f-twisted Alexander matrix と quandle homomorphism について

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Quandle

Definition (Joyce, Matveev, 1982)

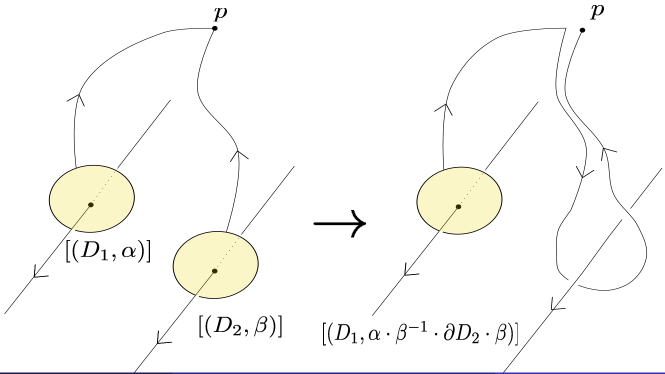
X: a set, $* : X \times X \to X$: a binary operation.
$X = (X, *)$: a *quandle*
$\Leftrightarrow$ $*$ satisfies the following three conditions:

1. $\forall x \in X$, $x * x = x$.
2. $\exists *^{-1} : X \times X \to X$: the binary operation s.t. $\forall x, y \in X$, $(x * y) *^{-1} y = x$.
3. $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

ex.
(1) M: a left $\mathbb{Z}[t^{\pm 1}]$-module, $x * y := tx + (1 - t)y$ $(x, y \in M)$.
Then, $M = (M, *)$: an *Alexander quandle*.
(2) G: a group, $x * y := y^{-1}xy(x, y \in G)$
Then, $\mathrm{Conj}(G) = (G, *)$: the *conjugation quandle* of G.

Fundamental quandle

L: an oriented link in $\mathbb{R}^3$. $p \in \mathbb{R}^3 \setminus \mathrm{int} N(L)$.
$Q(L,p) :=$
$\{(D,\alpha) \mid D : \text{a meridian disk of } L, \alpha : \text{a path from } \partial D \text{ to } p\}$/homotopy
$[(D_1,\alpha)], [(D_2,\beta)] \in Q(L,p)$
$[(D_1,\alpha)] * [(D_2,\beta)] := [(D_1, \alpha \cdot \beta^{-1} \cdot \partial D_2 \cdot \beta)]$.
$Q(L) := (Q(L,p), *)$: the *fundamental quandle* of L.



Quandle homomorphism to Alexander quandle

Notation X, Y: quandles.
$\mathrm{Hom}(X,Y) := \{f : X \to Y : \text{a quandle homomorphism}\}$

D: an oriented diagram of a knot K,
$G(K) := \pi_1(\mathbb{R}^3 \backslash K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ (Wirtinger presentation)
$\overset{\text{Fox derivative}}{\longrightarrow}$ A_D: the *Alexander matrix*
$\Delta_K^i(t) = \gcd(\{\text{all } (n-i)\text{-th minors of } A_D\})$: the *$i$-th Alexander polynomial*.

- $\Delta_K^0(t) = 0$ and $\Delta_K^i(t) \neq 0$ $(i > 0)$.
- $\Delta_K^i(t)$ is divisible by $\Delta_K^{i+1}(t)$ $(i > 0)$.

For each $i > 0$, $e_i(t) := \frac{\Delta_K^i(t)}{\Delta_K^{i+1}(t)}$.

Theorem (Inoue, 2001)

p: a prime number, J: an ideal of $\mathbb{Z}_p[t^{\pm 1}]$.
$|\mathrm{Hom}(Q(K), \mathbb{Z}_p[t^{\pm 1}]/J)| = |\mathbb{Z}_p[t^{\pm 1}]/J \oplus \{\oplus_{i=1}^{n-1} (\mathbb{Z}_p[t^{\pm 1}]/(e_i(t), J))\}|$

Alexander pair

X: a quandle, R: a ring with the identity element 1.
$f_1, f_2 : X \times X \to R$: maps

Definition (Ishii-Oshiro cf. Andruskiewitsch-Graña, 2003)

$f = (f_1, f_2)$: an *Alexander pair*
$\Leftrightarrow$ f satisfies the following three conditions.

1. $\forall x \in X,\ f_1(x,x) + f_2(x,x) = 1.$
2. $\forall x, y \in X,\ f_1(x,y) \in R^{\times}.$
3. $\forall x, y, z \in X,$

$$f_1(x*y,z)f_1(x,y) = f_1(x*z,y*z)f_1(x,z),$$
$$f_1(x*y,z)f_2(x,y) = f_2(x*z,y*z)f_1(y,z),$$
$$f_2(x*y,z) = f_1(x*z,y*z)f_2(x,z) + f_2(x*z,y*z)f_2(y,z).$$

ex.

(1) X: a quandle, $f_1, f_2 : X \times X \to \mathbb{Z}[t^{\pm 1}]$ defined by

$$f_1(x, y) = t, f_2(x, y) = 1 - t \ (x, y \in X)$$

Then, $f = (f_1, f_2)$: an Alexander pair.

(2) G: a group, $X = \mathrm{Conj}(G)$, $f_1, f_2 : X \times X \to \mathbb{Z}[G][t^{\pm 1}]$ defined by

$$f_1(x, y) = y^{-1}t, f_2(x, y) = y^{-1}x - y^{-1}t \ (x, y \in X)$$

Then, $f = (f_1, f_2)$: an Alexander pair.

Proposition (Andruskiewitsch-Graña, 2003)

X: a quandle, R: a ring, M: a left R-module.
$f = (f_1, f_2)$: an Alexander pair of maps $f_1, f_2 : X \times X \to R$.
$\Rightarrow V = (X \times M, \triangleleft)$: a quandle by

$$(x, a) \triangleleft (y, b) = (x * y, f_1(x, y)a + f_2(x, y)b) \ ((x, a), (y, b) \in X \times M).$$

f-derivative

$S = \{x_1, \ldots, x_n\}$: a set, $FQ(S)$: the free quandle on S.
$X = \langle x_1, \ldots x_n \mid r_1, \ldots, r_m \rangle$: a finitely presented quandle, R: a ring.
$f = (f_1, f_2)$: an Alexander pair of maps $f_1, f_2 : X \times X \to R$.

Definition (Ishii-Oshiro)

$1 \leq j \leq n$.
$\frac{\partial_f}{\partial x_j} : FQ(S) \to R$: the f-*derivative* w.r.t x_j

$$\Leftrightarrow \begin{cases} \frac{\partial_f}{\partial x_j}(x * y) = f_1(x, y)\frac{\partial_f}{\partial x_j}(x) + f_2(x, y)\frac{\partial_f}{\partial x_j}(y). \\ \frac{\partial_f}{\partial x_j}(x *^{-1} y) = f_1(x *^{-1} y, y)^{-1}\frac{\partial_f}{\partial x_j}(x) \\ \qquad\qquad -f_1(x *^{-1} y, y)^{-1} f_2(x *^{-1} y, y)\frac{\partial_f}{\partial x_j}(y) \\ \frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij} \ (1 \leq i \leq n). \end{cases}$$

f-twisted Alexander matrix

Remark.

Q, X: quandles, R: a ring.
$f = (f_1, f_2)$: an Alexander pair of maps $f_1, f_2 : X \times X \to R$.
$\rho : Q \to X$: a quandle homomorphism.
Then, $f \circ \rho = (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$: an Alexander pair.

Definition (Ishii-Oshiro)

$Q = \langle x_1, \ldots x_n \mid r_1, \ldots, r_m \rangle$: a finitely presented quandle.
X: a quandle, R: a ring.
$f = (f_1, f_2)$: an Alexander pair of maps $f_1, f_2 : X \times X \to R$,
$\rho : Q \to X$: a quandle homomorphism.

$$A(Q, \rho; f_1, f_2) := \begin{pmatrix} \frac{\partial_{f\circ\rho}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f\circ\rho}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f\circ\rho}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f\circ\rho}}{\partial x_n}(r_m) \end{pmatrix}$$

: the *f-twisted Alexander matix*

f-twisted Alexander matrix

$E_d(A(Q,\rho;f_1,f_2)) = (\{\text{all } (n-d)\text{-th minors of } A(Q,\rho;f_1,f_2)\})$
: the d*-th elementary ideal* of $A(Q,\rho;f_1,f_2)$ if R: a commutative ring.
$\Delta_d(A(Q,\rho;f_1,f_2)) = \gcd(\{\text{all } (n-d)\text{-th minors of } A(Q,\rho;f_1,f_2)\})$
: the d*-th Alexander invariant* of $A(Q,\rho;f_1,f_2)$ if R: a GCD domain.

Theorem (Ishii-Oshiro)

Q, Q': finitely presented quandles.
$\rho : Q \to X, \rho' : Q' \to X$: quandle homomorphisms.
If $\exists \psi : Q \to Q'$: a quandle isomorphism s.t. $\rho = \rho' \circ \psi$,
then $A(Q,\rho;f_1,f_2) \sim A(Q',\rho';f_1,f_2)$.
Futhermore, if R: a commutative ring,
$E_d(A(Q,\rho;f_1,f_2)) = E_d(A(Q',\rho';f_1,f_2))$,
and if R: GCD domain,
$\Delta_d(A(Q,\rho;f_1,f_2)) \doteq \Delta_d(A(Q',\rho';f_1,f_2))$.

Relation between quandle homomorphisms and f-twisted Alexander matrices

Proposition

M: a left R-module, $V = (X \times M, \triangleleft)$.
$\{\boldsymbol{u} \in M^n \mid A(Q, \rho; f_1, f_2)\boldsymbol{u} = \boldsymbol{0}\} \overset{1:1}{\longleftrightarrow} \{\varphi \in \mathrm{Hom}(Q, V) \mid \rho = \mathrm{pr}_1 \circ \varphi\}$

(Outline of proof)

$\boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in M^n$ such that $A(Q, \rho; f_1, f_2)\boldsymbol{u} = \boldsymbol{0}$.

$\varphi : S = \{x_1, \ldots, x_n\} \to V$: the map defined by $\varphi(x_i) = (\rho(x_i), u_i)$.
$\overset{\text{induce}}{\longrightarrow} \varphi : FQ(S) \to V$: the quandle homomorphism which satisfies
$\varphi(a) = (\rho(a), \sum_{i=1}^{n} \frac{\partial_{f \circ \rho}}{\partial x_i}(a) u_i)$ for any $a \in FQ(S)$.
Thus, $\varphi(r_1) = \varphi(r_2)$ for any relator $r = (r_1, r_2)$.
$\therefore \varphi : Q \to V$: well-defined.

Relation between quandle homomorphisms and Alexander invariants

Suppose that R: PID.

$$e_d(A(Q,\rho;f_1,f_2)) := \begin{cases} 0 & (\Delta_{d+1}(A(Q,\rho;f_1,f_2))=0) \\ \frac{\Delta_d(A(Q,\rho;f_1,f_2))}{\Delta_{d+1}(A(Q,\rho;f_1,f_2))} & (\text{otherwise}) \end{cases}.$$

Theorem

J: an ideal of R such that $|R/J|<\infty$, $V=(X\times R/J,\triangleleft)$.
Then, $|\{\varphi\in\mathrm{Hom}(Q,V)\mid \rho=\mathrm{pr}_1\circ\varphi\}|$
$=|\oplus_{i=0}^{n-1}R/(e_i(A(Q,\rho;f_1,f_2)),J)|$

Corollary

$|X|<\infty$
$\Rightarrow |\mathrm{Hom}(Q,V)|=\sum_{\rho\in\mathrm{Hom}(Q,X)}|\oplus_{i=0}^{n-1}R/(e_i(A(Q,\rho;f_1,f_2)),J)|.$

(Outline of proof) Since R: PID,

$$A(Q,\rho;f_1,f_2) \sim \begin{pmatrix} e_0(A(Q,\rho;f_1,f_2)) & & 0 \\ & \ddots & \\ 0 & & e_{n-1}(A(Q,\rho;f_1,f_2)) \end{pmatrix} =: A'.$$

- $|\mathrm{Ker}(\boldsymbol{u} \mapsto A(Q,\rho;f_1,f_2)\boldsymbol{u})| = |\mathrm{Ker}(\boldsymbol{u} \mapsto A'\boldsymbol{u})|$.
- $(R/J)^n/\mathrm{Ker}(\boldsymbol{u} \mapsto A'\boldsymbol{u}) \cong \mathrm{Im}(\boldsymbol{u} \mapsto A'\boldsymbol{u})$
  $\Rightarrow |(R/J)^n| = |\mathrm{Ker}(\boldsymbol{u} \mapsto A'\boldsymbol{u})| \cdot |\mathrm{Im}(\boldsymbol{u} \mapsto A'\boldsymbol{u})|$.
- $\oplus_{i=0}^{n-1} R/(e_i(A(Q,\rho;f_1,f_2)),J) \cong (R/J)^n/(\mathrm{Im}(\boldsymbol{u} \mapsto A'\boldsymbol{u}))$
  $\Rightarrow |(R/J)^n| = |\oplus_{i=0}^{n-1} R/(e_i(A(Q,\rho;f_1,f_2)),J)| \cdot |\mathrm{Im}(\boldsymbol{u} \mapsto A'\boldsymbol{u})|$.

$\therefore |\mathrm{Ker}(\boldsymbol{u} \mapsto A(Q,\rho;f_1,f_2)\boldsymbol{u})| = |\oplus_{i=0}^{n-1} R/(e_i(A(Q,\rho;f_1,f_2)),J)|$

Quandle 2-cocycle

Remark. D: a diagram of an oriented link L, X: a quandle.
$\mathrm{Hom}(Q(L), X) \overset{1:1}{\longleftrightarrow} \{c : \mathrm{Arc}(D) \to X : \textit{colorings}\}$

Definition (Carter-Jelsovsky-Kamada-Langford-Saito, 2003)

A: an abelian group
$\theta : X \times X \to A$: a *quandle 2-cocycle*
$\Leftrightarrow \theta$ satisfies the following conditions:

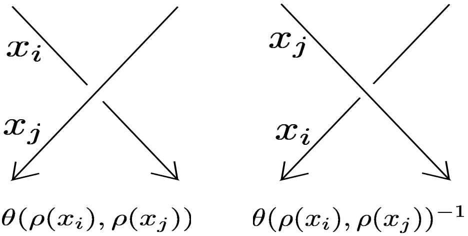
- $\forall x \in X$, $\theta(x, x) = 1_A$.
- $\forall x, y, z \in X$, $\theta(x * y, z)\theta(x, y) = \theta(x * z, y * z)\theta(x, z)$.

$\rho : \mathrm{Arc}(D) \to X$: a coloring ($\Leftrightarrow \rho : Q(L) \to X$: a quandle hom),

χ: a crossing of D. $\Phi_\theta(\chi, \rho) := \begin{cases} \theta(\rho(x_i), \rho(x_j)) & (\chi\text{: positive}) \\ \theta(\rho(x_i), \rho(x_j))^{-1} & (\chi\text{: negative}) \end{cases}$.

$\Phi_\theta(D, \rho) := \Pi_\chi \Phi_\theta(\chi, \rho)$.

Quandle 2-cocycle and Alexander pair



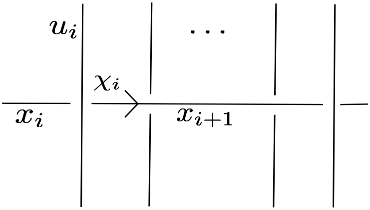
$f_\theta : X \times X \to \mathbb{Z}[A], 0 : X \times X \to \mathbb{Z}[A]$ defined by
$f_\theta(x, y) := \theta(x, y)$, $0(x, y) = 0$. Then, $(f_\theta, 0)$: an Alexander pair.

Theorem

D: a diagram of an oriented knot K, X: a quandle.
A: an abelian group, $\theta : X \times X \to A$: a quandle 2-cocycle.
$\rho : Q(K) \to X$: a quandle homomorphism. Then, we have

$$E_0(A(Q(K), \rho; f_\theta, 0)) = (\Phi_\theta(D, \rho) - 1) \subset \mathbb{Z}[A].$$

(Outline of the proof)



$$r_i := \begin{cases} (x_i * u_i, x_{i+1}) & (\chi_i\text{: positive}) \\ (x_i *^{-1} u_i, x_{i+1}) & (\chi_i\text{: negative}). \end{cases}$$

$Q(K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$, $f = (f_\theta, 0)$.

If χ_i is positive,

$$\begin{aligned} \frac{\partial_{f\circ\rho}}{\partial x_j}(r_i) &= \frac{\partial_{f\circ\rho}}{\partial x_j}(x_i * u_i) - \frac{\partial_{f\circ\rho}}{\partial x_j}(x_{i+1}) \\ &= \theta(\rho(x_i), \rho(u_i)) \frac{\partial_{f\circ\rho}}{\partial x_j}(x_i) - \frac{\partial_{f\circ\rho}}{\partial x_j}(x_{i+1}) \\ &= \Phi_\theta(\chi_i, \rho) \frac{\partial_{f\circ\rho}}{\partial x_j}(x_i) - \frac{\partial_{f\circ\rho}}{\partial x_j}(x_{i+1}). \end{aligned}$$

If χ_i is negative,

$$\begin{aligned}
\frac{\partial_{f\circ\rho}}{\partial x_j}(r_i) &= \frac{\partial_{f\circ\rho}}{\partial x_j}(x_i *^{-1} u_i) - \frac{\partial_{f\circ\rho}}{\partial x_j}(x_{i+1}) \\
&= \theta(\rho(x_i) *^{-1} \rho(u_i), \rho(u_i))^{-1} \frac{\partial_{f\circ\rho}}{\partial x_j}(x_i) - \frac{\partial_{f\circ\rho}}{\partial x_j}(x_{i+1}) \\
&= \theta(\rho(x_{i+1}), \rho(u_i))^{-1} \frac{\partial_{f\circ\rho}}{\partial x_j}(x_i) - \frac{\partial_{f\circ\rho}}{\partial x_j}(x_{i+1}) \\
&= \Phi_\theta(\chi_i, \rho) \frac{\partial_{f\circ\rho}}{\partial x_j}(x_i) - \frac{\partial_{f\circ\rho}}{\partial x_j}(x_{i+1}).
\end{aligned}$$

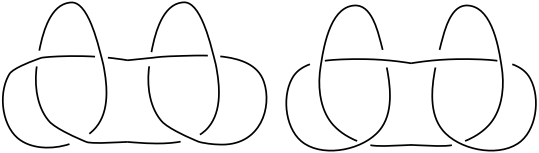
Then, we have

$$A(Q(K), \rho; f_\theta, 0) = \begin{pmatrix} \Phi_\theta(\chi_1, \rho) & -1 & & & \\ & \Phi_\theta(\chi_2, \rho) & -1 & & \\ & & \ddots & \ddots & \\ -1 & & & & \Phi_\theta(\chi_n, \rho) \end{pmatrix}$$

$$\therefore E_0(A(Q(K), \rho; f_\theta, 0)) = (\det(A(Q(K), \rho; f_\theta, 0))) = (\Phi_\theta(D, \rho) - 1).$$

Application

- We can distinguish these knots by using f-twisted Alexander matrices.



K_1 $\quad$ K_2

ex.) $X = \{g^{-1}(1234)g \in \mathfrak{S}_4 \mid g \in \mathfrak{S}_4\}$, $\theta : X \times X \to \mathbb{Z}_4$
(cf. [Carter-Saito-Satoh, 2006]).
$\to$ f-twisted Alexander matrices are NOT knot group invariants.

- We can determine the *deficiency* of $Q(K)$.

Theorem

If K is a nontrivial knot, then the deficiency of $Q(K)$ is 0.

Thank you for your attention.