4-manifolds admitting simplified $(2,0)$-trisections with prescribed vertical 3-manifolds

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結び目の数理 III
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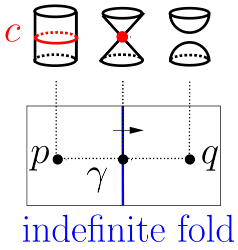
fold/cusp singularities

X: a closed oriented connected smooth 4-manifold,
$f : X \to \mathbf{R}^2$: smooth map and $\mathbf{p} \in \mathrm{Sing}(f)$,

- $\mathbf{p}$ is an indefinite fold singularity
  $\Leftrightarrow \exists$ local coord. (t,x,y,z) around $\mathbf{p}$ s.t. $f(t,x,y,z) = (t, -x^2 - y^2 + z^2)$.
- $\mathbf{p}$ is a definite fold singularity
  $\Leftrightarrow \exists$ local coord. (t,x,y,z) around $\mathbf{p}$ s.t. $f(t,x,y,z) = (t, -x^2 - y^2 - z^2)$.
- $\mathbf{p}$ is a cusp singularity
  $\Leftrightarrow \exists$ local coord. (t,x,y,z) around $\mathbf{p}$ s.t. $f(t,x,y,z) = (t, x^3 - 3xt + y^2 - z^2)$.



- c is called a vanishing cycle associated with γ.
- γ is called a reference path.

fold/cusp singularities

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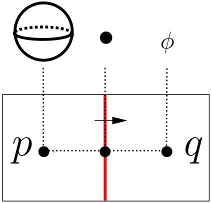

definite fold

fold/cusp singularities

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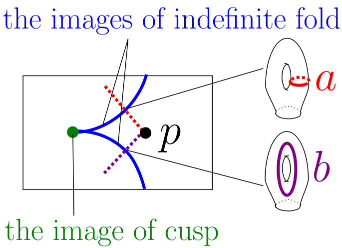
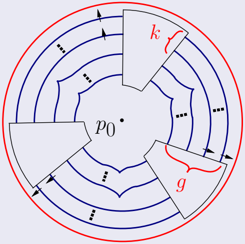
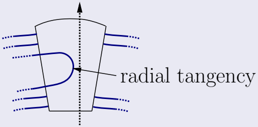


- the vanishing cycles a and b intersect at one point transversely.

(g,k)-trisection mappings [Gay-Kirby, 2016]

a smooth mapping $f : X \to \mathbf{R}^2$ is a (g,k)-trisection map. $(g \geq k \geq 0)$
$\Leftrightarrow f(\mathrm{Sing}(f))$ is as in the figure.





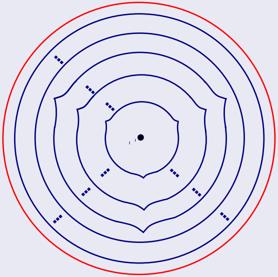
- the most outer red circle$\cdots$definite fold singular value.
- the solid curves$\cdots$indefinite fold singular value.
- the cusped points$\cdots$cusp singular value.
- three white boxes$\cdots$consists of indefinite fold images with transverse double points but without “radial tangencies".
- $f^{-1}(p_0)$ is a closed oriented surface of genus g.

Theorem (Gay-Kirby, 2016)

For any 4-manifolds X, there is a (g,k)-trisection map.

Simplified (g,k)-trisection mappings [Baykur-Saeki, 2017]

$f : X \to \mathbf{R}^2$: a (g,k)-trisection map is simplified.
$\Leftrightarrow f(\mathrm{Sing}(f))$ is as in the figure.



Theorem (Baykur-Saeki, 2017)

For any 4-manifold X, there is a simplified (g,k)-trisection map.

Classification

$\underline{g=1}$(easy)

$X \simeq \pm\mathbf{CP}^2, S^1 \times S^3$.

$\underline{g=2}$ [Meier-Zupan, 2017]

X has a $(2,0)$-trisection. $\Leftrightarrow X \simeq S^2 \times S^2, \pm\mathbf{CP}^2 \sharp \pm \mathbf{CP}^2$
X has a $(2,1)$-trisection. $\Leftrightarrow X \simeq S^1 \times S^3 \sharp \pm \mathbf{CP}^2$.
X has a $(2,2)$-trisection. $\Leftrightarrow X \simeq \sharp^2(S^1 \times S^3)$.

[Meier-Zupan, 2017] [Baykur-Saeki, 2017]

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X has a simplified $(2,2)$-trisection. $\Leftrightarrow X \simeq \sharp^2(S^1 \times S^3)$.

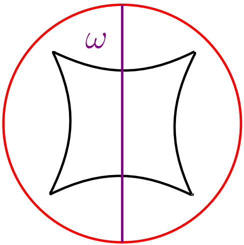
Theorem (Kobayashi, 2013)

(1) There exists a stable map $f_p : S^2 \times S^2 \to \mathbf{R}^2$ with the following singular value set such that
- $f_p^{-1}(\omega) = L(2p,1)$

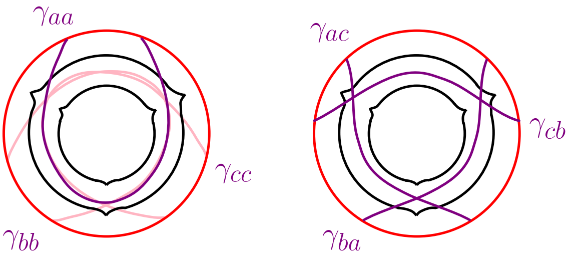
In particular, $p \neq q \Longrightarrow f_p$ and f_q are NOT right-left equivalent.

(2) There exists a stable map $g_p : \mathbf{CP}^2 \# \overline{\mathbf{CP}^2} \to \mathbf{R}^2$ with the following singular value set such that
- $g_p^{-1}(\omega) = L(2p-1,1)$

In particular, $p \neq q \Longrightarrow g_p$ and g_q are NOT right-left equivalent.



Simplified $(2,0)$-trisection



Question.

- What is the 3-manifold $V_{ij} = f^{-1}(\gamma_{ij})$?
- What is the 4-manifold ?

Note: The 6-tuple $\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix} = \begin{pmatrix} S^1 \times S^2 & S^1 \times S^2 & S^1 \times S^2 \\ S^3 & S^3 & S^3 \end{pmatrix}$ is called a trivial 6-tuple. We exclude the trivial 6-tuple from our discussion.

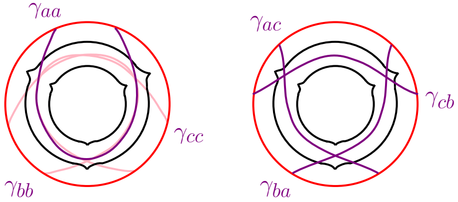
Theorem (A. 2019)

The non-trivial 6-tuple $\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix}$ is one of the following up to reflection:

$$\begin{pmatrix} S^3 & S^3 & L((q-1)^2, q-1+\varepsilon) \\ S^1\times S^2 & L(q-2,\varepsilon) & L(q,-\varepsilon) \end{pmatrix}, \begin{pmatrix} S^3 & L(9,2\varepsilon) & L(4,\varepsilon) \\ L(2,1) & L(5,\varepsilon) & S^3 \end{pmatrix},$$

$$\begin{pmatrix} S^1\times S^2 & S^3 & S^3 \\ S^3 & L(1+\varepsilon,1) & S^3 \end{pmatrix}, \begin{pmatrix} S^1\times S^2 & L(4,1) & L(4,1) \\ S^3 & L(4+\varepsilon,1) & S^3 \end{pmatrix},$$

where $q \neq 1$ and $\varepsilon \in \{-1,1\}$.



Main Theorem

$f : X \to \mathbf{R}^2$: a simplified $(2,0)$-trisection map. The 4-manifold X is determined by the 6-tuples as follows:

(1) $\begin{pmatrix} S^3 & S^3 & L((q-1)^2, q-1+\varepsilon) \\ S^1 \times S^2 & L(q-2,\varepsilon) & L(q,-\varepsilon) \end{pmatrix}$

$$\Longrightarrow X = \begin{cases} S^2 \times S^2 & \text{if } q \text{ is even} \\ \mathbf{CP}^2 \# \overline{\mathbf{CP}^2} & \text{if } q \text{ is odd and } q \neq 1 \end{cases}.$$

(2) $\begin{pmatrix} S^3 & L(9,2\varepsilon) & L(4,\varepsilon) \\ L(2,1) & L(5,\varepsilon) & S^3 \end{pmatrix} \Longrightarrow X = \begin{cases} \mathbf{CP}^2 \# \mathbf{CP}^2 & \text{if } \varepsilon = -1 \\ \overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2} & \text{if } \varepsilon = 1 \end{cases}$

(3) $\begin{pmatrix} S^1 \times S^2 & L(4,1) & L(4,1) \\ S^3 & L(4+\varepsilon,1) & S^3 \end{pmatrix} \Longrightarrow X = \begin{cases} \mathbf{CP}^2 \# \mathbf{CP}^2 & \text{if } \varepsilon = 1 \\ \mathbf{CP}^2 \# \overline{\mathbf{CP}^2} & \text{if } \varepsilon = -1. \end{cases}$

(4) $\begin{pmatrix} S^1 \times S^2 & S^3 & S^3 \\ S^3 & S^1 \times S^2 & S^3 \end{pmatrix} \Longrightarrow \mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$.

(5) $\begin{pmatrix} S^1 \times S^2 & S^3 & S^3 \\ S^3 & L(2,1) & S^3 \end{pmatrix} \Longrightarrow \mathbf{CP}^2 \# \mathbf{CP}^2$ or $\overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2}$.

(1) $$\begin{pmatrix} S^3 & S^3 & L((q-1)^2, q-1+\varepsilon) \\ S^1\times S^2 & L(q-2,\varepsilon) & L(q,-\varepsilon) \end{pmatrix}$$
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(2) $$\begin{pmatrix} S^3 & L(9,2\varepsilon) & L(4,\varepsilon) \\ L(2,1) & L(5,\varepsilon) & S^3 \end{pmatrix} \Longrightarrow X = \begin{cases} \mathbf{CP}^2\#\mathbf{CP}^2 & \text{if } \varepsilon=-1 \\ \overline{\mathbf{CP}^2}\#\overline{\mathbf{CP}^2} & \text{if } \varepsilon=1 \end{cases}$$

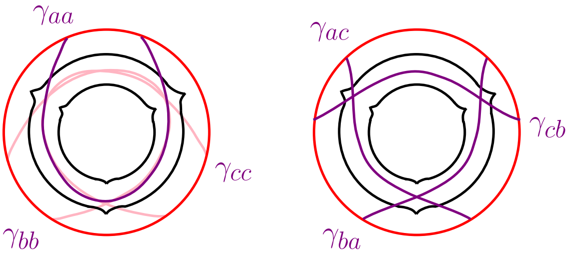
(3) $$\begin{pmatrix} S^1\times S^2 & L(4,1) & L(4,1) \\ S^3 & L(4+\varepsilon,1) & S^3 \end{pmatrix} \Longrightarrow X = \begin{cases} \mathbf{CP}^2\#\mathbf{CP}^2 & \text{if } \varepsilon=1 \\ \mathbf{CP}^2\#\overline{\mathbf{CP}^2} & \text{if } \varepsilon=-1. \end{cases}$$

(4) $$\begin{pmatrix} S^1\times S^2 & S^3 & S^3 \\ S^3 & S^1\times S^2 & S^3 \end{pmatrix} \Longrightarrow \mathbf{CP}^2\#\overline{\mathbf{CP}^2}.$$

(5) $$\begin{pmatrix} S^1\times S^2 & S^3 & S^3 \\ S^3 & L(2,1) & S^3 \end{pmatrix} \Longrightarrow \mathbf{CP}^2\#\mathbf{CP}^2 \text{ or } \overline{\mathbf{CP}^2}\#\overline{\mathbf{CP}^2}.$$

Corollary.

The non-trivial 6-tuple determines the 4-manifold unless it is $\begin{pmatrix} S^1\times S^2 & S^3 & S^3 \\ S^3 & L(2,1) & S^3 \end{pmatrix}$. In this exceptional case, the 4-manifold is determined up to orientation reversing diffeomorphisms.



Remark.

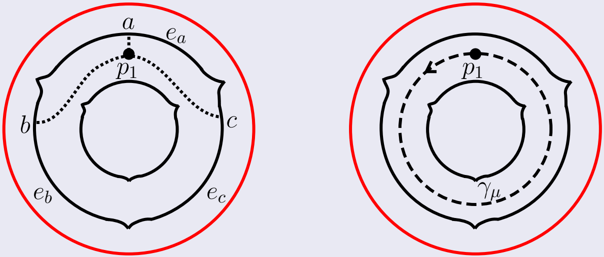
The trivial 6-tuple $\begin{pmatrix} S^1 \times S^2 & S^1 \times S^2 & S^1 \times S^2 \\ S^3 & S^3 & S^3 \end{pmatrix}$

$\Longrightarrow X = \mathbf{CP}^2 \# \mathbf{CP}^2$, $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ or $\overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2}$.

In particular, the 4-manifold of a trivial 6-tuple is NOT uniquely determined even if it is up to orientation reversing diffeomorphisms.

$p_1 \in \mathbf{R}^2, f^{-1}(p_1) \simeq$ a torus T^2
γ_μ : the circle as shown in the figure.
$\mu : T^2 \to T^2$:monodromy associated with γ_μ.

Lemma (Hayano, 2017)



μ is either (1), (2) or (3).

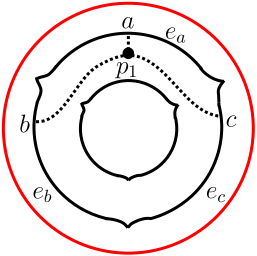
(1) $\mu = \mathrm{id}_{T^2}$
(2) $\mu =$ (a Dehn twist)
(3) $\mu =$ (a Dehn twist)^4

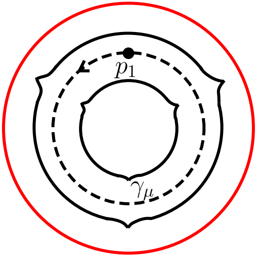
Proof of the Main Theorem

The proof is done according to the following steps.

1. Classification of the 6-tuples
   - (A) d is not parallel to any of a, b and c
   - (B) d is parallel to one of a, b and c

   where d is the curve for the monodromy along γ_μ
2. Draw a Kirby diagram for each 6-tuple

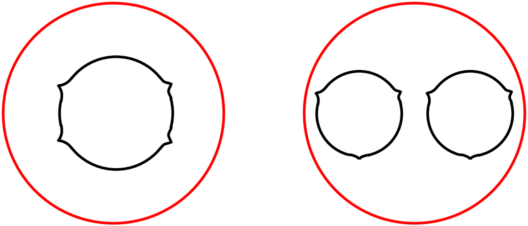




If μ is not the identity map, then we can divide the discussion into the following two cases depending on the mutual positions of vanishing cycles:

(A) d is not parallel to any of a, b and c.

(B) d is parallel to one of a, b and c.



Lemma (Hayano, 2017)

Suppose that μ is not the identity map. Then the following hold:

- If d is in case (A), then one of a, b, c intersects d once transversely.
- If $\mu = t_d^{\pm 4}$, then d is in case (B).

(1) $\begin{pmatrix} S^3 & S^3 & L((q-1)^2, q-1+\varepsilon) \\ S^1\times S^2 & L(q-2,\varepsilon) & L(q,-\varepsilon) \end{pmatrix}$

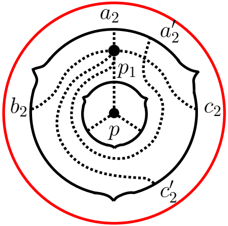
$$\Longrightarrow X = \begin{cases} S^2\times S^2 & \text{if } q \text{ is even} \\ \mathbf{CP}^2\#\overline{\mathbf{CP}^2} & \text{if } q \text{ is odd and } q\neq 1 \end{cases}.$$

Sketch of the proof :

The 6-tuple can happen when it is in Case (A) and $V_{ba} = S^1\times S^2$.

Since it is in Case (A), $\mu = t_d^{\pm 1}$ and d intersects a_2 once.

If $\mu = t_d$, we can take vanishing cycles as follows :

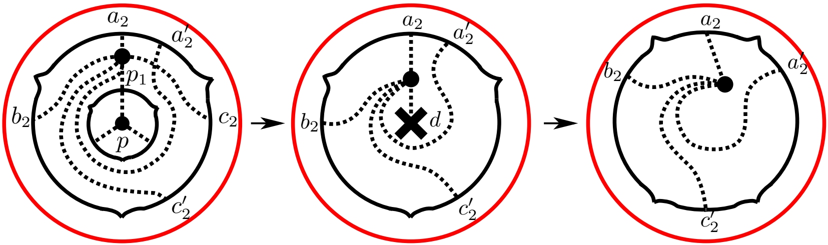


$$[a_2] = \begin{pmatrix}1\\0\end{pmatrix},\quad [b_2] = \begin{pmatrix}0\\1\end{pmatrix},\quad [c_2'] = \begin{pmatrix}-1\\q\end{pmatrix},$$

$$[a_2'] = [t_d^{\mp 1}(a_2)] = \begin{pmatrix}0\\ \pm 1\end{pmatrix},\quad [d] = \begin{pmatrix}r\\1\end{pmatrix}.$$

($\because$ a_2 and b_2 (resp. b_2 and c_2', etc...) vanish at the same cusp.)

Since this case is Case (A), we can deform f so that the singular value set has a four cusped circle as follows.

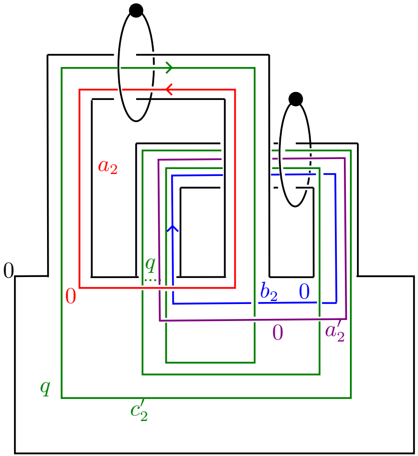


$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ q \end{pmatrix},$$

$$[a_2'] = [t_d^{\mp 1}(a_2)] = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}, \quad [d] = \begin{pmatrix} r \\ 1 \end{pmatrix}.$$

$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ q \end{pmatrix}, \quad [d_2'] = [t_d^{\mp}(a_2)] = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}.$$

We obtain the following Kirby diagram of X (Case : $q > 0$):



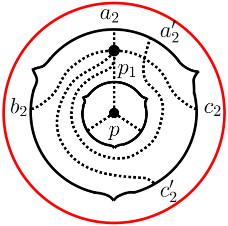
$$\text{(2)}\quad \begin{pmatrix} S^3 & L(9,2\varepsilon) & L(4,\varepsilon) \\ L(2,1) & L(5,\varepsilon) & S^3 \end{pmatrix} \Longrightarrow X = \begin{cases} \mathbf{CP}^2 \# \mathbf{CP}^2 & \text{if } \varepsilon = -1 \\ \overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2} & \text{if } \varepsilon = 1 \end{cases}$$

Sketch of the proof :

The 6-tuple can happen when it is in Case (A) and V_{ba} and V_{ac} are not $S^1 \times S^2$.

Since it is in Case (A), $\mu = t_d^{\pm 1}$ and d intersects a_2 once.

If $\mu = t_d$, we can take vanishing cycles as follows :



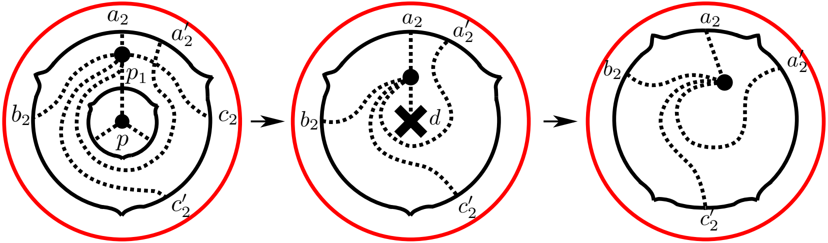
$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ q \end{pmatrix},$$

$$[a_2'] = [t_d^{\mp 1}(a_2)] = \begin{pmatrix} p \\ \pm 1 \end{pmatrix}, \quad [d] = \begin{pmatrix} r \\ 1 \end{pmatrix}.$$

From the fact that V_{ba} and V_{ac} are not $S^1 \times S^2$, it is either

$$\text{(i) } p = -1 \text{ and } [c_2'] = \begin{pmatrix} -1 \\ \pm 2 \end{pmatrix} \quad \text{or} \quad \text{(ii) } p = -2 \text{ and } [c_2'] = \begin{pmatrix} -1 \\ \pm 1 \end{pmatrix}.$$

Apply the deformation as in (1).



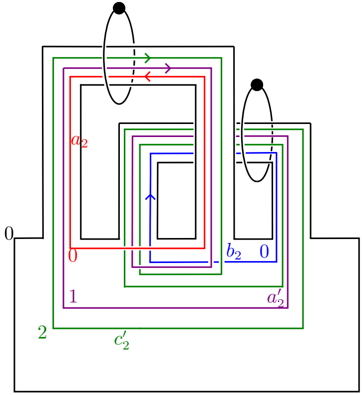
Case (i) :

$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ \pm 2 \end{pmatrix},$$

$$[a_2'] = [t_d^{\mp 1}(a_2)] = \begin{pmatrix} -1 \\ \pm 1 \end{pmatrix}, \quad [d] = \begin{pmatrix} \mp 2 \\ 1 \end{pmatrix}.$$

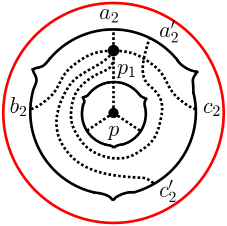
$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ \pm 2 \end{pmatrix}, \quad [a_2'] = [t_d^{\mp 1}(a_2)] = \begin{pmatrix} -1 \\ \pm 1 \end{pmatrix}.$$

We obtain the following Kirby diagram of X:



(3) $\begin{pmatrix} S^1 \times S^2 & L(4,1) & L(4,1) \\ S^3 & L(4+\varepsilon,1) & S^3 \end{pmatrix} \Longrightarrow X = \begin{cases} \mathbf{CP}^2 \# \mathbf{CP}^2 & \text{if } \varepsilon = 1 \\ \mathbf{CP}^2 \# \overline{\mathbf{CP}^2} & \text{if } \varepsilon = -1. \end{cases}$

Sketch of the proof :
The 6-tuple in Case (3) only appears in Case (B) with $\mu = t_d^{\pm 4}$. We have.
Since it is in Case (B), d and a_2 are disjoint on $f^{-1}(p_1)$.
We can take vanishing cycles as follows :



$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ \varepsilon_2 \end{pmatrix}$$

where $\varepsilon_2 \in \{-1, 1\}$, and $\varepsilon_2 = \mp\varepsilon$.

$$\mu = t_d^4 \leftrightarrow \mathbf{CP}^2 \qquad \varepsilon_2 = -1 \leftrightarrow \mathbf{CP}^2$$

$$\mu = t_d^{-4} \leftrightarrow \overline{\mathbf{CP}^2} \qquad \varepsilon_2 = 1 \leftrightarrow \overline{\mathbf{CP}^2}$$

(4) $\begin{pmatrix} S^1\times S^2 & S^3 & S^3 \\ S^3 & S^1\times S^2 & S^3 \end{pmatrix} \Longrightarrow \mathbf{CP}^2\#\overline{\mathbf{CP}^2}$.

(5) $\begin{pmatrix} S^1\times S^2 & S^3 & S^3 \\ S^3 & L(2,1) & S^3 \end{pmatrix} \Longrightarrow \mathbf{CP}^2\#\mathbf{CP}^2$ or $\overline{\mathbf{CP}^2}\#\overline{\mathbf{CP}^2}$.

Sketch of the proofs of (4) and (5):
In Case (4), the 6-tuple appears in Case (B) with $\mu = t_d^{\pm 1}$ and $\pm\varepsilon_2 = 1$.
Hence X is $\mathbf{CP}^2\#\overline{\mathbf{CP}^2}$ by the same observation as in Case (3).

In Case (5), the 6-tuple appears in Case (B) with $\mu = t_d^{\pm 1}$ and $\pm\varepsilon_2 = -1$.
Hence X is $\mathbf{CP}^2\#\mathbf{CP}^2$ if $\varepsilon_2 = -1$, where $\mu = t_d$, and $\overline{\mathbf{CP}^2}\#\overline{\mathbf{CP}^2}$ if $\varepsilon_2 = 1$, where $\mu = t_d^{-1}$.

□