Classfication of handlebody decompositions of the 3-sphere and lens spaces

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December 25, 2020

A Heegaard splitting

Definition 1 (Heegaard splitting).

M : closed, orientable, 3-manifold.
H_i : genus g handlebody (i=1, 2)

$$M = H_1 \cup_f H_2$$

This is a genus g Heegaard splitting of M where $f : \partial H_2 \to \partial H_1$ is an orientation reversing homeomorphism.

It is well-known that any closed, orientable 3-manifolds have Heegaard splittings (Moise).

A Heegaard splitting

Heegaard splitting of the 3-sphere and lens spaces are classified by Waldhausen, Bonahon and Otal.

Theorem 1 (Waldhausen, 1968).

The 3-sphere admits unique Heegaard splitting up to isotopy for each genus.

Theorem 2 (Bonahon-Otal, 1983).

Lens spaces admit unique Heegaard splitting up to isotopy for each genus $g \geq 1$.

Remark 1.

Above theorems are classifications of Heegaard surfaces of manifolds up to ambient isotopy.

A handlebody decomposition

Definition 2.

M : a closed, orientable 3-manifold.
H_i : a genus g_i handlebody for $i = 1, 2, 3$.
$M = H_1 \cup H_2 \cup H_3$ is a handlebody decomposition if it satisfies the following conditions:

1. *$H_i \cap H_j = \partial H_i \cap \partial H_j$ is a (possibly disconnected) compact surface with boundary. It is denoted by F_{ij}; and*
2. *$H_1 \cap H_2 \cap H_3$ is a set of simple closed curves in M. We call a component of this set a branched locus.*

We call this decomposition a type-$(g_1, g_2, g_3; b)$ decomposition where b is the number of branched loci.
We name $F_{12} \cup F_{13} \cup F_{23}$ a branched surface of the handlebody decomposition.

Proposition 1 (Ito-O).

A type-$(0,0,1)$ decomposition of the 3-sphere and lens spaces has exactly 2 branched loci and satisfies $F_{12} \cong D_1 \cup D_2$ and $F_{23} \cong F_{31} \cong A$.

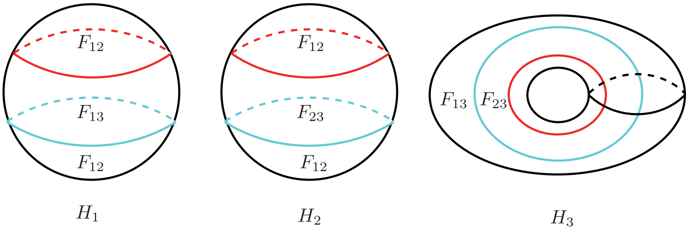


Figure: A type-$(0,0,1)$ decomposition of the 3-sphere.

We classified above decomposition up to isotopy.

Theorem 3 (O, 2020).

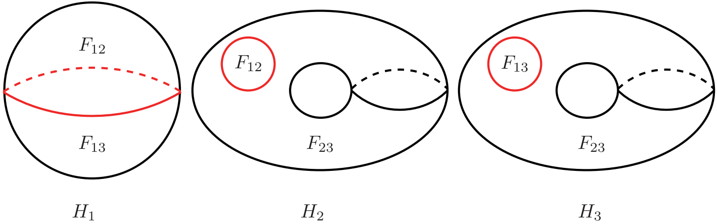
There is a unique ambient isotopy class of the branched surface of a type-$(0,0,1)$ decomposition if M is a 3-sphere or a lens space with $(p-1)q \equiv \pm 1 \pmod p$. Otherwise, there are exactly two ambient isotopy classes of the branched surface of a type-$(0,0,1)$ decomposition.

Proposition 2 (Ito-O).

A type-$(0,1,1)$ decomposition of the 3-sphere and lens spaces satisfies one of the followings.

(1) *It has exactly one branched locus and satisfies $F_{12} \cong F_{31} \cong D$ and $F_{23} \cong T^\circ$.*

(2) *It has exactly three branched loci and satisfies $F_{12} \cong F_{31} \cong D \cup A$ and $F_{23} \cong P$.*



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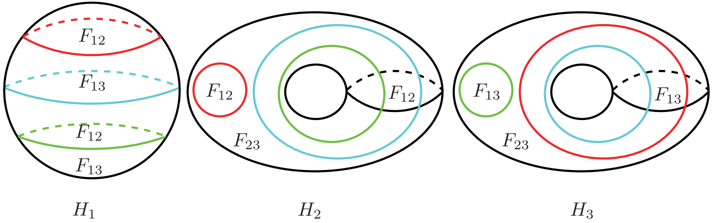
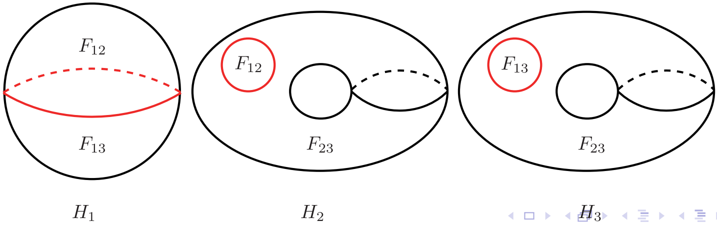


Theorem 4 (O. 2020).

M: The 3-sphere or a lens space $L(p, q)$.

- *Type-$(0,1,1)$ decompositions of M of Proposition 3 (1) has exactly one isotopy class.*
- *Type-$(0,1,1)$ decompositions of M of Proposition 3 (2) is classified as follows:*

$$\begin{cases} \text{exactly 1 ambient isotopy class} & (\text{the 3-sphere or } p = 2) \\ \text{exactly 2 ambient isotopy classes} & (p \neq 2,\ (p-1)q \equiv \pm 1 \pmod p) \\ \text{exactly 4 ambient isotopy classes} & (p \neq 2,\ (p-1)q \not\equiv \pm 1 \pmod p) \end{cases}$$



Proposition 4 (Ito-O).

A type-$(1,1,1)$ decomposition of a 3-sphere and a lens space M satisfies one of the following, where $\{i,j,k\} = \{1,2,3\}$.

(1) *It has exactly two branched loci and satisfies $F_{ij} \cong F_{jk} \cong F_{ki} \cong A$.*

(2) *It has exactly two branched loci and satisfies $F_{ij} \cong D \cup T^\circ$ and $F_{jk} \cong F_{ki} \cong A$.*

(3) *It has exactly four branched loci and satisfies $F_{ij} \cong D \cup P$ and $F_{jk} \cong F_{ki} \cong A_1 \cup A_2$.*

(4) *It has exactly four branched loci and satisfies $F_{ij} \cong A_1 \cup A_2$ and $F_{jk} \cong F_{ki} \cong D \cup P$.*

(5) *It has exactly four branched loci and satisfies $F_{ij} \cong F_{jk} \cong F_{ki} \cong D \cup P$.*

Furthermore, the following holds.

(6) *$M \cong L(4,1)$. In this case, M also has a decomposition that satisfies $F_{12} \cong F_{13} \cong F_{23} \cong A \cup A$ and it has four branched loci.*

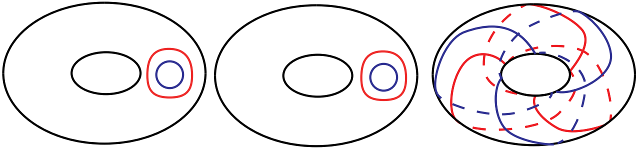
Theorem 5 (O, 2020).

There is exactly one ambient isotopy class of embeddings of the branched surface of a type-$(1,1,1)$ *decomposition of a 3-sphere that satisfies one of conclusions (2), (3), (4), or (5) of Proposition above.*

Theorem 6 (O. 2020).

M: a lens space $L(p,q)$.
Type-$(1,1,1)$ decompositions of M of Proposition 4 (2) is classified as follows:

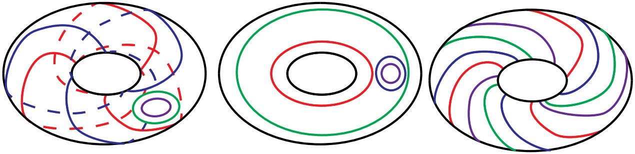
$$\begin{cases} \textit{exactly 2 ambient isotopy class} & (p=2) \\ \textit{exactly 2 ambient isotopy classes} & (p \neq 2,\ (p-1)q \equiv \pm 1 \ (mod\ p)) \\ \textit{exactly 4 ambient isotopy classes} & (p \neq 2,\ (p-1)q \not\equiv \pm 1 \ (mod\ p)) \end{cases}$$



Theorem 6.

Type-$(1,1,1)$ decompositions of M of Proposition 4 (3) is classified as follows:

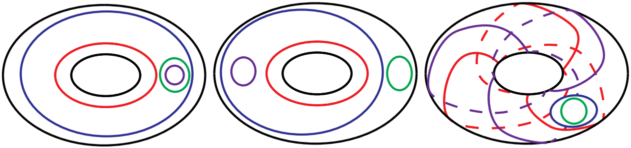
$$\begin{cases} \textit{exactly 2 ambient isotopy class} & (p=2) \\ \textit{exactly 3 ambient isotopy classes} & (p \neq 2,\ (p-1)q \equiv \pm 1\ (\textit{mod}\ p)) \\ \textit{exactly 6 ambient isotopy classes} & (p \neq 2,\ (p-1)q \not\equiv \pm 1\ (\textit{mod}\ p)) \end{cases}$$



Theorem 6.

Type-$(1,1,1)$ decompositions of M of Proposition 4 (4) is classified as follows:

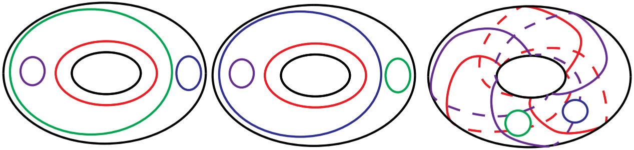
$$\begin{cases} \textit{exactly 2 ambient isotopy class} & (p=2) \\ \textit{exactly 4 ambient isotopy classes} & (p \neq 2,\ (p-1)q \equiv \pm 1 \ (mod\ p)) \\ \textit{exactly 8 ambient isotopy classes} & (p \neq 2,\ (p-1)q \not\equiv \pm 1 \ (mod\ p)) \end{cases}$$



Theorem 6.

Type-$(1,1,1)$ decompositions of M of Proposition 4 (5) is classified as follows:

$$\begin{cases} \textit{exactly 1 ambient isotopy class} & (p=2) \\ \textit{exactly 1 ambient isotopy classes} & (p \neq 2,\ (p-1)q \equiv \pm 1 \ (mod\ p)) \\ \textit{exactly 2 ambient isotopy classes} & (p \neq 2,\ (p-1)q \not\equiv \pm 1 \ (mod\ p)) \end{cases}$$



Outline of proof

Let $H_1 \cup H_2 \cup H_3$ and $H'_1 \cup H'_2 \cup H'_3$ be two handlebody decompositions and $V_1 \cup V_2$ be the genus one Heegaard splitting.

1. We construct a genus one Heegaard splitting by handlebody decompositions.

Outline of proof

Let $H_1 \cup H_2 \cup H_3$ and $H'_1 \cup H'_2 \cup H'_3$ be two handlebody decompositions and $V_1 \cup V_2$ be the genus one Heegaard splitting.

1. We construct a genus one Heegaard splitting by handlebody decompositions.
2. We see the embedding of F_{ij} in a solid torus of genus one Heegaard splitting.

Outline of proof

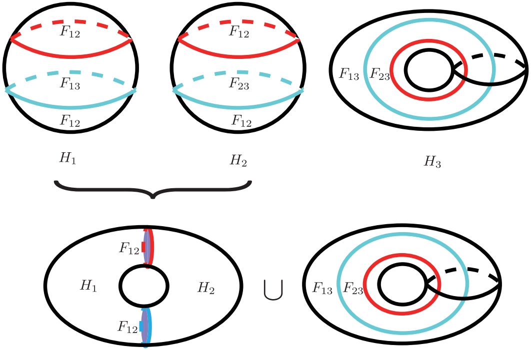
Let $H_1 \cup H_2 \cup H_3$ and $H_1' \cup H_2' \cup H_3'$ be two handlebody decompositions and $V_1 \cup V_2$ be the genus one Heegaard splitting.

1. We construct a genus one Heegaard splitting by handlebody decompositions.
2. We see the embedding of F_{ij} in a solid torus of genus one Heegaard splitting.
3. We consider the ambient isotopy with keeping Heegaard surface.

Handlebody decompositions of the 3-sphere and lens spaces can be seen as genus one Heegaard splittings.

Lemma 1.

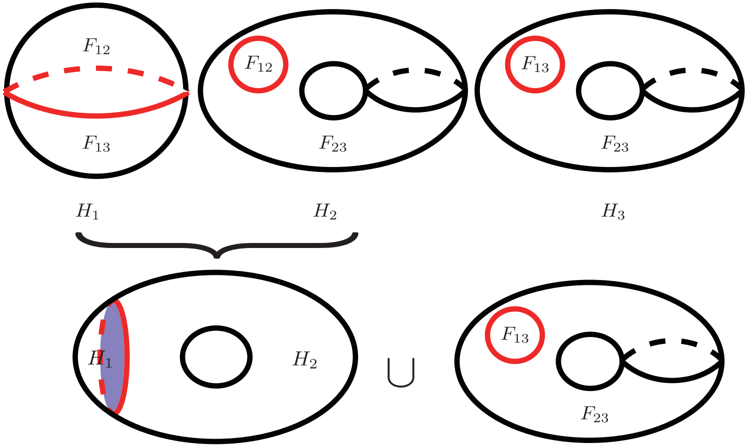
If $H_1 \cup H_2 \cup H_3$ is a type-$(0,0,1)$ decomposition of the 3-sphere or a lens space M, ∂H_3 is a genus-one Heegaard surfaces of M.



Lemma 2.

Let $H_1 \cup H_2 \cup H_3$ *be a type-*$(0,1,1)$ *decomposition of the 3-sphere or a lens space* M *that satisfies either case (1) or case (2) of Proposition 2. Then, the following hold:*

- *If* $H_1 \cup H_2 \cup H_3$ *is a type-*$(0,1,1)$ *decomposition of* M *that satisfies the conclusion of Proposition 2 (1), both* ∂H_2 *and* ∂H_3 *are genus-one Heegaard surfaces of* M*.*
- *If* M *is a lens space and* $H_1 \cup H_2 \cup H_3$ *is a type-*$(0,1,1)$ *decomposition of* M *that satisfies the conclusion of Proposition 2 (2), exactly either* ∂H_2 *or* ∂H_3 *is a genus-one Heegaard surface of* M
- *If* M *is a 3-sphere and* $H_1 \cup H_2 \cup H_3$ *is a type-*$(0,1,1)$ *decomposition of* M *that satisfies the conclusion of Proposition 2 (2), both* ∂H_2 *and* ∂H_3 *are genus-one Heegaard surfaces of* M*.*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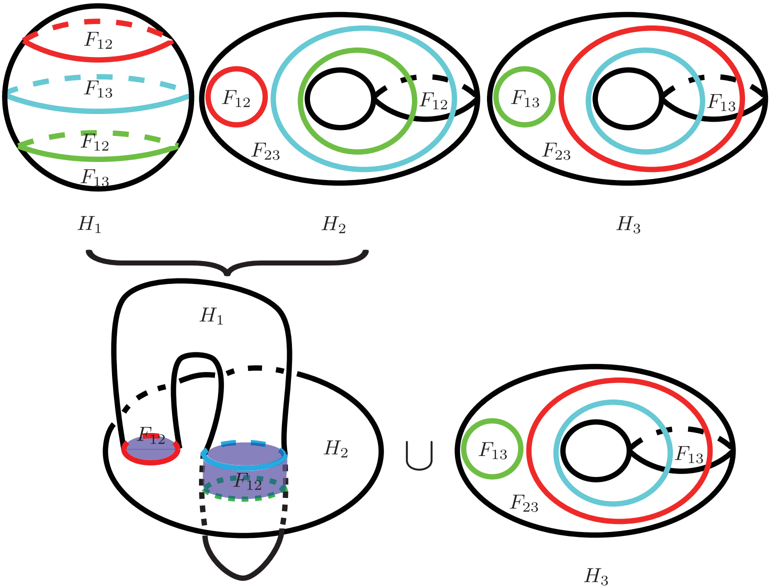
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Lemma 3.

Let $H_1 \cup H_2 \cup H_3$ be a type-$(1,1,1)$ decomposition of a 3-sphere M that satisfies one of conclusion (2), (3), (4), or (5) of Proposition 3. Then, each of ∂H_i is a genus-one Heegaard surface for $i = 1, 2, 3$.

Lemma 4.

Let $H_1 \cup H_2 \cup H_3$ be a type-$(1,1,1)$ decomposition of a lens space M that satisfies conclusion (2), (3), (4), or (5) of Proposition 3. Then, the following are satisfied.

- *If $H_1 \cup H_2 \cup H_3$ is a type-$(0,1,1)$ decomposition of M that satisfies conclusion (3), (4), or (5) of Proposition 3, exactly one of ∂H_i is a genus-one Heegaard surface of M, where $i \in \{1, 2, 3\}$.*
- *If $H_1 \cup H_2 \cup H_3$ is a type-$(0,1,1)$ decomposition of M that satisfies conclusion (2) of Proposition 3, at most two of ∂H_i are genus-one Heegaard surfaces of M, where $i \in \{1, 2, 3\}$.*

We consider the type-$(0, 1, 1)$ decomposition such that $F_{12} \cong F_{13} \cong D \cup A$ and $F_{23} \cong P$. Let $V_1 \cup V_2$ be the genus one Heegaard splitting of the 3-sphere or lens space.

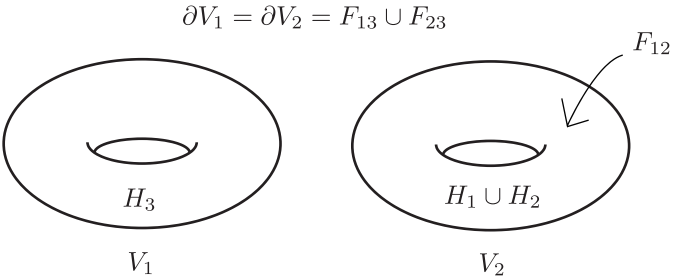


Figure: $H_3 = V_1$

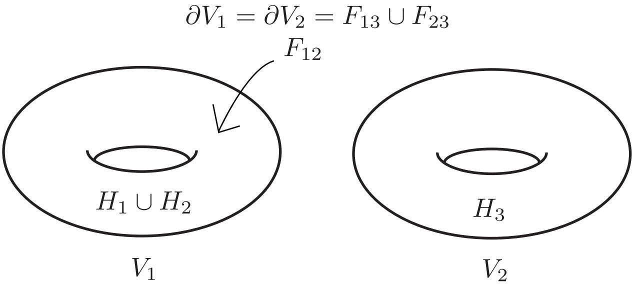


Figure: $H_3 = V_2$

If $H_3 = V_1$ and $H_3' = V_2$, we have to consider whether V_1 is isotopic to V_2 or not.

We can switch the solid tori of Heegaard splitting by ambient isotopy if the ambient manifold satisfies the following condition.

Lemma 5 (2).

Let M be a 3-sphere or a lens space $L(p,q)$ and $V_1 \cup_\varphi V_2$ be a genus-one Heegaard splitting of M, where $\varphi : \partial V_2 \to \partial V_1$ is an orientation-reversing homeomorphism. The core of V_1 is isotopic to the core of V_2 in M if and only if M is either the 3-sphere or $L(p,q)$ with $(p-1)q \equiv \pm 1 \pmod p$.

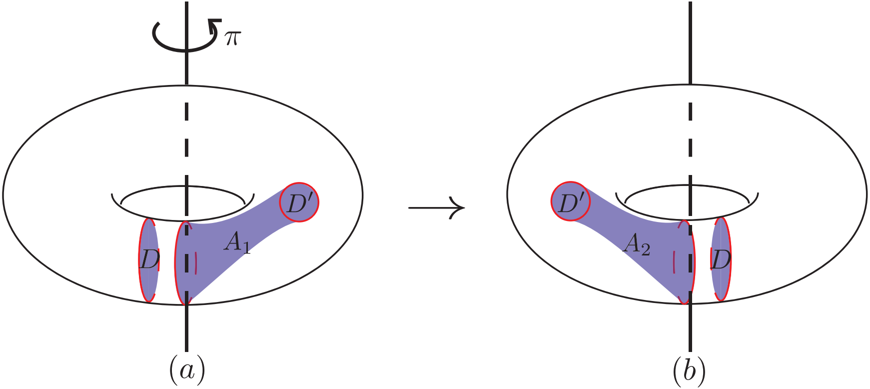
We shall have to consider the embedding of $F_{12} \cong D \cup A$ to V_1 or V_2. We prepare the following lemma

Lemma 6 (6).

Let V be a solid torus, D be a meridian disk of V, and A be an annulus properly embedded in V, which does not intersect D. Then, there are two embeddings of $D \cup A$ up to ambient isotopy if $D \cup A$ satisfies the following.

- *$D \cup A$ cuts open V into a 3-ball B and a solid torus V' such that each of them does not have a self-intersection.*
- *$\partial(D \cup A)$ cuts open ∂V into an annulus A', a disk D', and a thrice-punctured sphere P.*

Furthermore, these ambient isotopy classes are taken to each other by hyperelliptic involution on V.



If there is an ambient isotopy which send V_1 to its-self by the hyperelliptic involution, two of the isotopy classes in a lemma above are isotopic each other in the ambient manifold.

The diffeotopy groups of lens spaces

The diffeotopy group $\mathcal{D}(M)$ of a 3-manifold M is the quotient of the diffeomorphism group $\mathrm{Diff}(M)$ by its normal subgroup $\mathrm{Diff}_0(M)$.

$V_1 \cup V_2$: a genus-one Heegaard splitting of a lens space $L(p, q)$.
$V_i = S^1 \times D^2 \subset \mathbb{C} \times \mathbb{C}$ for $i = 1, 2$.
Then, we consider the following three diffeomorphisms preserving the Heegaard surface $\partial V_1 = \partial V_2$.

- τ : an involution fixing each solid torus, defined as $\tau(u, v) = (\bar{u}, \bar{v})$ in each solid torus , where $\bar{z}$ denotes the complex conjugated of z;
- σ_+ : an involution exchanging V_1 and V_2, given by $\sigma_+ : V_i \ni (u, v) \mapsto (u, v) \in V_j$ for $i \neq j$;
- σ_- : a diffeomorphism exchanging V_1 and V_2, given by $\sigma_- : V_1 \ni (u, v) \mapsto (\bar{u}, v) \in V_2$ and $\sigma_- : V_2 \ni (u, v) \mapsto (u, \bar{v}) \in V_1$.

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- σ_- : a diffeomorphism exchanging V_1 and V_2, given by
  $\sigma_- : V_1 \ni (u, v) \mapsto (\bar{u}, v) \in V_2$ and
  $\sigma_- : V_2 \ni (u, v) \mapsto (u, \bar{v}) \in V_1$.

Remark 2.

$\tau|_{V_i}$ *is the hyperelliptic involution.* $\sigma_-^2 = \tau$.

Hodgeson, Rubstein and Bonahon determined such groups of lens spaces.

Theorem 7 (Hodgeson-Rubstein, Bonahon (1983)).

The diffeotopy group of lens space $L(p, q)$ is isomorphic to the following:

- $\mathbb{Z}_2$, *with generator* σ_-, *if* $p = 2$;
- $\mathbb{Z}_2$, *with generator* τ, *if* $q \equiv \pm 1$ *mod* p *and* $p \neq 2$;
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, *with generator* τ *and* σ_+, *if* $q^2 \equiv +1$ *mod* p *and* $q \not\equiv \pm 1$ *mod* p;
- $\mathbb{Z}_4$, *with generator* σ_-, *if* $q^2 \equiv -1$ *mod* p *and* $p \neq 2$;
- $\mathbb{Z}_2$, *with generator* τ, *if* $q^2 \not\equiv \pm 1$ *mod* p.

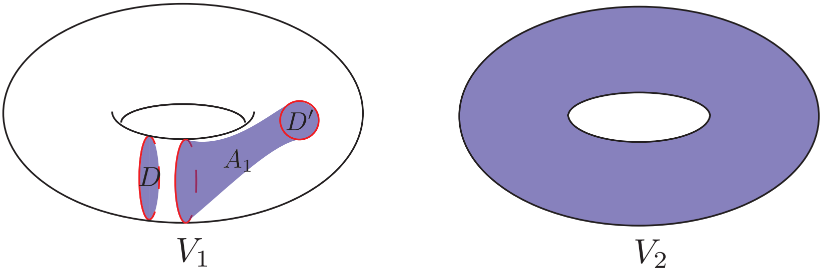
We obtain the following lemma by a theorem above.

Lemma 7.

Let M be a 3-sphere or a lens space $L(p, q)$ with a genus-one Heegaard splitting $V_1 \cup V_2$. M is a 3-sphere or a lens space $L(p, q)$ with $p = 2$ if and only if M admits an ambient isotopy $F : M \times [0, 1] \to M$ that satisfies the following:

- $F(V_1, 1) = V_1$ *and* $F(V_2, 1) = V_2$.
- *Let* $f_t(x) = F(x, t)$. *Then,* $f_1|_{\partial V_1} : \partial V_1 \to \partial V_1$ *is a hyperelliptic involution in a mapping class group of a torus* ∂V_1.

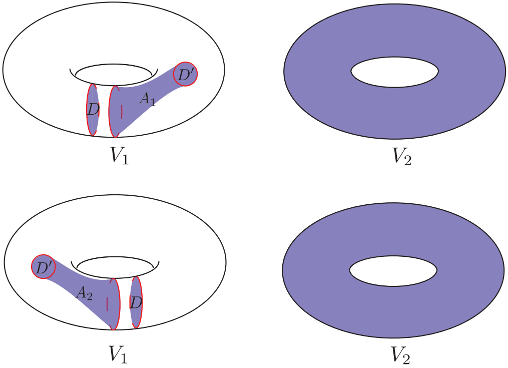
the 3-sphere or $p = 2$



- we can assume $H_3 = V_2$ since the ambient manifold is the 3-sphere and $L(2,1)$.
- we can isotope two embedding of F_{12} into V_1 to each other by Lemma 7.

$\Longrightarrow$ exactly one ambient isotopy classe.

$p \neq 2$ and $(p-1)q \equiv \pm 1 \ (mod\ p)$

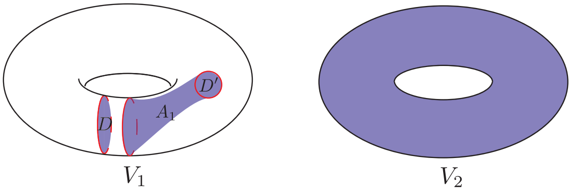


- we can assume $H_3 = V_2$ since the ambient manifold satisfies $(p-1)q \equiv \pm 1 \ (mod\ p)$.
- we <u>can not</u> isotope two embedding of F_{12} into V_1 to each other by Lemma 7.

$\Longrightarrow$ exactly two ambient isotopy classes.

$p \neq 2$ and $(p-1)q \not\equiv \pm 1 \ (mod\ p)$

(1)



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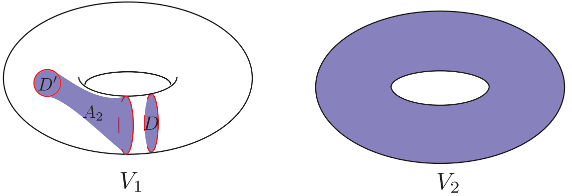

$p \neq 2$ and $(p-1)q \not\equiv \pm 1$ $(mod\ p)$

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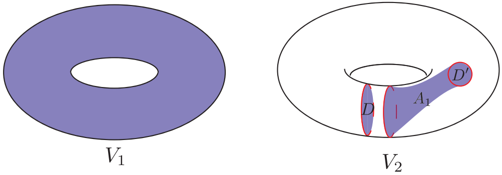

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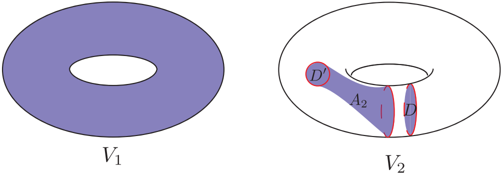

- we have to consider the cases $H_3 = V_2$ and $H_3 = V_1$ since the ambient manifold does not satisfies $(p-1)q \equiv \pm 1$ $(mod\ p)$.
- we can not isotope two embedding of F_{12} into V_1 to each other by Lemma 7.

$\Longrightarrow$ exactly four ambient isotopy classes.

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