1}(R_4)}_{(R_4,f_1)}(D)$



$VQ^{\mathrm{Com}_{f_2}(R_4)}_{(R_4,f_2)}(D)$

つまり $(R_4, f_1) \not\sim (R_4, f_2)$ $\square$

Thank you for your attention.