Thick isotopy property and the mapping class groups of Heegaard splittings

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The mapping class groups of Heegaard splittings

Let S be a Heegaard surface of a closed orientable 3-manifold M.
Let $\mathrm{Diff}(M, S)$ denote the group of diffeomorphisms of M that preserve S setwise.

Definition.

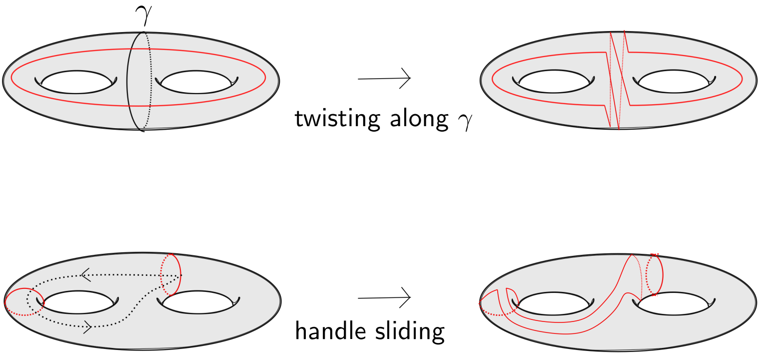
The mapping class group of the Heegaard splitting is defined by

$$\mathrm{MCG}(M, S) := \pi_0(\mathrm{Diff}(M, S)).$$

There is a natural homeomorphism

$$\begin{array}{ccc} \mathrm{MCG}(M, S) & \to & \mathrm{MCG}(S) \\ \Downarrow & & \Downarrow \\ [\varphi] & \mapsto & [\varphi|_S] \end{array}$$

Example Some elements of the mapping class group of the genus-2 Heegaard splitting of S^3.



The mapping class groups of Heegaard splittings

Question. (Minsky 2007)

When is the mapping class group of a Heegaard splitting finitely generated?

- The mapping class group of the genus-g Heegaard splitting of S^3 is finitely generated if $g = 2$ [Goeritz 1933] or $g = 3$ [Freedman-Scharlemann 2018].
- The mapping class group of the genus-3 Heegaard splitting of the 3-torus is finitely generated [Johnson 2011].
- For any genus-2 weakly reducible Heegaard splitting, a finite presentation of the mapping class group is known [Cho-Koda 2019].

Main result

Theorem. (I. 2020)

Let N be a closed, orientable, hyperbolic 3-manifold. Let S be a strongly irreducible Heegaard surface of N. Then $\mathrm{MCG}(N, S)$ is finitely generated.

Remark

- There are infinitely many examples of strongly irreducible Heegaard splittings with the mapping class group of infinite-order.
- The proof is based on the method due to Colding-Gabai-Ketover “On the classification of Heegaard splittings” Duke Math. J. (2018).

The space of Heegaard surfaces

Let S be a Heegaard surface of a closed orientable 3-manifold M.

Definition. (Johnson-McCullough 2013)

The space of Heegaard surfaces $\mathcal{H}(M,S)$ is defined by

$$\begin{aligned}\mathcal{H}(M,S) &:= \mathrm{Diff}(M)/\mathrm{Diff}(M,S)\\ &= \{\text{The images of } S \text{ under diffeomorphisms of } M\}.\end{aligned}$$

We will take S (which corresponds to the left coset $\mathrm{id}_M \cdot \mathrm{Diff}(M,S)$) as the basepoint of $\mathcal{H}(M,S)$ unless otherwise stated.

$$\begin{array}{ccc}\mathcal{H}(M,S) & \longleftrightarrow & \{\psi(S) \mid \psi \in \mathrm{Diff}(M)\}\\ \Cup & & \Cup\\ \varphi \cdot \mathrm{Diff}(M,S) & \longmapsto & \varphi(S)\end{array}$$

Remark

- Each path in $\mathcal{H}(M, S)$ determines an isotopy of Heegaard surfaces, and vice versa.
  In particuler, we can think of the fundamental group $\pi_1(\mathcal{H}(M, S))$ as the group of (smooth) motions of the Heegaard surface S.
- There is the following exact sequence:

$$\pi_1(\mathcal{H}(M, S)) \to \mathrm{MCG}(M, S) \to \mathrm{MCG}(M).$$

The thick isotopy property

In what follows, suppose that M is a closed, orientable, irreducible 3-manifold with a Reimannian metric.
Let S be a Heegaard surface of M.

Definition.

Let $\delta > 0$. An embedded surface T in M is said to be δ-compressible if there exists a compression disk D of T such that $\mathrm{diam}(\partial D) < \delta$. Otherwise T is said to be δ-locally incompressible.

Definition. (Thick isotopy property)

We say that S satisfies the thick isotopy property if there exist constants $C = C(S)$ and $\delta = \delta(S)$ with the following property. For any loop $\{T_t\}_{t \in I}$ in $\mathcal{H}(M, S)$, $\{T_t\}_{t \in I}$ can be homotoped (rel. its basepoint) so that

- $\mathrm{Area}(T_t) \leq C$ for ${}^{\forall} t \in I$, and
- T_t is δ-locally incompressible for ${}^{\forall} t \in I$.

The thick isotopy property

Theorem. (I. 2020)

Let M be a closed, orientable, irreducible, Reimannian 3-manifold. Let S be a Heegaard surface of M. Then the following two statements are equivalent.

(1) *The surface S satisfies the thick isotopy property.*

(2) *The group $\pi_1(\mathcal{H}(M,S))$ is finitely generated.*

$(2) \Rightarrow (1)$ is straightforward: if $\left[\{T^i_t\}_{t\in I}\right]$ $(1 \leq i \leq n)$ is a finite generating set of $\pi_1(\mathcal{H}(M,S))$, then we may choose

$$C := \max_{1\leq i\leq n,\, t\in I} \mathrm{Area}(T^i_t),$$

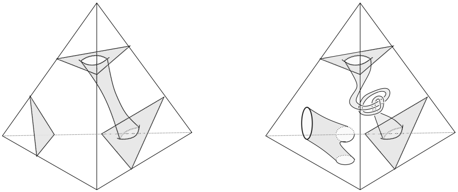
$$\delta := \min_{1\leq i\leq n,\, t\in I} \min\{\mathrm{diam}(\partial D) \mid D \text{ is a compression disk of } T^i_t\}.$$

Sketch of proof: $(1) \Rightarrow (2)$

Fix a triangulation Δ of M s.t. $\operatorname{diam}\sigma \ll \delta$ for any 3-simplex σ of Δ.
A surface T that is transverse to Δ is said to be crudely almost normal (w.r.t. Δ) if

- for each 2-simplex τ, no component of $T \cap \tau$ is a circle,
- for each 3-simplex σ, any component of $T \cap \sigma$ is either a disk or an unknotted annulus, and
- there is at most one annulus component of $T \cap \sigma$.

The weight of T is defined to be $|T \cap \Delta^1|$.

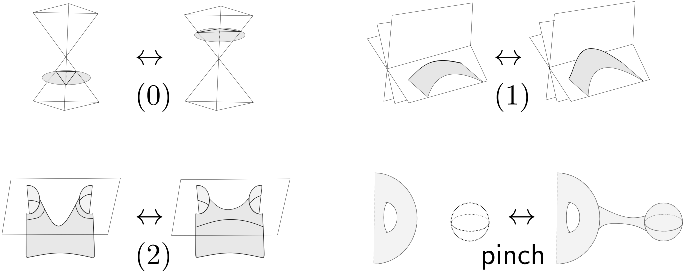


crudely almost normal　　　　not crudely almost normal

Sketch of proof: $(1) \Rightarrow (2)$

We construct a graph $\mathcal{G}$ as follows.

- Each vertex of $\mathcal{G}$ is a crudely almost normal surface (up to "transverse isotopy") of weight at most $K(C+1)$, where K is some uniform constant.
- Two vertices of $\mathcal{G}$ spans an edge if and only if they are related either by one of three elementary moves (0), (1), (2) or a pinch.



Sketch of proof: $(1) \Rightarrow (2)$

Lemma.

1. *The graph $\mathcal{G}$ is a finite graph.*
2. *The natural homomorphism $\eta : \pi_1(\mathcal{G}, S) \to \pi_1(\mathcal{H}(M, S))$ is surjective.*

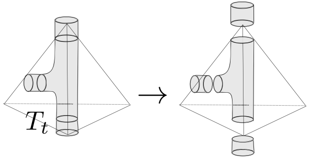
<u>**Proof of the item 2**</u> Suppose that $[\{T_t\}_{t\in I}] \in \pi_1(\mathcal{H}(M, S))$.

Step 1: We may assume that $\{T_t\}_{t\in I}$ is a " thick isotopy " .

Step 2: We may perturb $\{T_t\}_{t\in I}$ so that T_t is transverse to Δ and $|T_t \cap \Delta^1| \leq K(C+1)$ for all but finitely many $t \in I$.

Step 3: For generic $t \in I$, T_t can be changed into a crudely almost normal surface by applying pinches.

Thus $[\{T_t\}_{t\in I}]$ is contained by the image of η. □



Proof of the main theorem

Lemma.

If S is a strongly irreducible Heegaard surface of a closed, orientable, hyperbolic 3-manifold N, then S satisfies the thick isotopy property.

Key tools are the following:

- the min-max theory due to Almgren-Pitts, and
- an area bound for embedded minimal surfaces in hyperbolic 3-manifolds in terms of the genus.

Proof of the main result There is the following exact sequence:

$$\pi_1(\mathcal{H}(N, S)) \to \mathrm{MCG}(N, S) \to \mathrm{MCG}(N).$$

We have already seen that $\pi_1(\mathcal{H}(N, S))$ is finitely generated.
On the other hand, $\mathrm{MCG}(N)$ is a finite group since N is hyperbolic.
Therefore $\mathrm{MCG}(N, S)$ is finitely generated. $\square$

Problem

Remark.

In fact, a similar argument as above shows the following: if S is a strongly irreducible Heegaard surface of a closed, orientable, spherical 3-manifold N, then $\mathrm{MCG}(N, S)$ is finitely generated.

Question.

Which Heegaard surfaces satisfy the thick isotopy property?
For example,

- a weakly reducible Heegaard surface of a hyperbolic/spherical 3-manifold, or
- a strongly irreducible Heegaard surface of a 3-manifold with a geometric structure other than a hyperbolic/spherical structure.

ご清聴ありがとうございました