Extension of Tong-Yang-Ma Representation

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結び目の数理 III

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B_n : braid group

$$\left\langle \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \geq 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (i = 1, \cdots, n-2) \end{array} \right\rangle$$

[Tong-Yang-Ma] (1996)

$\exists$ only two non-trivial representations of B_n s.t.

$$\sigma_i \longmapsto I_{i-1} \oplus \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus I_{n-i-1}.$$

$$\begin{cases} \cdot\ \beta_n : B_n \longrightarrow GL_n(\mathbb{Z}[t^{\pm 1}]) : \text{Burau representation} \\ \cdot\ TYM_n : B_n \longrightarrow GL_n(\mathbb{Z}[t^{\pm 1}]) : \text{Tong-Yang-Ma representation} \end{cases}$$

$$\beta_n : \begin{pmatrix} 0 & t \\ 1 & 1-t \end{pmatrix} \qquad TYM_n : \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$$

Extensions for string links

[Le Dimet] (1992)

Gassner representation (Fox derivatives)

[Lin-Tian-Wang] (1998)

Burau representation (combinatorially)

[Kirk-Livingston-Wang] (2001)

Gassner representation ((co) homologically)

[Silver-Williams] (2001)

two-variable Burau representation (combinatorially)

In this talk

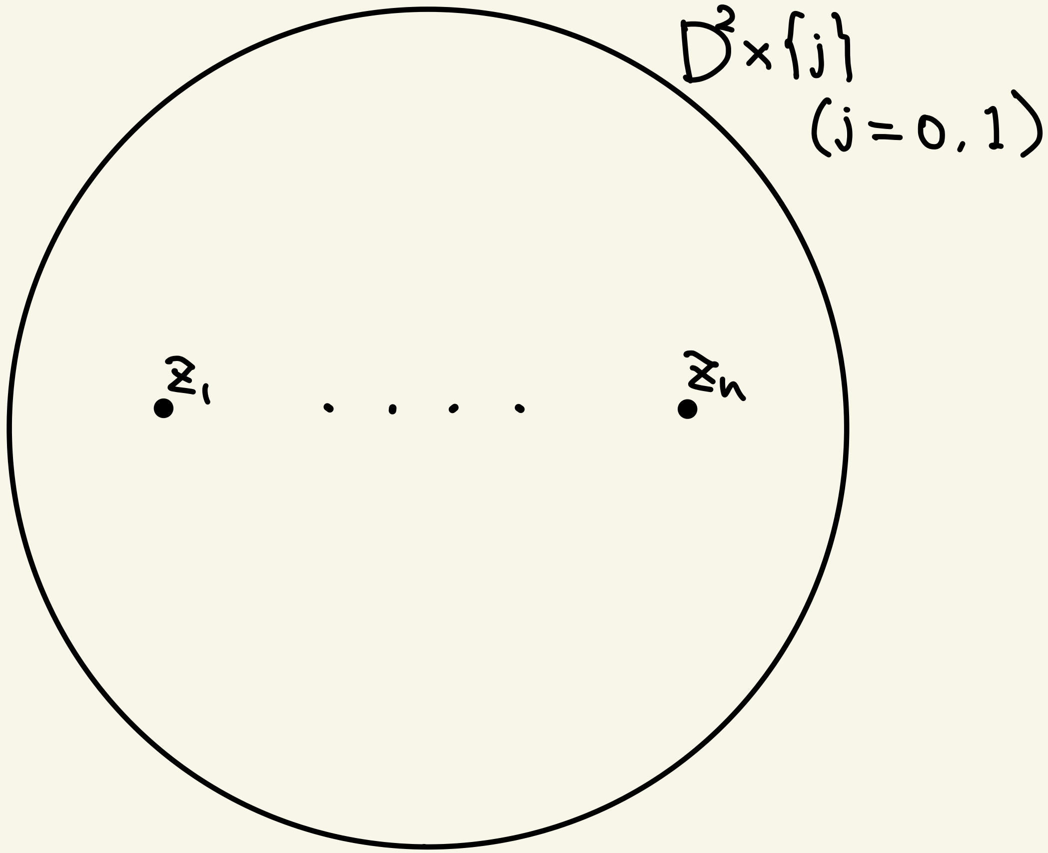
- $\mathrm{TYM}_n : \underline{B_n} \longrightarrow GL_n(\mathbb{Z}[t^{\pm 1}])$

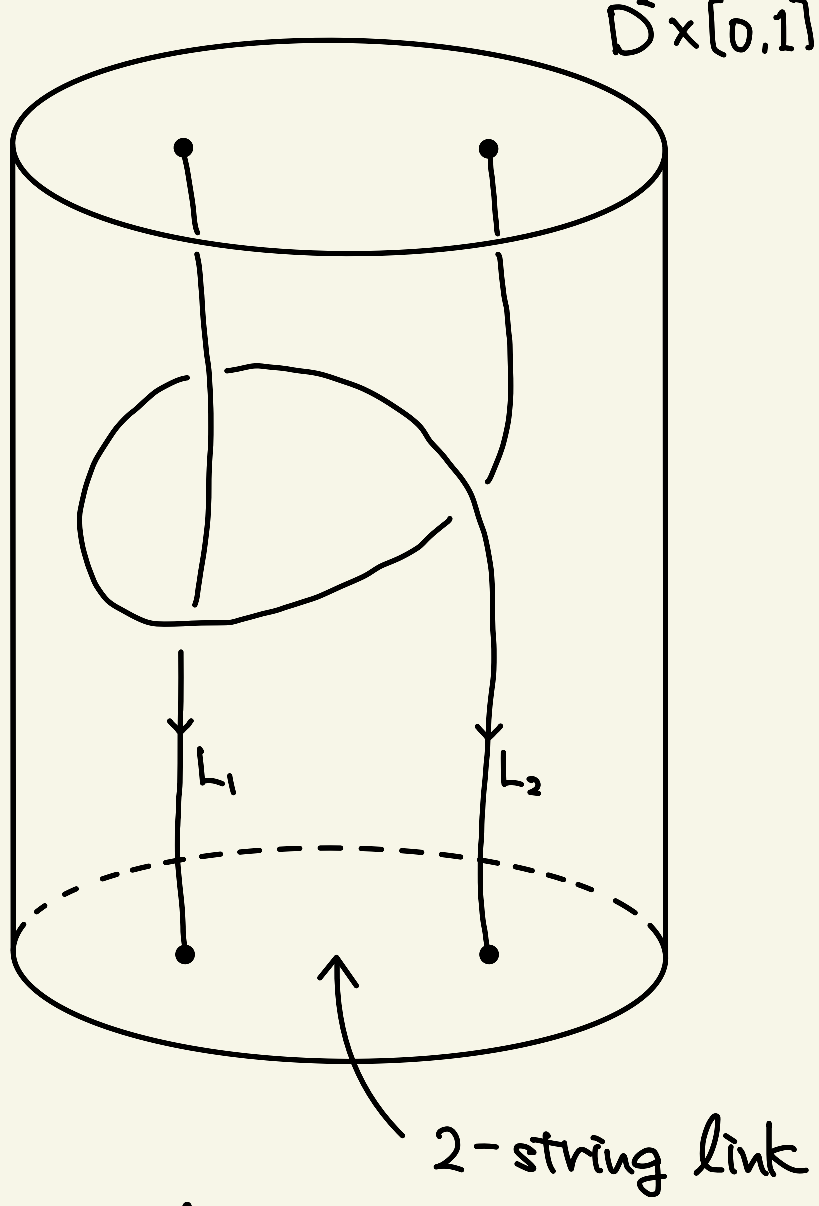
  string link

- $\mathrm{Ker}\,\mathrm{TYM}_n$ is written by linking numbers.

  (cf. [Massuyeau-Oancea-Salamon])

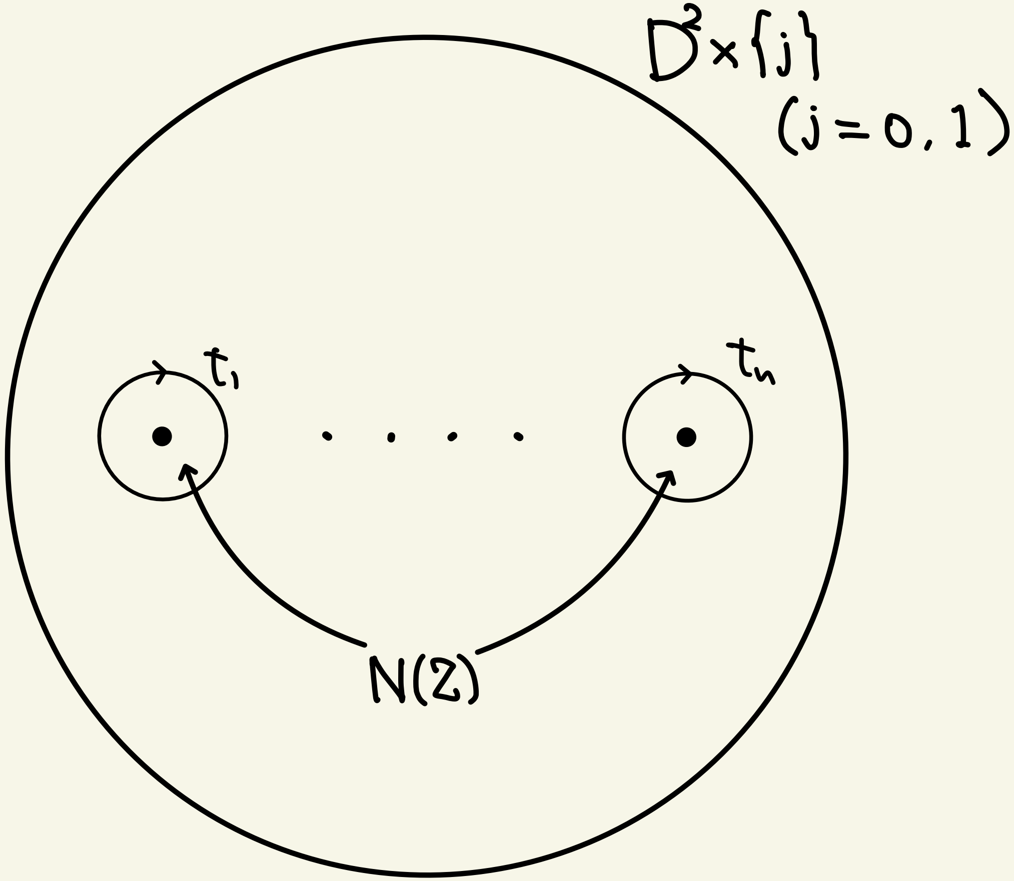
Definition

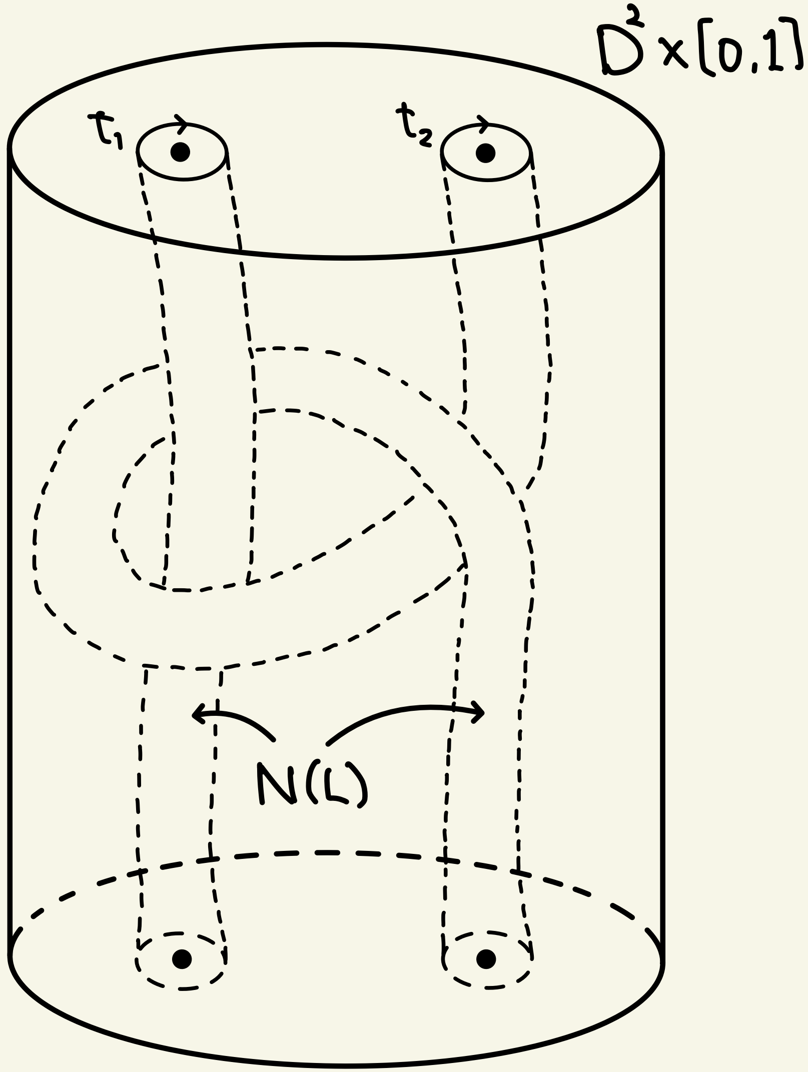




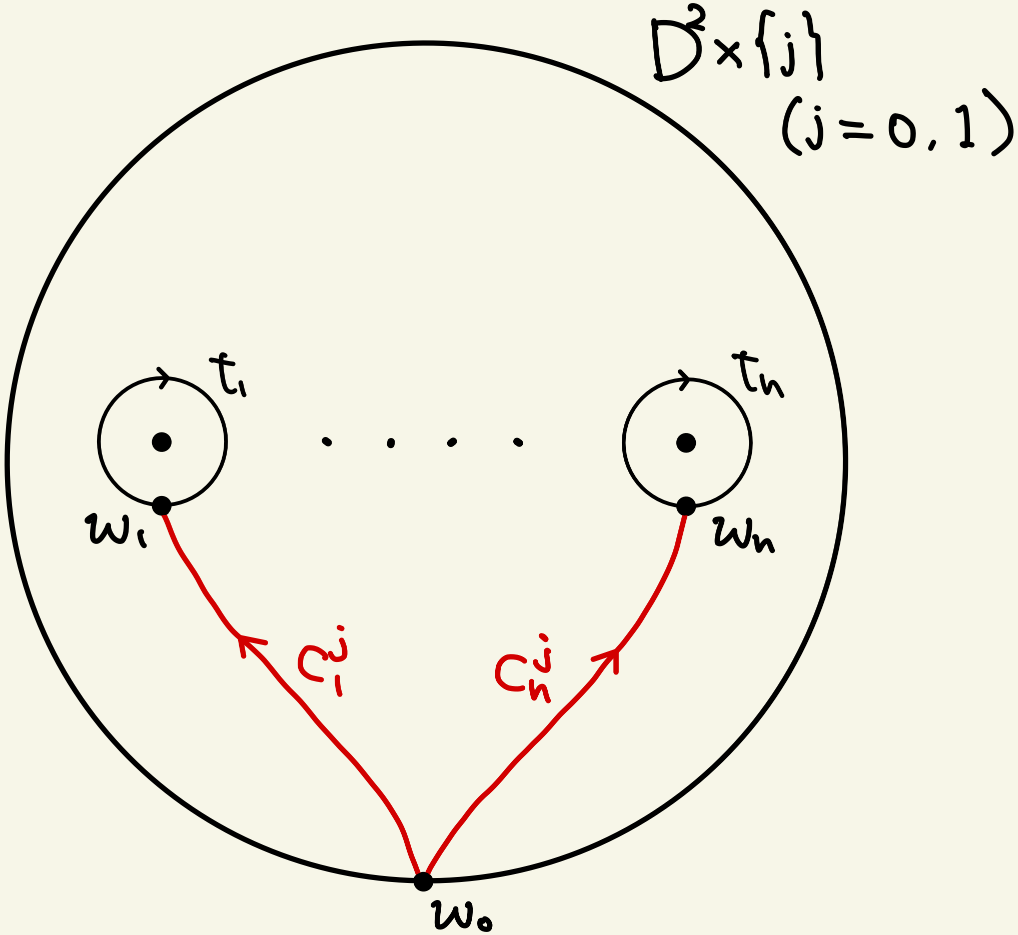
- $\mathcal{SL}_n := \{$n-string links$\}$
- $\mathcal{PSL}_n := \{$pure n-string links$\}$

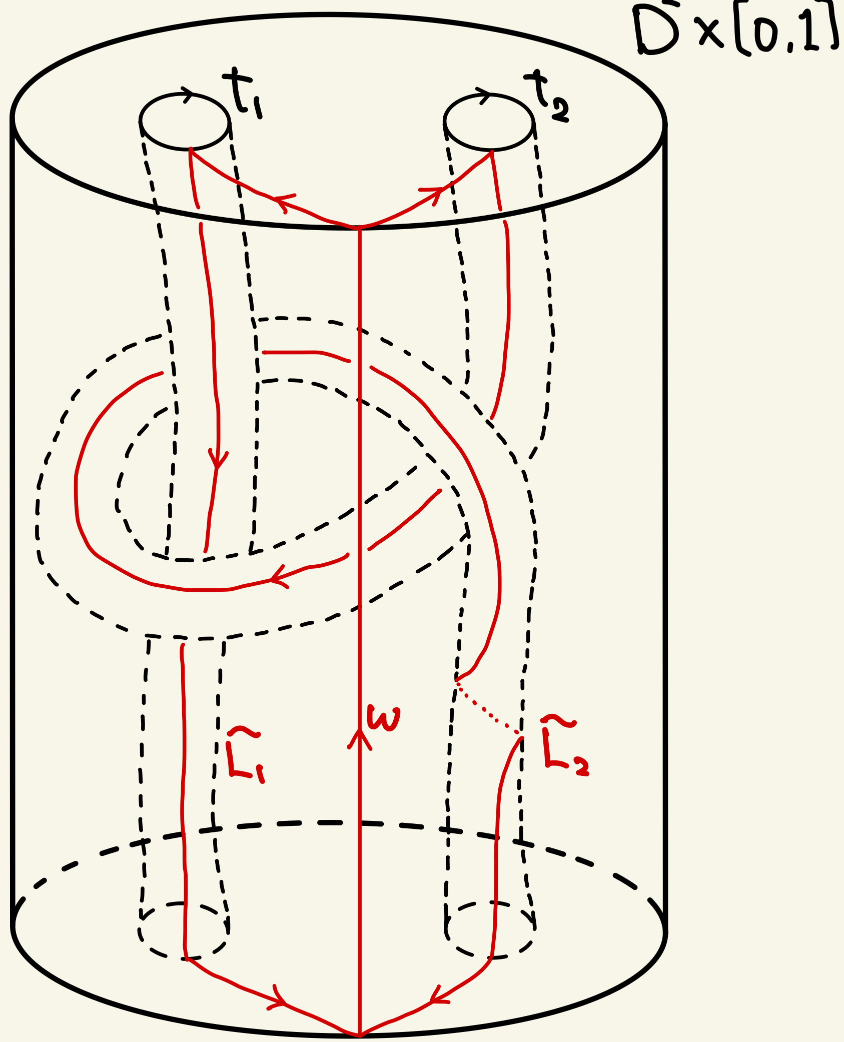
$\rightsquigarrow$ monoid





- $N(L) \cap (D^2 \times \{j\}) = N(Z) \times \{j\}$
- $X := (D^2 \times [0,1]) \setminus \mathrm{int}(N(L))$, $X_0 := X \cap (D^2 \times \{0\})$
- $H_1(X_0; \mathbb{Z}) \cong \mathbb{Z}^{\oplus n} \cong \langle t_1, \cdots, t_n \mid t_i t_j = t_j t_i \rangle \xhookrightarrow[i_*]{\cong} H_1(X; \mathbb{Z})$



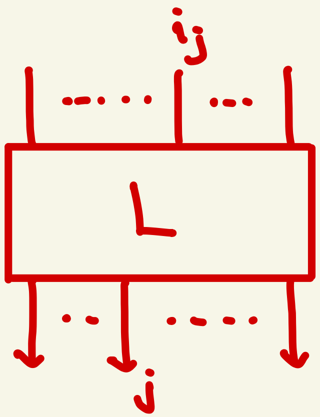


· $\tilde{L}_i$ is parallel to L_i s.t. $lk(L_i, \tilde{L}_i) = 0$.

Def (multi-variable Tong-Yang-Ma map)

$$\mathcal{T}: \mathcal{SL}_n \longrightarrow GL_n(\mathbb{Z}[H_1(X_0;\mathbb{Z})]) \cong GL_n(\mathbb{Z}[t_1^{\pm1},\dots,t_n^{\pm1}])$$

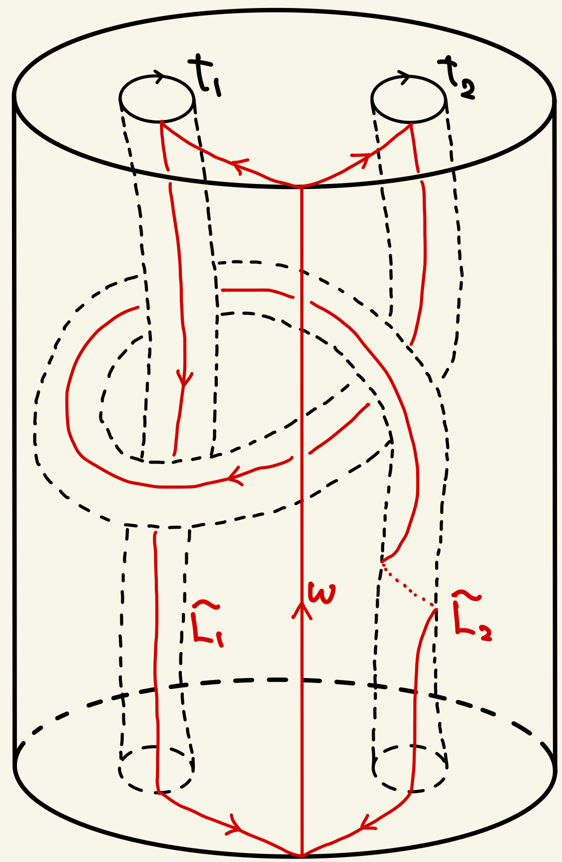
$$(\mathcal{T}(L))_{ij} := \begin{cases} i_*^{-1}(c_i^0\cdot \tilde{L}_i \cdot (c_j^1)^{-1}\cdot w) & (\tau_L(j)=i) \\ 0 & (\text{otherwise}) \end{cases}$$



$\rightsquigarrow$ $\tau_L \in S_n$ is defined by $\tau_L(j) := i_j$

Ex

$$\mathcal{T}(L) = \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}$$



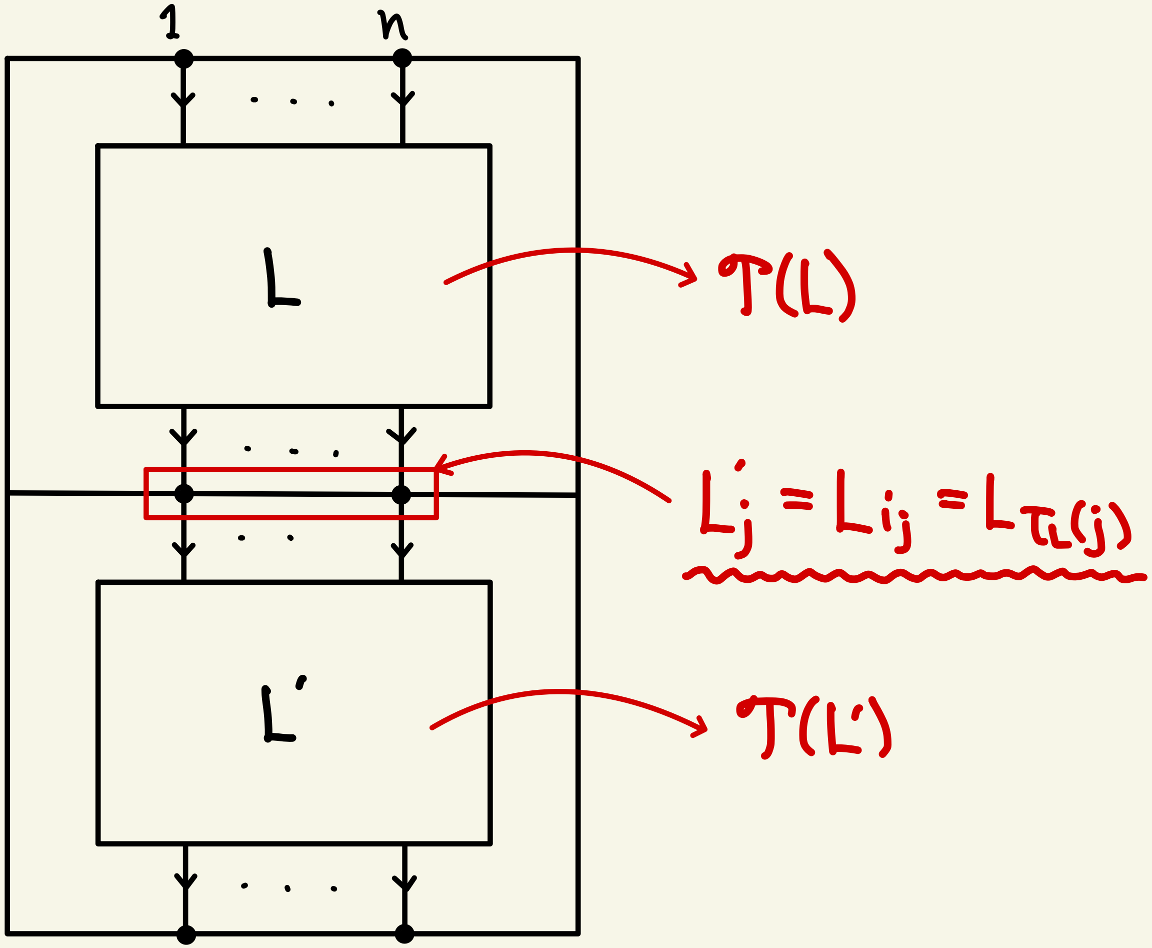
Thm L, L' : n-string links

$$\mathcal{T}(LL') = \mathcal{T}(L)\,(\underline{L\cdot\mathcal{T}(L')})$$

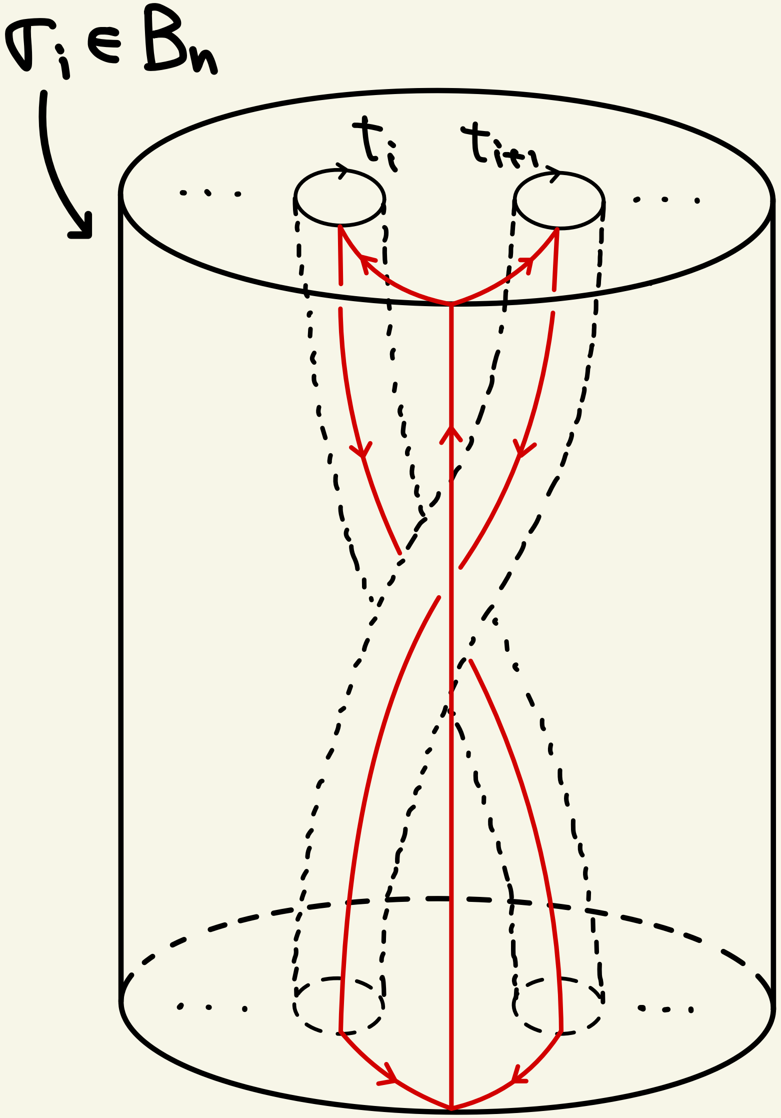
$$\mathcal{T} : S\mathcal{L}_n \longrightarrow GL_n(\mathbb{Z}[t_1^{\pm1},\dots,t_n^{\pm1}])$$

$$(\mathcal{T}(L))_{ij} := \begin{cases} i_*^{-1}(c_i^0\cdot\tilde{L}_i\cdot(c_j^1)^{-1}\cdot w) & (\tau_L(j)=i) \\ 0 & (\text{otherwise}) \end{cases}$$

($S\mathcal{L}_n$ acts on $H_1(X_0;\mathbb{Z})$. $L\cdot t_j := t_{\tau_L(j)}$)



Relation to the Tong-Yang-Ma representation



$$\rightsquigarrow \mathcal{T}(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 0 & t_{i+1} \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

$$\Big\downarrow t_1 = \cdots = t_n = t$$

$$I_{i-1} \oplus \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1} = TYM_n(\sigma_i)$$

Kernel of the Tong-Yang-Ma map

- $\operatorname{Ker}\mathcal{T} = \{ L \in \mathcal{PSL}_n \mid 1 \le {}^\forall i < j \le n,\ lk(L_i, L_j) = 0 \}.$

⊙ $\hat{L}$: closure of L in S^3.

$H_1(S^3 \setminus \operatorname{int}(N(\hat{L})); \mathbb{Z}) \cong \mathbb{Z}^{\oplus n} \cong \langle t_1, \cdots, t_n \mid t_i t_j = t_j t_i \rangle$

($\hookleftarrow t_i$: i-th meridian of $\hat{L}$)

$(\mathcal{T}(L))_{ii}$ = i-th longitude of $\hat{L}$ in $H_1(S^3 \setminus \operatorname{int}(N(\hat{L})); \mathbb{Z})$

$\rightsquigarrow$ exponent of t_j of $(\mathcal{T}(L))_{ii} = lk(\hat{L}_i, \hat{L}_j) = lk(L_i, L_j)$ $\square$

- $\operatorname{Ker}(\mathcal{T}|_{t_i = t}) = \{ L \in \mathcal{PSL}_n \mid 1 \le {}^\forall j \le n,\ \sum_{i=1}^{n} lk(L_i, L_j) = 0 \}.$

Further works

- $\begin{cases} \text{Homological definition of the } \mathbb{Z}/2\mathbb{Z}\text{-reduction} \\ \text{Another extension (to string links and welded string links)} \end{cases}$

- The Long-Moody construction

References

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  "Lefschetz fibrations, intersection numbers and representation of the framed braid group"