On relations between difference on the singularity of fold maps and that on information of the manifolds
(折り目写像の特異性に関する違いと多様体の情報の違いの関係について)

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The main theme and remarks.

The main theme.

- Difference on singularities of fold maps, which are **smooth maps regarded as higher dimensional versions of Morse functions**, and difference of information of the manifolds are closely related.
- **Restrictions on topologies and differentiable structures of manifolds admitting specific fold maps**.

Morse functions?

Fundamental tools in so-called Morse theory and its application to geometry of manifolds.
$\Rightarrow$ Singular points, appearing discretely, tell us information of homology groups and some information on homotopy of a manifold.

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Notation, terminologies and remarks.

$m \geq n \geq 1$: integers
M : a closed, connected and smooth manifold of dimension m.
$f : M \to \mathbb{R}^n$: a smooth map.
A singular point p of f. $\leftrightarrow$ A point p at which the rank of differential df_P drops or the rank of this is smaller than $\min\{m, n\}$.
$S(f)$: the set of all singular points (the singular set) of f.
$f(S(f))$ $(\mathbb{R}^n - f(S(f)))$: the singular (resp. regular) value set.
A singular (regular) value of f. $\leftrightarrow$ A point (resp. not) in $f(S(f))$.

Manifolds, maps between them and bundles whose fibers are manifolds are smooth (of class C^∞) and (boundary) connected sums etc. are discussed in the smooth category unless otherwise stated.

Fold maps.

Fold maps.

Definition 1

f is a fold map.
$\leftrightarrow$ At each singular point p f is of the form
$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i(p)} {x_k}^2 - \sum_{k=m-i(p)+1}^{m} {x_k}^2)$
for an integer $0 \leq i(p) \leq \frac{m-n+1}{2}$.

$n = 1$ case (where the target space is the line). $\leftrightarrow$ f is a Morse function.

Proposition 1

1. *The integer $i(p)$ is unique (we call $i(p)$ the index of p).*
   (f : special generic $\leftrightarrow$ *f is a fold map s.t. $i(p) = 0$ for each p*)
2. *The set of all singular points of an index is a closed submanifold of dimension $n - 1$ with no boundary and $f|_{S(f)}$ is an immersion.*

The function case. $\rightarrow$ The number of singular points of an index (defined by respecting the orientation of the target and defined uniquely) tells us about homology groups (the classical theory of Morse functions).

Fundamental remarks on fold maps.

A fold map from a surface into the plane.

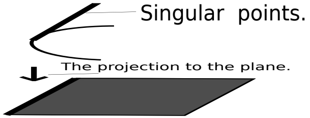

Figure 1: A local form of a fold map from a surface into the plane.

Existence of fold maps into general Euclidean spaces.

- A closed manifold whose dimension is greater than or equal to 2 admits a fold map into the plane **if and only** if the Euler number is even (1950–60s: Thom and Whitney (Levine)).
- A closed manifold of dimension m admits a fold map into $\mathbb{R}^n$ if the Whitney sum of the tangent bundle and a trivial real line bundle over the manifold is trivial for $m \geq n \geq 1$ ("if" is replaced by "if and only if" for the case $m = n$) (1970s: Eliashberg).

Special generic maps restrict the topologies and the differentiable structures of the manifolds.

Fundamental notes on special generic maps.

Special generic functions ($n = 1$ case where the target space is the line).

Special generic functions on closed and connected manifolds characterize spheres topologically except 4-dim. cases. In 4-dim. cases, these maps characterize S^4 as smooth manifolds. $\Rightarrow$ So-called **Reeb's theorem**.

Simplest special generic maps.

・ Canonical projections of unit spheres are special generic: the singular sets are spheres and the restrictions to the singular sets are embeddings.

・ A manifold M represented as a connected sum of the manifolds in $\{S^{l_j} \times S^{m-l_j}\}_{j=1}^{l}$ satisfying $1 \leq l_j \leq n-1$ admits a special generic map f into $\mathbb{R}^n$ such that $f|_{S(f)}$ is an embedding and that $f(M)$ is represented as a boundary connected sum of manifolds in a family $\{S^{l_j} \times D^{n-l_j}\}$.

Figure 2: A canonical projection of a unit sphere and the image of a special generic map into $\mathbb{R}^2$ (the restrictions to the singular sets are embeddings).

Special generic maps on homotopy spheres.

Fact 1

1. (*1993 Saeki.*)
   Homotopy spheres whose dimensions are greater than or equal to 2 *which are not diffeomorphic to* 4*-dim. homotopy spheres non-diffeomorphic to* S^4 *admit special generic maps into the plane.*
2. (*1960s Calabi, 1993 Saeki, and so on.*) **m-dim. homotopy spheres non-diffeomorphic to standard spheres** *admit* **no** *special generic map into* $\mathbb{R}^n$ *for* $n = m-3, m-2, m-1$ *where* $m \geq 4$*.*
3. (*Wrazidlo 2017–8.*) 7*-dim. oriented homotopy spheres of at least* 14 *types of all the* 28 *types admit no special generic map into* $\mathbb{R}^3$*.*

Special generic maps **restrict the differentiable structures of the homotopy spheres** in considerable cases.

Special generic maps restrict the topologies and the differentiable structures of the manifolds.

Fact 2

1. (*Saeki 1993.*) *Let $m \geq 2$. An m-dim. closed and connected manifold admits a special generic map into* $\mathbb{R}^2$ **iff** *it is represented as a connected sum of the total space of a smooth bundle over the circle whose fiber is an $(m-1)$-dim. homotopy sphere admitting a special generic function.*
2. (*2000s- Saeki, Sakuma, and so on.*) *There exist pairs of 4-dimensional closed and connected manifolds satisfying the following three.*
   - 2.1 *Two manifolds of each pair are mutually homeomorphic.*
   - 2.2 *Both of each pair admit fold maps into* $\mathbb{R}^3$.
   - 2.3 **Exactly one manifold** *of each pair admits a special generic map into* $\mathbb{R}^3$.

Special generic maps **restrict the topologies and the differentiable structures of the manifolds** in considerable cases.

Round fold maps and round fold maps into $\mathbb{R}^4$ on 7-dimensional homotopy spheres.

Round fold maps.

$x \in \mathbb{R}^k$: $||x||$ denotes the distance between x and the origin 0 where the space is endowed with the Euclidean metric.

Definition 2 (2013– K)

A fold map $f : M \to \mathbb{R}^n$ is round. $\leftrightarrow$ Either of the following holds.

1. $n = 1$, $f|_{S(f)}$ is an embedding and $\exists a \in \mathbb{R}^n$ a regular value $\exists(\Phi : f^{-1}((-\infty, a]) \to f^{-1}([a, \infty)), \phi : (-\infty, a] \to [a, \infty))$ a pair of diffeomorphisms s.t. $f|_{f^{-1}([a,\infty))} \circ \Phi = \phi \circ f|_{f^{-1}((-\infty,a])}$.
2. $n \geq 2$, $f|_{S(f)}$ is an embedding and $\exists \phi$ a diffeomorphism on $\mathbb{R}^n$, $\exists l > 0$ an integer s.t. $\phi(f(S(f))) = \{||x|| = r \mid \ r \in \mathbb{N}, 1 \leq r \leq l\}$.

- $n = 1$. $\to$ So-called (twisted) doubles of Morse functions: e. g. the Morse functions characterizing spheres topologically.
- $n \geq 2$. $\to$ A fold map whose singular value set is a disjoint union of spheres embedded concentrically.
  $\to$ Canonical projections of unit spheres, special generic maps into the plane whose singular value sets are embedded circles (, **which were constructed in Fact 1** (1),) etc. are round.

Monodromies for round fold maps.

Definition 3 (2013– K)

$f : M \to \mathbb{R}^n$: a round fold map.

1. f is said to <u>have a globally trivial monodromy</u>.
   $\leftrightarrow$ $n = 1$
   or
   $n \geq 2$ and for a diffeomorphism ϕ on $\mathbb{R}^n$ and an integer $l > 0$ s.t. $\phi(f(S(f))) = \{||x|| = r \mid \ r \in \mathbb{N}, 1 \leq r \leq l\}$, the composition of the restriction of $\phi \circ f$ to $\phi \circ f^{-1}(\{||x|| = r \mid \frac{1}{2} \leq r\})$ with a canonical map defined by $x \mapsto \frac{1}{||x||}x$ gives a trivial bundle over the unit sphere.
2. f is said to <u>have a componentwisely trivial monodromy</u>.
   $\leftrightarrow$ $n = 1$
   or
   $n \geq 2$, and for a diffeomorphism ϕ on $\mathbb{R}^n$ and an integer $l > 0$ s.t. $\phi(f(S(f))) = \{||x|| = r \mid \ r \in \mathbb{N}, 1 \leq r \leq l\}$, the composition of the restriction of $\phi \circ f$ to $\phi \circ f^{-1}(\{||x|| = r \mid k - \frac{1}{2} \leq r \leq k + \frac{1}{2}\})$ for each integer $1 \leq k \leq l$ with a canonical map defined by $x \mapsto \frac{1}{||x||}x$ gives a trivial bundle over the unit sphere.

Monodromies for round fold maps (figures).

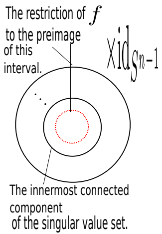

Figure 3: The image of a round fold map $f : M \to \mathbb{R}^n$ having a **globally trivial monodromy**: the restriction of f to the preimage of the complementary set of the interior of an n-dim. standard closed disc in the connected component of the regular value set in the center is represented as a product map of a Morse function and $\mathrm{id}_{S^{n-1}}$.

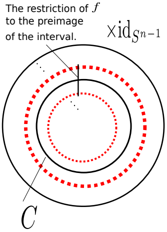

Figure 4: The image of a round fold map f having a **componentwisely trivial monodromy**: the restriction of f to the preimage of a small closed tubular neighborhood (,represented as the annulus bounded by the disjoint union of two dotted red circles,) of each connected component C of the singular value set is represented as a product map of a Morse function and $\mathrm{id}_{S^{n-1}}$.

Examples on the monodromies.

- Canonical projections of unit spheres, special generic maps into the plane whose singular value sets are embedded circles etc. have globally trivial monodromies.
- (2014– K)
  $\exists$ infinitely many round fold maps on infinitely many manifolds

  not having **globally trivial monodromies**

  &

  having **componentwisely trivial monodromies**.

Round fold maps on manifolds represented as connected sums of total spaces of S^k-bundles over S^n ($k > 0$).

Theorem 1 (2013–4 K)

$m > n \geq 1$.
M : an m-dim. manifold represented as a connected sum of $l > 0$ total spaces of S^{m-n}-bundles over S^n.
$\rightarrow \exists\ f : M \rightarrow \mathbb{R}^n$: a round fold map having a componetwisely trivial monodromy s. t.

1. *The index of each singular point is 0 or 1. The number of singular points is $2(l+1)$ for $n = 1$ and that of connected components of the singular set is $l+1$ for $n \geq 2$.*
2. *Preimages of regular values are disjoint unions of copies of S^{m-n} and the numbers of the connected components of preimages of regular values in the center are $l+1$.*

Remark 1 (2013–4 K)

In Theorem 1 "$\leftarrow$" also holds for "$m \geq 2n$" or "$l = 1$ and round fold maps having a globally trivial monodromy".

A sketch of a proof of Theorem 1 for "$l = 1$ and round fold maps having a globally trivial monodromy".

More generally, we can take the fiber as a homotopy sphere Σ admitting a special generic function.

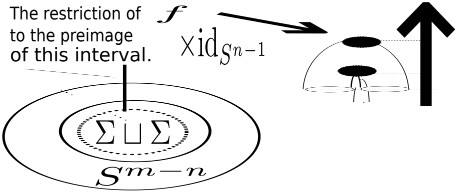

Figure 5: The image (singular value set), preimages of regular values, etc. of a round fold map in Theorem 1 for "$l = 1 \cdots$".

- On the preimage of the straight line in the figure, it is a Morse function with exactly two singular points on a cylinder.
- M is regarded as a maifold obtained by gluing the two manifolds $D^n \times (\Sigma \sqcup \Sigma)$ (the preimage of the disc bounded by the dotted circle in the figure) and $S^{n-1} \times \Sigma \times I$ via a bundle isomorphism between the trivial bundles over $\partial D^n = S^{n-1}$ (∂D^n and S^{n-1} are identified in a canonical way) whose fibers are $\Sigma \sqcup \Sigma$.

An interesting example closely related to 7-dim. homotopy spheres.

Thanks to the classical theory of exotic spheres of Milnor, related theory of Eells and Kuiper, more general theory of Kervaire etc. $\cdots$.

1. There are exactly 28 types of 7-dim. oriented homotopy spheres.
2. These oriented homotopy spheres of 16 types of the 28 types including the standard sphere are represented as total spaces of S^3-bundles over S^4.
3. Oriented homotopy spheres of the remaining 12 types cannot be represented as before and are represented as connected sums of two of these spheres.

Corollary 1 (2013–4 K)

M : a 7-dim. homotopy sphere. $\rightarrow$ We can apply Theorem 1 for $(m, .n) = (7, 4)$.

Remark 2

14 of the 28 types in Fact 1 presented before come from a different viewpoint.

Remarks etc. on Corollary 1.

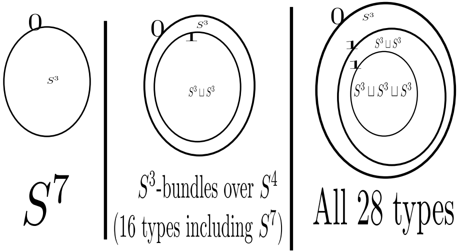

Figure 6: The images of round fold maps of Corollary 1: circles represent connected components of singular value sets and copies of S^3, S^3 etc. represent preimages of regular values and 0 and 1 represent indices of singular points. The descriptions of the 7-dim. manifolds are for manifolds admitting the presented maps.

Differentiable structures are affected by topological properties of maps of an explicit class (**in the class of special generic maps these phenomena have been explicitly well-known**).

Explicit fold maps on explicit m-dim. closed and simply-connected manifolds and their singularities $(m \geq 7)$.

Let's construct explicit fold maps on 7-dim. closed and simply-connected manifolds.

Problem 1

Can we obtain 7-dim. and general higher dimensional closed and simply-connected manifolds of wider classes and understand them in more geometric and constructive ways via fold maps?

Existing algebraic and abstract studies.

- The classification via sophisticated algebraic topological methods such as homotopy theory, surgery theory, and so on (around the 1950s–60s).
- Recent studies on 7-dim. closed and simply-connected manifolds (**whose second integral homology groups are free**) via concrete algebraic topology: the 2000s–2010s Crowley, 2018 Kreck, Wang etc..

It is difficult to understand these manifolds in constructive ways due to the constraint that the dimensions are high and general.

A result on fold maps and cohomology rings of manifolds.

Theorem 2 (2019–20 K)

A, B, C: free commutative groups of rank a,b and c.
$\{a_{i,j}\}_{j=1}^{a}$: a sequence of integers ($1 \leq i \leq b$: integer). $p \in B \oplus C$.
$(h_{i,j})$: a symmetric $b \times b$ matrix such that the (i,j)-th component is an integer satisfying $h_{i,i} = 0$ for $1 \leq i \leq b$.
$\rightarrow \exists M$: a 7-dim. closed and simply-connected manifold whose total SW class is $1 \in H^0(M; \mathbb{Z}/2\mathbb{Z})$ $\exists f : M \rightarrow \mathbb{R}^4$: a fold map s.t.

1. *$H_*(M;\mathbb{Z})$ is free. $H^2(M;\mathbb{Z}) \cong A \oplus B$ and $H^4(M;\mathbb{Z}) \cong B \oplus C$ (we fix suitable identifications) and the following properties hold.*
   - 1.1 *Products of elements in $A \oplus \{0\} \subset H^2(M;\mathbb{Z})$ vanish.*
   - 1.2 *Consider a suitable basis $\{({a_j}^*, 0)\}_{j=1}^{a}$ of $A \oplus \{0\} \subset H^2(M;\mathbb{Z})$ and a suitable basis $\{(0, {b_j}^*)\}_{j=1}^{b}$ of $\{0\} \oplus B \subset H^2(M;\mathbb{Z})$. The product of $({a_{j_1}}^*, 0)$ and $(0, {b_{j_2}}^*)$ is regarded as $(a_{j_2,j_1}{b_{j_2}}^*, 0) \in B \oplus \{0\} \subset H^4(M;\mathbb{Z})$. The product of $(0, {b_{j_1}}^*)$ and $(0, {b_{j_2}}^*)$ is regarded as $(h_{j_1,j_2}{b_{j_1}}^* + h_{j_2,j_1}{b_{j_2}}^*, 0) \in H^4(M;\mathbb{Z})$.*
2. *The 1st Pontryagin class of M is $4p \in H^4(M;\mathbb{Z})$.*
3. *The index of each singular point of f is 0 or 1 and preimages of regular values are disjoint unions of at most 3 copies of S^3.*

Remarks etc. on Theorem 2.

We can change the **3rd** property if (h_{j_1,j_2}) is the zero matrix as follows: $f|_{S(f)}$ **is an embedding, and preimages of regular values are disjoint unions of at most** 2 **copies of** S^3.

Corollary 2 (2020 K)

The class of the 7-dim. manifolds obtained in Theorem 2 *is wider than the class of the 7-dim. manifolds obtained above.*

- (2020 K) Similar theorems where the pairs of the dimensions are general ($\neq (7,4)$) assuming additional suitable conditions.

Theorem 3 (2020 K)

$\exists \{M_j\}_{j \in \mathbb{N}}$: *a family of infinitely many 7-dim. spin, closed and simply-connected manifolds whose integral cohomology rings are isomorphic to* $H^*(\mathbb{C}P^2 \times S^3; \mathbb{Z})$, *which are* **mutually non-homeomorphic**, *which* **we cannot obtain in Theorems 1–2** *and which* **admit round fold maps into** $\mathbb{R}^4$.

The main result of **Wang's preprint in 2018** improves as follows: $\{M_j\}$ contains all 7-dim. spin, closed and simply-connected manifolds whose integral cohomology rings are isomorphic to $H^*(\mathbb{C}P^2 \times S^3; \mathbb{Z})$.

Special generic maps revisited (known results).

Theorem 4 (1993 Saeki, 2015 Nishioka. etc..)

Let $m = 4, 5$. m-dim. manifolds represented as connected sums of total spaces of S^{m-2}-bundles over S^2 are characterized as closed and simply-connected ones admitting special generic maps into $\mathbb{R}^3$ (if $m = 5$ we can replace $3 \to 4$).
Furthermore, closed and simply-connected manifolds of dim. $m > 4$ whose 2nd integral homology groups are **not** *free admit no special generic map into $\mathbb{R}^4$.*

How special generic maps restrict the topologies.

Theorem 5 (2019–20 K.)

In Theorem 2, if at least one number of $\{a_{i,j}\}$ is non-zero, the matrix is not zero, or $p \neq 0$ holds, then M admits no special generic map into $\mathbb{R}^4$.

Theorem 6 (2020 K)

$\exists\{(M_{j.1}, M_{j,2})\}$: *infinitely many pairs of 9-dimensional closed and simply-connected manifolds satisfying the following properties.*

1. *The integral homology groups of $M_{j,1}$ and $M_{j,2}$ are isomorphic. For distinct j's, the corresponding homology groups are not isomorphic.*
2. *The k-th Stiefel Whitney classes and the k-th Pontryagin classes of $M_{j,i}$ vanish for $k \geq 1$.*
3. *The rational cohomology rings of $M_{j,1}$ and $M_{j,2}$ are isomorphic to that of a manifold represented as a connected sum of manifolds represented as products of two spheres.*
4. *$M_{j,1}$ admits a special generic map into $\mathbb{R}^5$, admitting no special generic map into $\mathbb{R}^{n'}$ for $1 \leq n' \leq 4$. $M_{j,2}$ admits a special generic map into $\mathbb{R}^6$, admitting no special generic map into $\mathbb{R}^{n'}$ for $1 \leq n' \leq 5$.*

Conclusions and future problems.

Conclusions and future problems.

- Difference on singularities of fold maps and difference of information of the manifolds are closely related.
- Topologies and differentiable structures of manifolds admitting specific fold maps such as some special generic maps are restricted.

Problem 2

Find meaningful examples more. Moreover, study such phenomena from a general viewpoint.

Related to this problem, it is still a fundamental and difficult problem and also a new and interesting problem to understand higher dimensional closed and connected manifolds in constructive ways.

Problem 3

Can we do more effective construction of explicit fold maps and more general generic smooth maps (into lower dimensional spaces) and manifolds admitting them?

Thank you.

Appendices.

A sketch of a proof of Theorem 2 for the case where (h_{j_1,j_2}) is the zero matrix (for $b = 1$).

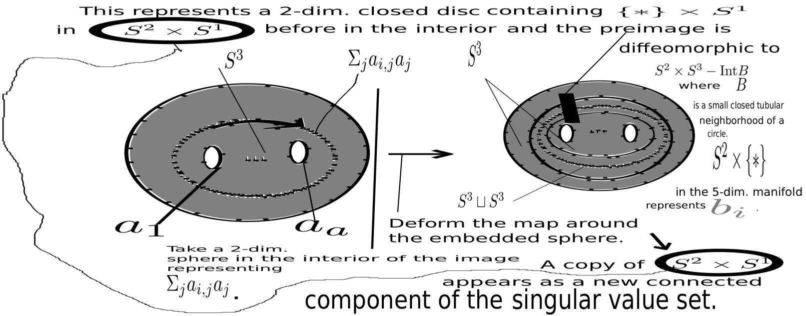

Figure 7: A proof of Theorem 2 where (h_{j_1,j_2}) is the zero matrix: the left figure shows the image of a special generic map f_0 on a manifold M_0 represented as a connected sum of a copies of $S^2 \times S^5$ into $\mathbb{R}^4$: $f_0(M_0)$ is represented as a boundary connected sum of a copies of $S^2 \times D^2$ where $\{a_j\}$ generates $H_2(M_0; \mathbb{Z})$ and also $H_2(f_0(M_0); \mathbb{Z})$ in a canonical way (${a_i}^*$ is regarded as the dual of a_i and ${b_i}^*$ is regarded as the dual of b_i). S^3, $S^3 \sqcup S^3$ etc. represent preimages of regular values.

A round fold map in Theorem 3.

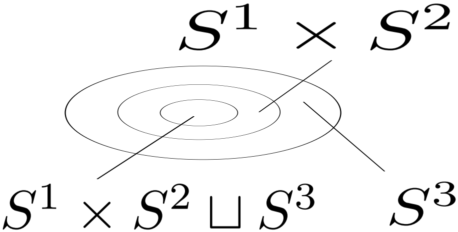

Figure 8: The image (singular value set) and preimages of a round fold map into $\mathbb{R}^4$ of Theorem 3.

To obtain such maps on all spin, closed and simply-connected manifolds whose integral cohomology rings are isomorphic to that of $\mathbb{C}P^2 \times S^3$, we need a result of Wang's preprint in 2018 (we do not need this to obtain some).