Coloring links by Symmetric group of order 3

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Introduction

We consider a coloring links by the symmetric group S_3 of order 3.

Definition 1

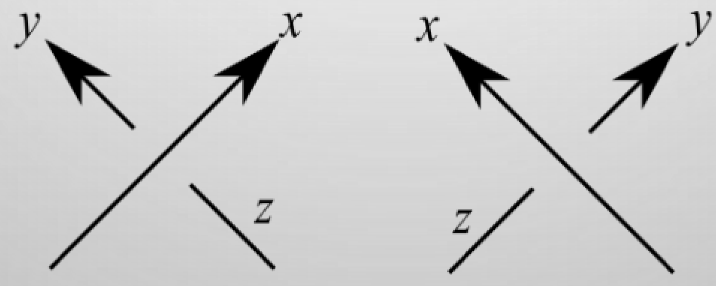
Let D be an oriented diagram of a link L.
A map C:{arcs of D} $\to \{S_3 - e\}$ is called an **S_3-coloring** on D if it satisfies the following conditions at each crossings.

①On a positive crossing

$$C(x)C(y) = C(z)C(x)$$

②On a negative crossing

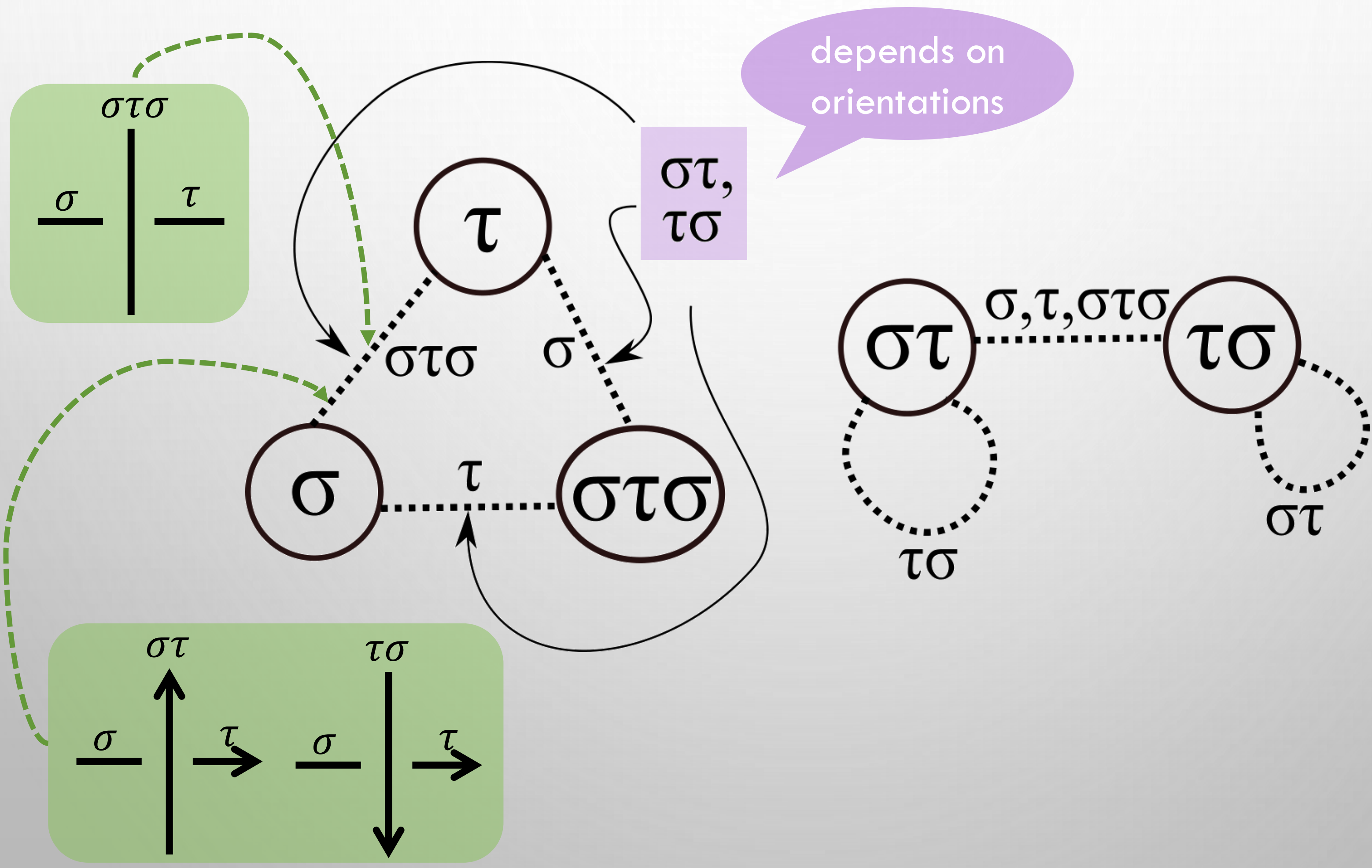
$$C(y)C(x) = C(x)C(z)$$



$C(x)C(y)=C(z)C(x)$ $C(y)C(x)=C(x)C(z)$

Introduction

We see the palette graph of this coloring as follows. ($\sigma = (1\ 2)$, $\tau = (2\ 3)$)

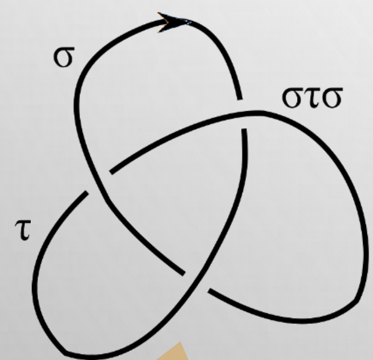


Introduction

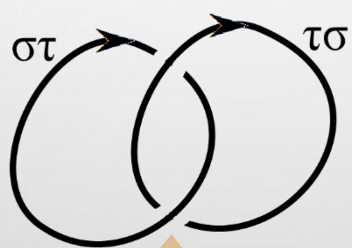
Definition 2

① L is an ***S_3*-colorable** link if ∃ a diagram of L admits a non-trivial S_3-coloring, i.e., an S_3-coloring with at least two colors.

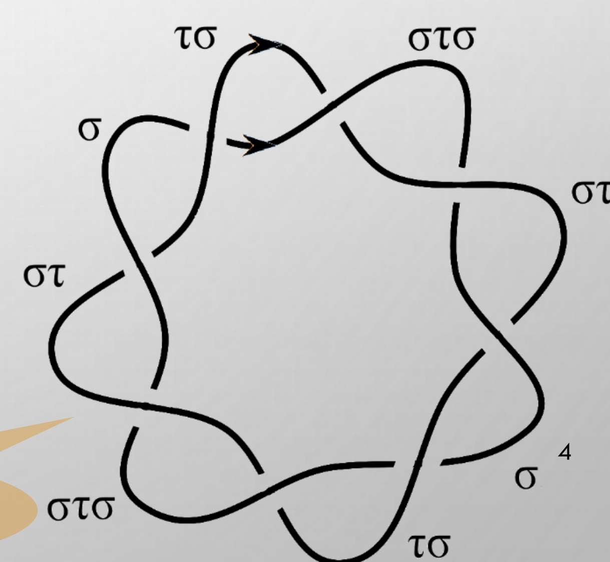
② L is an **(S_3, n)-colorable** link if ∃ a diagram of L admits a S_3-coloring with n colors.



$(S_3, 3)$-colorable



$(S_3, 2)$-colorable



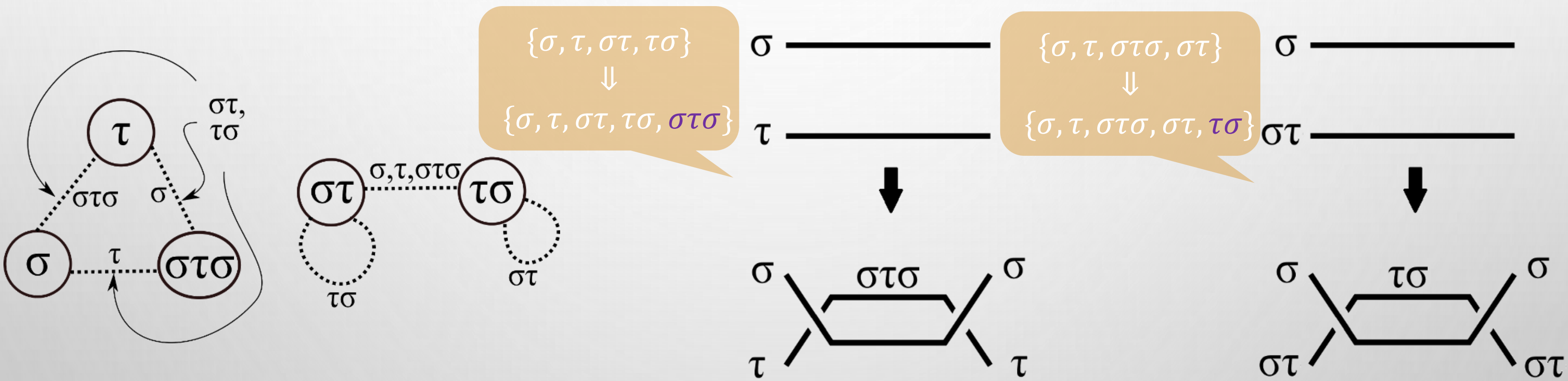
$(S_3, 4)$-colorable

Introduction

We can see the following.

Proposition

Any $(S_3, 4)$-colorable link is $(S_3, 5)$-colorable.



Here, we have the next question.

Question 1

For any $(S_3, 5)$-colorable link L, is L $(S_3, 4)$-colorable？

Main theorem

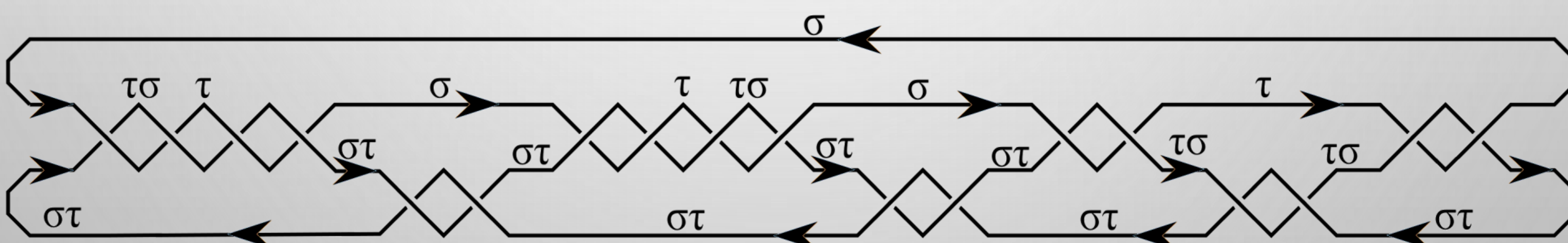
Theorem 1 [Ichihara – M.]

①A 2-bridge link L is $(S_3, 4)$-colorable if and only if L has a Conway diagram $D = C(2a_1,\ \ 2b_1,\ \ 2a_2,\ \ 2b_2, \dots, 2b_m, 2a_{m+1})$ such that D satisfies the following.

$$\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2$$

②Any $(S_3, 5)$-colorable 2-bridge link L is $(S_3, 4)$-colorable.

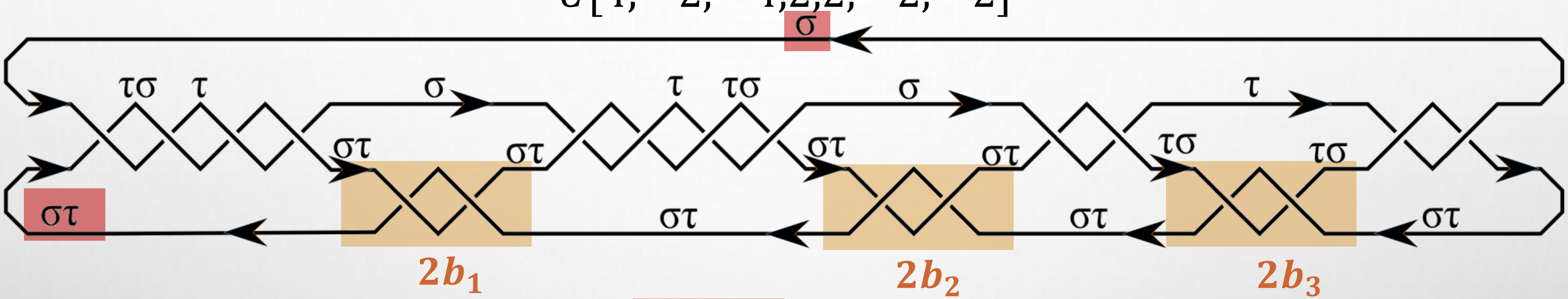
(Example 1) $C[4, -2, -4, 2, 2, -2, -2]$



Main theorem

[Outline of proof ①] ($\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2 \Rightarrow (S_3, 4)$-colorable)

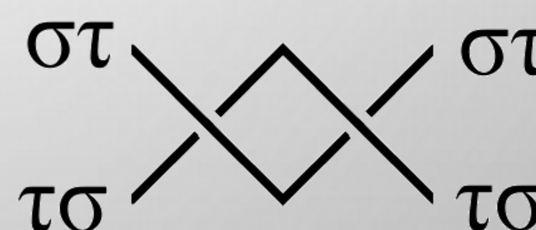
$$C[4, -2, -4, 2, 2, -2, -2]$$



First, we can fix the colors on the arcs in the figure to be σ and $\sigma\tau$.

Here, any twists with $2b_i$ does not change the colors on parallel arcs.

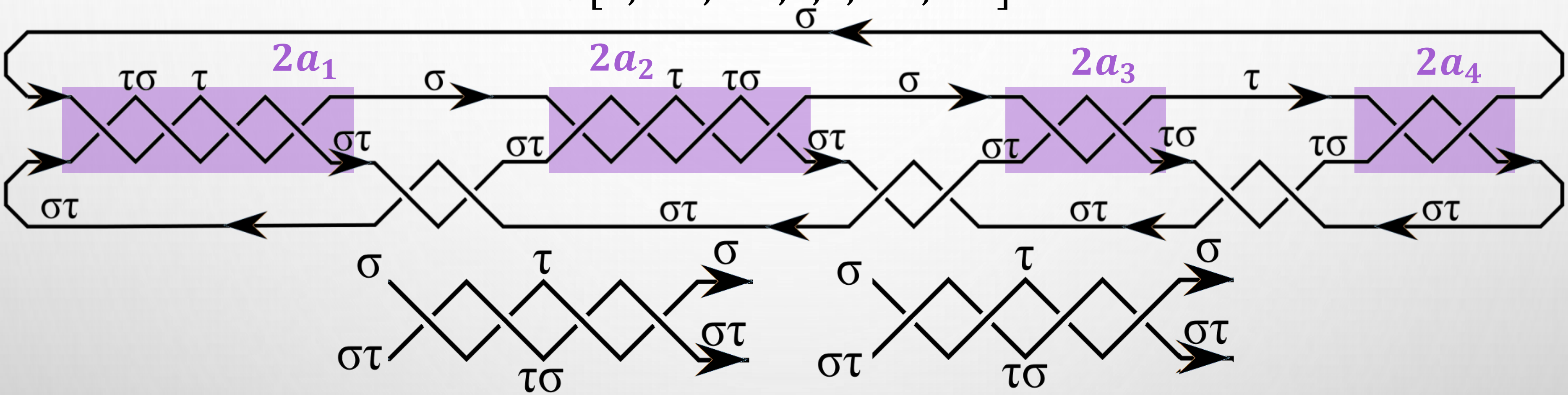
We consider only the colors on twists with $2a_i$.



Main theorem

[Outline of proof ①] ($\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2 \Rightarrow (S_3, 4)$-colorable)

$$C[4,-2,-4,2,2,-2,-2]$$



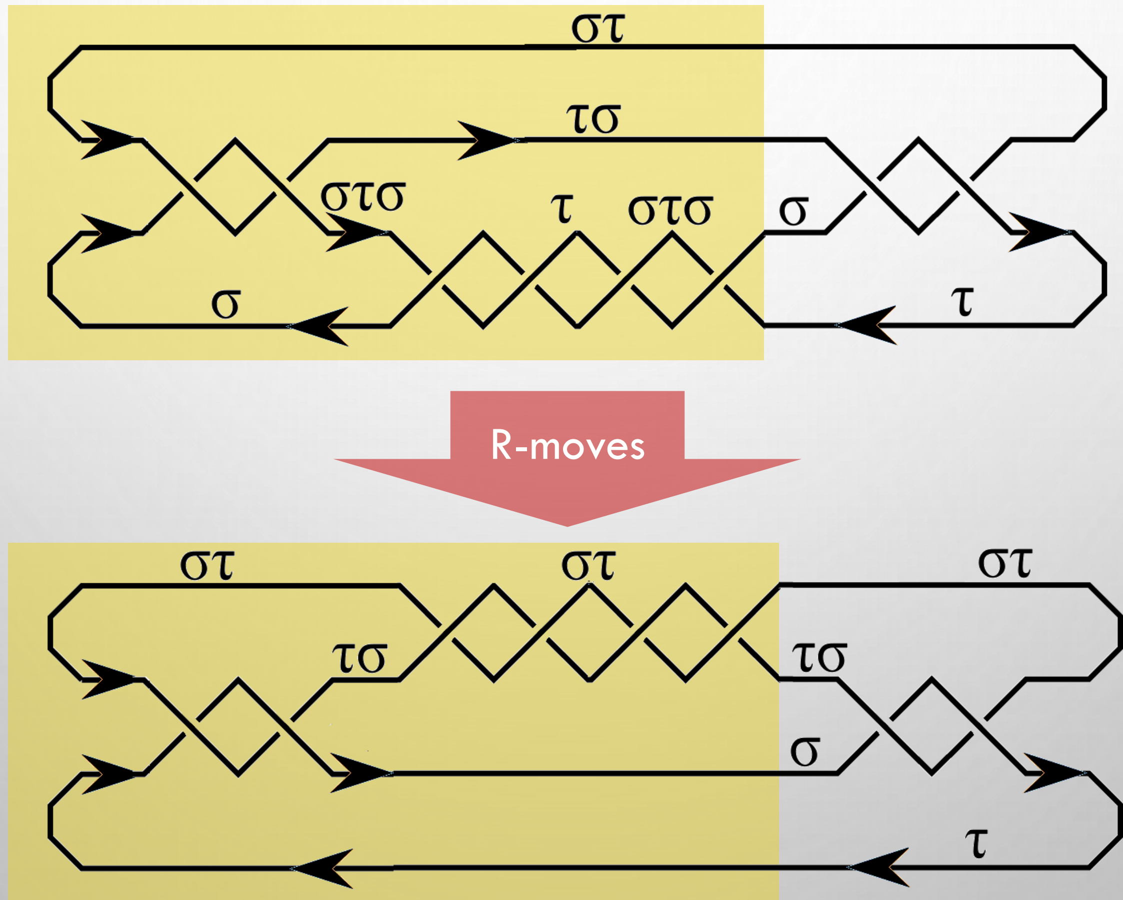
Here the color “$\sigma\tau\sigma$” does not apper.

In the same way, if $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2$ holds, then the diagram admits an S_3-coloring with 4 colors.

Main theorem

[Outline of proof ②] (($(S_3, 5)$-colorable ⇒ $(S_3, 4)$-colorable)

$C[2,4,2]$



Main theorem

We obtain the following results about the minimal coloring number for torus links and double twist links.

Theorem 2 [Ichihara – M.]

$T(2,q)$: a torus link

$T(2,q)$ is $(S_3, 4)$-colorable but not $(S_3, 3)$-colorable

$\Leftrightarrow q \equiv 0\ (mod\ 4)$ and $q \not\equiv 0\ (mod\ 3)$

Theorem 3 [Ichihara – M.]

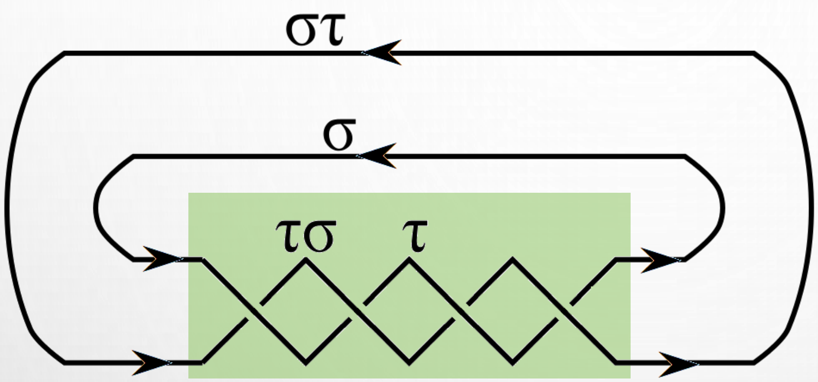
$J(k,l)$: a double twist link

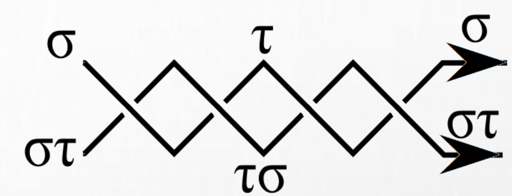
$J(k,l)$ is $(S_3, 4)$-colorable but not $(S_3, 3)$-colorable

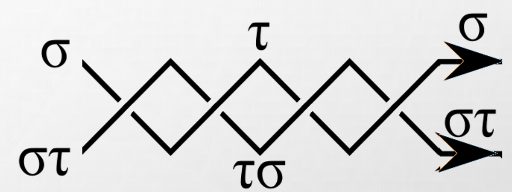
$\Leftrightarrow kl \equiv 3(mod\ 4)$ and $kl \not\equiv 2\ (mod\ 3)$

Main theorem

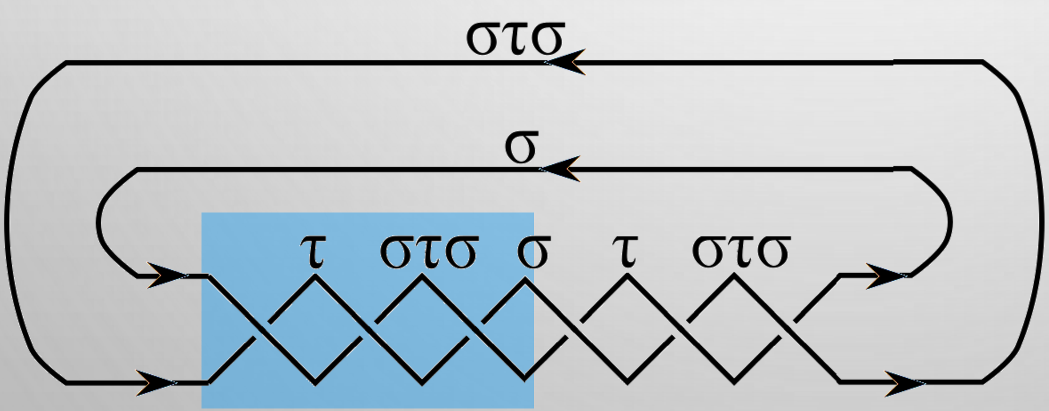
(Example 2) $T(2,4)$

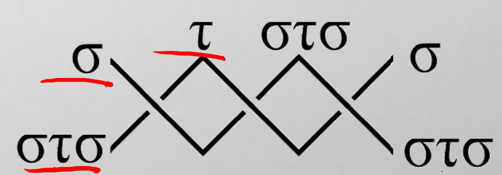






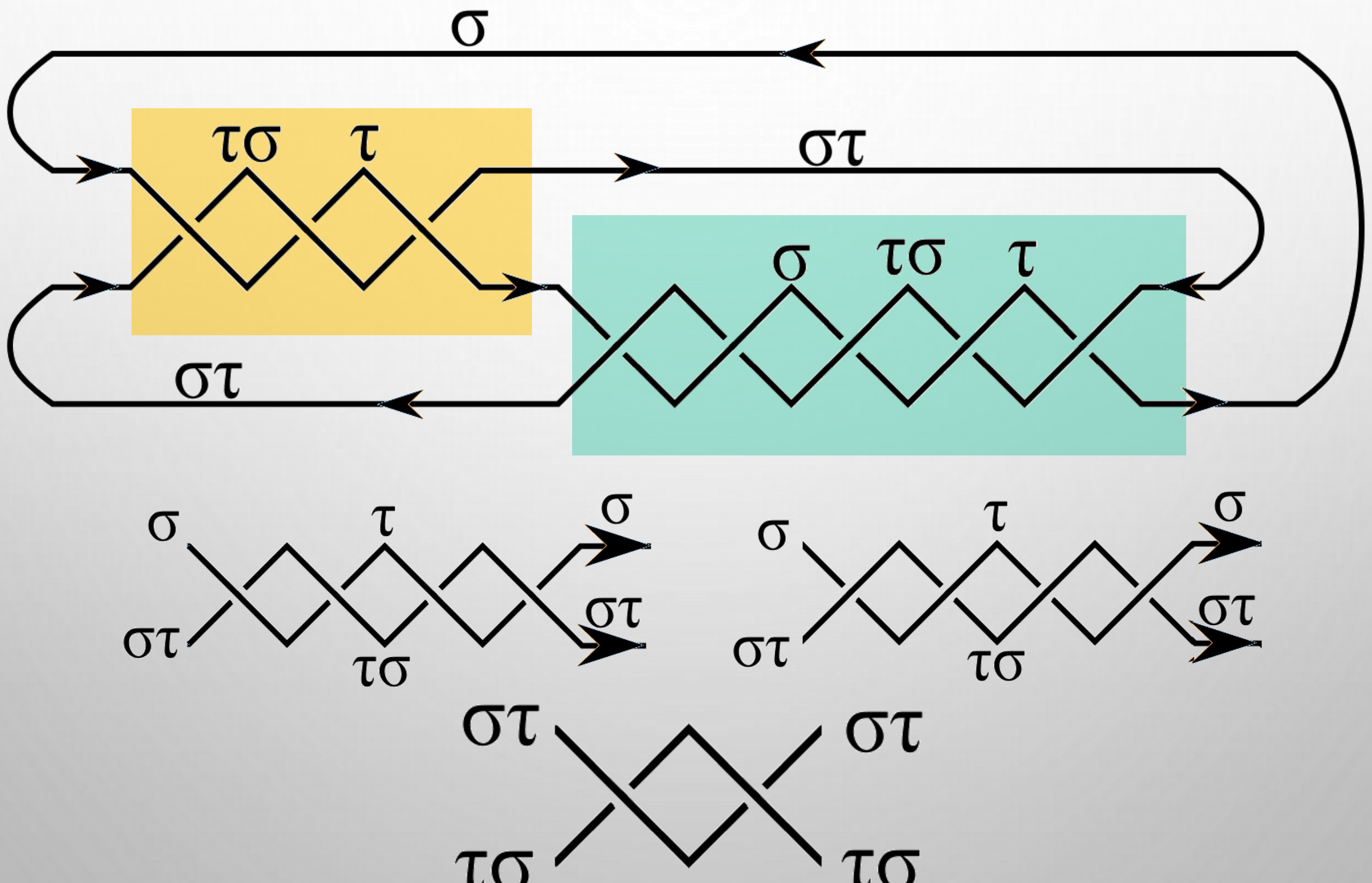
(Example 3) $T(2,6)$





Main theorem

(Example 4) $J(3,5)$



Thank you
for your attention.