ハンドル体結び目のMCQねじれAlexander不変量

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結び目の数理III
オンライン会議システム「Zoom」による開催

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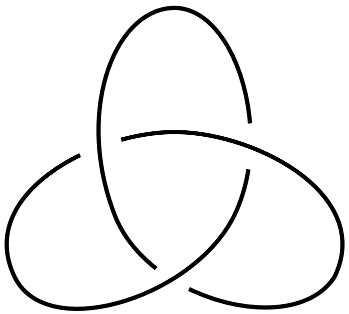
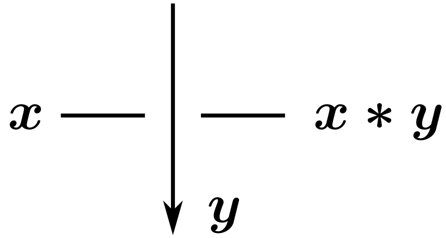


knot

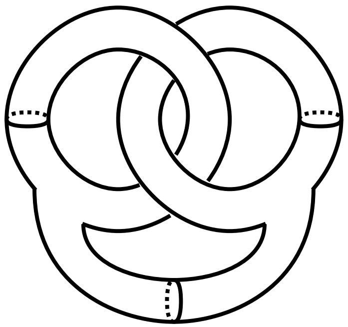
$(Q, *)$: quandle

$\rightsquigarrow$ The universal algebra for defining an arc coloring invariant for oriented knot diagrams.

quandle coloring:



$\rightsquigarrow$
- quandle coloring number
- quandle cocycle invariant
- etc.

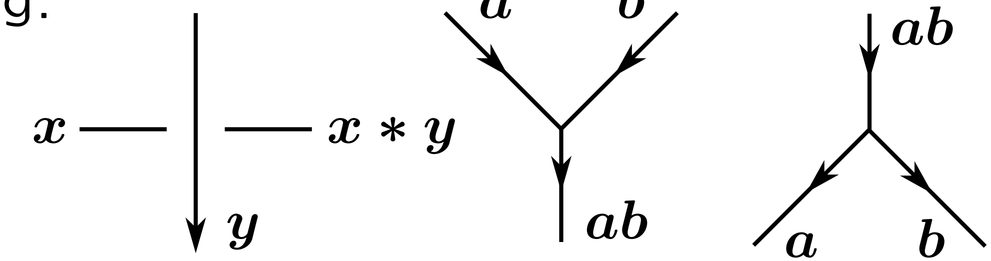


handlebody-knot

$(\bigsqcup_{\lambda \in \Lambda} G_\lambda, *)$: **multiple conjugation quandle**
$(G_\lambda$: group) (MCQ)

$\rightsquigarrow$ The universal algebra for defining an arc coloring invariant for handlebody-knot diagrams.

MCQ coloring:



$\rightsquigarrow$
- MCQ coloring number
- MCQ cocycle invariant
- etc.

(Twisted) Alexander matrix

Grp [Alexander '23]

$\rho : G_K = \langle x \mid r \rangle \to GL(k;\mathbb{Z})$: grp. rep.

$$\mathbb{Z}F_{\mathsf{Grp}}(x) \xrightarrow{\frac{\partial}{\partial x_j}} \mathbb{Z}F_{\mathsf{Grp}}(x) \xrightarrow{\mathrm{pr}} \mathbb{Z}G_K \xrightarrow{\alpha} \mathbb{Z}[t^{\pm 1}]$$

$$\rightsquigarrow \left(\alpha \circ \mathrm{pr} \circ \frac{\partial}{\partial x_j}(r_i)\right)$$: Alexander matrix

Qnd [Ishii–Oshiro '20]

$\rho : Q_K = \langle x \mid r \rangle \to Q$: qnd. rep.

$f := (f_1, f_2)$: Alexander pair of maps $f_1, f_2 : Q \times Q \to R$

$$F_{\mathsf{Qnd}}(x) \xrightarrow{\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}} R \qquad \rightsquigarrow \left(\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(r_i)\right)$$: quandle twisted Alex. mtx.

MCQ [Ishii–M '20]

$\rho : MCQ_H = \langle x \mid r \rangle \to X$: MCQ rep.

$f := (f_1, f_2)$: MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$

$$F_{\mathsf{MCQ}}(x) \xrightarrow{\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}} R \qquad \rightsquigarrow \left(\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_j}(r_i)\right)$$: MCQ twisted Alex. mtx.

(Twisted) Alexander matrix

Grp [Alexander '23]

$\rho : G_K = \langle x \mid r \rangle \to GL(k;\mathbb{Z})$: grp. rep.

$$\mathbb{Z}F_{\mathsf{Grp}}(x) \xrightarrow{\frac{\partial}{\partial x_j}} \mathbb{Z}F_{\mathsf{Grp}}(x) \xrightarrow{\mathrm{pr}} \mathbb{Z}G_K \xrightarrow{\rho\otimes\alpha} M(k;\mathbb{Z}[t^{\pm 1}])$$

$\rightsquigarrow \left((\rho\otimes\alpha)\circ \mathrm{pr} \circ \frac{\partial}{\partial x_j}(r_i) \right)$: twisted Alexander matrix

Qnd [Ishii–Oshiro '20]

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Def [Joyce, Matveev '82]
$(Q, *)$: quandle
$\overset{\text{def}}{\iff}$
$\forall a, b, c \in Q$,
- $a * a = a$
- $*a : Q \to Q; x \mapsto x * a$: bijection
- $(a * b) * c = (a * c) * (b * c)$

Def [Ishii–Oshiro '20]
Q : quandle, R : ring, $f_1, f_2 : Q \times Q \to R$
(f_1, f_2) : **Alexander pair**
$\overset{\text{def}}{\iff}$
$\forall a, b, c \in Q$,
- $f_1(a, a) + f_2(a, a) = 1$
- $f_1(a, b)$ is invertible
- $f_1(a * b, c) f_1(a, b) = f_1(a * c, b * c) f_1(a, c)$
  $f_1(a * b, c) f_2(a, b) = f_2(a * c, b * c) f_1(b, c)$
  $f_2(a * b, c) = f_1(a * c, b * c) f_2(a, c) + f_2(a * c, b * c) f_2(b, c)$

$\iff$
$\forall M$: left R-module,
$Q \times M$: quandle by $(a, x) * (b, y) = (a * b, f_1(a, b)x + f_2(a, b)y)$

Def [Ishii '15]
G_λ : group w/ the identity e_λ ($\forall \lambda \in \Lambda$), $X := \bigsqcup_{\lambda \in \Lambda} G_\lambda$
$(X, *)$: **multiple conjugation quandle (MCQ)**
$\overset{\text{def}}{\Longleftrightarrow}$
$\forall a, b \in G_\lambda,\ \forall x, y, z \in X,$

- $a * b = b^{-1}ab$
- $x * e_\lambda = x,\ \ x * (ab) = (x * a) * b$
- $(x * y) * z = (x * z) * (y * z)$
- $(ab) * x = (a * x)(b * x)$

Rem
An MCQ itself is a quandle.

Prop
$(Q, *)$: quandle

$$k := \mathrm{type}\, Q := \min \left\{ n \in \mathbb{Z}_{>0} \;\middle|\; \begin{array}{l} \forall x, y \in Q, \\ x *^n y := (\cdots((x \underbrace{*y) * y) * \cdots * y}_{n}) = x \end{array} \right\}$$

Then,
$\mathbb{Z}_k \times Q = \bigsqcup_{x \in Q} (\mathbb{Z}_k \times \{x\})$: MCQ with $(a, x) * (b, y) := (a, x *^b y),$
$(a, x) \cdot (b, x) := (a + b, x).$

Def [M '19]
$X := \bigsqcup_{\lambda \in \Lambda} G_\lambda$: MCQ, R : ring, $f_1, f_2 : X \times X \to R$
(f_1, f_2) : **MCQ Alexander pair**
$\overset{\text{def}}{\Longleftrightarrow}$

- $\forall a, b \in G_\lambda$,
  $$f_1(a,b) + f_2(a,b) = f_1(a, a^{-1}b).$$
- $\forall a, b \in G_\lambda, \forall x \in X$,
  $$f_1(a,x) = f_1(b,x),$$
  $$f_2(ab,x) = f_2(a,x) + f_1(b*x, a^{-1}*x) f_2(b,x).$$
- $\forall x \in X, \forall a, b \in G_\lambda$,
  $$f_1(x, e_\lambda) = 1,$$
  $$f_1(x, ab) = f_1(x*a, b) f_1(x,a),$$
  $$f_2(x, ab) = f_1(x*a, b) f_2(x,a).$$
- $\forall x, y, z \in X$,
  $$f_1(x*y, z) f_1(x,y) = f_1(x*z, y*z) f_1(x,z),$$
  $$f_1(x*y, z) f_2(x,y) = f_2(x*z, y*z) f_1(y,z),$$
  $$f_2(x*y, z) = f_1(x*z, y*z) f_2(x,z) + f_2(x*z, y*z) f_2(y,z).$$

$\Longleftrightarrow$

$\forall M$: left R-module,
$\widetilde{X} := \bigsqcup_{\lambda \in \Lambda} (G_\lambda \times M)$: MCQ with

$$(x,u) * (y,v) := (x*y, f_1(x,y)u + f_2(x,y)v) \quad ((x,u),(y,v) \in \widetilde{X}),$$
$$(a,u) \cdot (b,v) := (ab, u + f_1(a, a^{-1})v) \quad ((a,u),(b,v) \in G_\lambda \times M).$$

Free MCQ

$\boldsymbol{S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}}$: set of pairwise disjoint sets

- $\boldsymbol{F_{\mathsf{MCQ}}(S_\Lambda)}$: **free MCQ** on $\boldsymbol{S_\Lambda}$
  (the free object in the category of MCQs.)

MCQ presentation

$\boldsymbol{R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)}$

- $\boldsymbol{\langle S_\Lambda \mid R\rangle := \overline{F_{\mathsf{MCQ}}(S_\Lambda)/\sim_R}}$
- $\boldsymbol{\langle S_\Lambda \mid R\rangle}$: a **presentation** of an MCQ $\boldsymbol{X} \overset{\text{def}}{\Longleftrightarrow} \boldsymbol{\langle S_\Lambda \mid R\rangle \cong X}$
- Any MCQ has a presentation.

	Grp	Qnd	MCQ
Free object	$F_{\mathsf{Grp}}(S)$ (S : set)	$F_{\mathsf{Qnd}}(S)$ (S : set)	$F_{\mathsf{MCQ}}(S_\Lambda)$ ($S_\Lambda = \{S_\lambda \,\vert\, \lambda \in \Lambda\}$: set of sets)
Presentation	$\forall$grp $\cong \langle S \,\vert\, R\rangle$ $:= F_{\mathsf{Grp}}(S)/N(R)$ $(N(R) \lhd F_{\mathsf{Grp}}(S))$	$\forall$qnd $\cong \langle S \,\vert\, R\rangle$ $:= F_{\mathsf{Qnd}}(S)/\sim_R$ $(R \subset F_{\mathsf{Qnd}}(S) \times F_{\mathsf{Qnd}}(S))$	$\forall$MCQ $\cong \langle S_\Lambda \,\vert\, R\rangle$ $:= \overline{F_{\mathsf{MCQ}}(S_\Lambda)/\sim_R}$ $(R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda))$

Def [Ishii–M '20]

$S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$: finite set of pairwise disjoint finite sets so that $\bigcup S_\Lambda = \{x_1, \ldots, x_n\}$

$F_{\mathsf{MCQ}}(S_\Lambda) = \bigsqcup_{\mu \in M} G_\mu$: free MCQ on S_Λ

$X = \langle S_\Lambda \mid \{r_1, \ldots, r_m\} \rangle$: finitely presented MCQ

$\mathrm{pr} : F_{\mathsf{MCQ}}(S_\Lambda) \to X$: canonical projection

$f = (f_1, f_2)$: MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$

$\frac{\partial_f}{\partial x_j} : F_{\mathsf{MCQ}}(S_\Lambda) \to R$: **f-derivative (twisted derivative)**

$\overset{\text{def}}{\Longleftrightarrow}$

$\forall x, y \in F_{\mathsf{MCQ}}(S_\Lambda)$, $\forall a, b \in G_\mu$,

- $\frac{\partial_f}{\partial x_j}(x * y) = f_1(\mathrm{pr}(x), \mathrm{pr}(y)) \frac{\partial_f}{\partial x_j}(x) + f_2(\mathrm{pr}(x), \mathrm{pr}(y)) \frac{\partial_f}{\partial x_j}(y)$
- $\frac{\partial_f}{\partial x_j}(ab) = \frac{\partial_f}{\partial x_j}(a) + f_1(\mathrm{pr}(a), \mathrm{pr}(a)^{-1}) \frac{\partial_f}{\partial x_j}(b)$
- $\frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}$ (Kronecker delta)

Def [Ishii–M '20]

H : handlebody-knot

$MCQ_H = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$: fund. MCQ of H

$\rho : MCQ_H \to X$: MCQ rep.

$f = (f_1, f_2)$: MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$

$f \circ (\rho \times \rho) := (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$: MCQ Alexander pair of maps $f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho) : MCQ_H \times MCQ_H \to R$

For a relator $r = (r', r'')$, $\quad \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_j}(r) := \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_j}(r') - \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_j}(r'')$

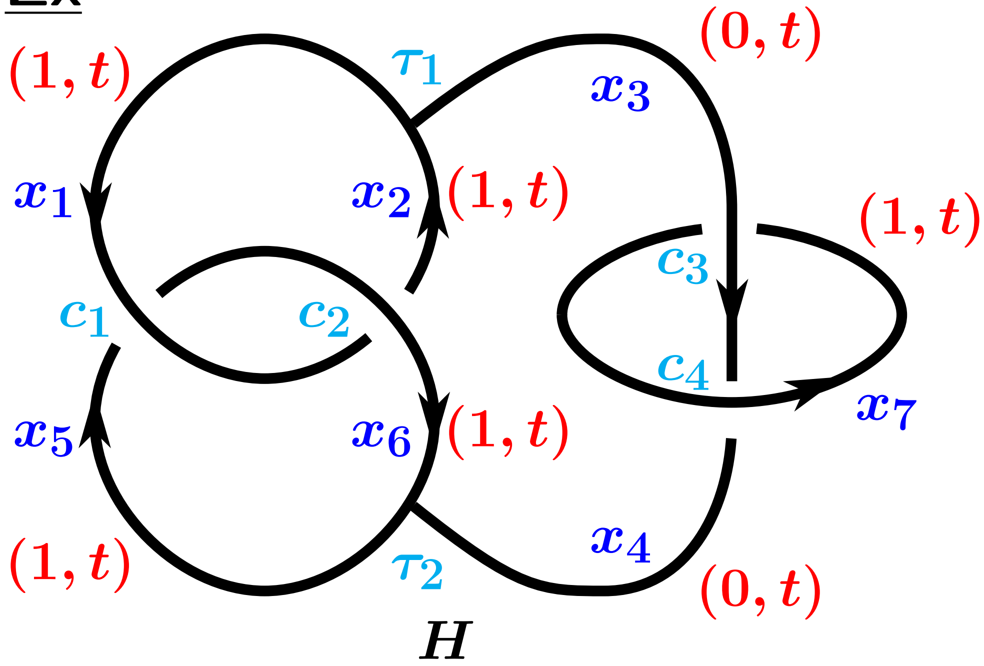
- $A(H, \rho; f) := \left(\frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_j}(r_i) \right)$: f**-twisted Alexander matrix** of (H, ρ) (**MCQ twisted Alexander matrix**)

- R : GCD domain

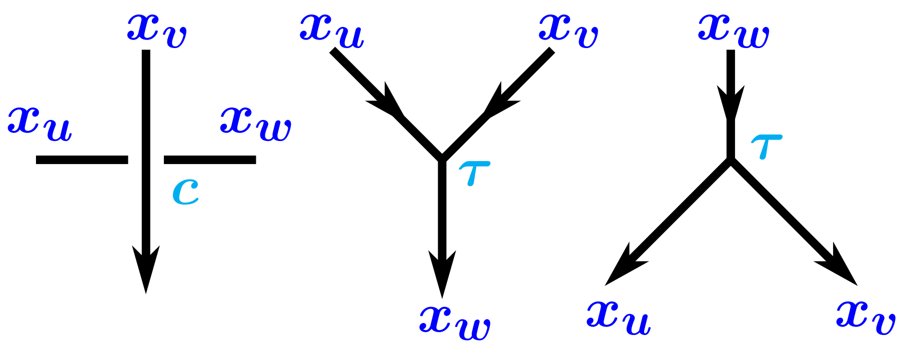
$$\Delta_d(H, \rho; f) := \begin{cases} 0 & (d < n - m) \\ \gcd(\{(n-d)\text{-minors of } A(H, \rho; f)\}) & (n - m \leq d < n) \\ 1 & (n \leq d) \end{cases}$$

: d**-th** f**-twisted Alexander invariant** of (H, ρ) (**MCQ twisted Alexander invariant**)

Ex



- relators



$r_c : (x_u * x_v, x_w)$

$r_\tau : (x_u x_v, x_w)$

$$MCQ_H = \left\langle \begin{array}{c} x_1, x_2, x_3; \\ x_4, x_5, x_6; x_7 \end{array} \middle| \begin{array}{ll} r_{c_1} : (x_5 * x_1, x_6), & r_{c_2} : (x_1 * x_6, x_2), \\ r_{c_3} : (x_7 * x_3, x_7), & r_{c_4} : (x_4 * x_7, x_3), \\ r_{\tau_1} : (x_3 x_1, x_2), & r_{\tau_2} : (x_6 x_4, x_5) \end{array} \right\rangle$$

$X := \mathbb{Z}_2 \times \langle t \mid t^3 \rangle$: MCQ, $\rho : MCQ_H \to X$: MCQ rep.

$f_1, f_2 : X \times X \to \mathbb{Z}[t^{\pm 1}]/(t^3 - 1)$: MCQ Alexander pair by

$$\begin{cases} f_1((a,x),(0,y)) = 1, \\ f_1((a,x),(1,y)) = -x^{-1}y, \end{cases} \qquad \begin{cases} f_2((0,x),(b,y)) = 0, \\ f_2((1,x),(0,y)) = -1 - xy^{-1}, \\ f_2((1,x),(1,y)) = 1 + x^{-1}y. \end{cases}$$

(memo) $f_2((1,x),(1,y)) = 1 + x^{-1}y$

$$\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}((x_5 * x_1, x_6)) = \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(x_5 * x_1) - \frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(x_6)$$
$$= f_1(\rho(x_5),\rho(x_1))\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(x_5) + f_2(\rho(x_5),\rho(x_1))\frac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(x_1)$$
$$= f_2(\rho(x_5),\rho(x_1)) = f_2((1,t),(1,t)) = 1 + t^{-1}t = 2$$

$$A(H,\rho;f) = \begin{array}{c} \\ r_{c_1} \\ r_{c_2} \\ r_{c_3} \\ r_{c_4} \\ r_{\tau_1} \\ r_{\tau_2} \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix} \end{array} \sim \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

$$\therefore \Delta_d(H,\rho;(f_1,f_2)) = \begin{cases} 0 & (d < 2) \\ 4 & (d = 2) \\ 1 & (d > 2) \end{cases}$$

$\therefore H \not\cong$ trivial handlebody-link