Introduction to framed vertex operator algebras

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1 Preliminaries

1.1 Notation

In this article, \( \mathbb{N} \) denotes the set of non-negative integers, i.e. \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( \mathbb{Z}_2 \) denotes \( \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \).

A linear space \( V \) is called a superspace if it has a \( \mathbb{Z}_2 \)-grading \( V = V^0 \oplus V^1 \). For a superspace \( V = V^0 \oplus V^1 \), we define the parity function on \( V \) by means of the map \( \epsilon : (V^0 \sqcup V^1) \times (V^0 \sqcup V^1) \to \mathbb{Z}_2 \) defined by \( \epsilon(a, b) = ij \) for \( a \in V^i, b \in V^j, i, j \in \mathbb{Z}_2 \).

For any algebra \( A \) and any group \( G \) acting on \( A \), we denote by \( A^G \) the fixed-point subalgebra of \( A \) under \( G \).

A code means a linear subspace of \( \mathbb{Z}_2^n \). For an element \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_2^n \), its support is defined to be \( \{i \mid \alpha_i \neq 0\} \subset \{1, 2, \ldots, n\} \) and denoted by \( \text{Supp}(\alpha) \), and its weight is defined to be \( |\text{Supp}(\alpha)| \in \mathbb{N} \) and denoted by \( |\alpha| \). An even code \( D \) is called doubly even if its minimum weight \( \min\{|\alpha| \mid \alpha \in D\} \) is greater than or equal to 4. We refer to \( D \) as self-orthogonal if \( D \subset D^\perp \) and self-dual if \( D^\perp = D \), where the inner product on \( \mathbb{Z}_2^n \) is given by \( \langle \alpha, \beta \rangle = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n \in \mathbb{Z}_2 \) for \( \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n \).
Let $V$ be a $\mathbb{C}$-linear space and $z$ a formal variable. We introduce the following linear spaces:

\[
V[z] := \{ \sum_{n \in \mathbb{N}} v_n z^n \mid v_n \in V, \ v_n = 0 \text{ for all but finitely many } n \},
\]

\[
V[z, z^{-1}] := \{ \sum_{n \in \mathbb{Z}} v_n z^n \mid v_n \in V \},
\]

\[
V((z)) := \{ \sum_{n \in \mathbb{Z}} v_n z^n \mid v_n \in V, \ v_n = 0 \text{ if } n \ll 0 \},
\]

\[
V\{z\} := \{ \sum_{r \in \mathbb{Z}} v_r z^r \mid v_r \in V \}.
\]

1.2 Formal residue

For $f(z) = \sum_{n \in \mathbb{Z}} v_n z^n \in V[z, z^{-1}]$, we define the formal residue by

\[
\text{Res}_z f(z) := v_{-1},
\]

the coefficient of $z^{-1}$ in $f(z)$.

1.3 Formal binomial expansion

For $r \in \mathbb{C}$, we define the formal binomial expansion by

\[
(z + w)^r := \sum_{i \geq 0} \binom{r}{i} z^{r-i} w^i, \quad \text{where } \binom{r}{i} := \frac{r(r-1) \cdots (r-i+1)}{i!}.
\]

Note that $(z+w)^r \neq (w+z)^r$ unless $r \in \mathbb{N}$.

2 Vertex operator superalgebras

In this article we give two different, but equivalent definitions of a vertex superalgebra.

2.1 Vertex superalgebras

The first definition, which is more axiomatic, is as follows:

**Definition 2.1.** A vertex superalgebra is a quadruple $(V, Y(\cdot, z), 1, \partial)$ where $V = V^0 \oplus V^1$ is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space, $Y(\cdot, z)$ is a linear map called vertex operator map from $V \otimes \mathbb{C} V$ to $V((z))$, i.e., $V \otimes V \ni a \otimes b \mapsto Y(a, z)b = \sum_{n \in \mathbb{Z}} a(m) b z^{-n-1} \in V((z))$, where $z$ is a formal variable, $1$ is the specified element of $V$ called the vacuum vector and $\partial$ is a parity preserving endomorphism of $V$ such that the following conditions hold:

(i) $Y(1, z)a = a$ for any $a \in V$.
(ii) $Y(a, z)1 \in a + V[z]z$ for any $a \in V$;

(iii) $[\partial, Y(a, z)]b = Y(\partial a, z)b = \frac{d}{dz}Y(a, z)b$ for any $a, b \in V$;

(iv) For $a \in V^i$ and $b \in V^j$, $Y(a, z)b \in V^{i+j}(z)$, where $i, j \in \mathbb{Z}_2$;

(v) For any $\mathbb{Z}_2$-homogeneous $a, b \in V$, there exists an integer $N$ such that the following commutativity holds for any $v \in V$:

$$(z_1 - z_2)^N Y(a, z_1)Y(b, z_2)v = (-1)^{\epsilon(a,b)}(z_1 - z_2)^N Y(b, z_2)Y(a, z_1)v.$$ 

We usually denote $(V, Y(\cdot, z), 1, \partial)$ simply by $V$. In the case of $V^1 = 0$, $V$ is called a vertex algebra. If a vertex superalgebra $V$ has a $\frac{1}{2}\mathbb{Z}$-graded decomposition $V = \oplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$ such that $V^0 = \oplus_{n \in \mathbb{Z}} V^0 \cap V_n$, $V^1 = \oplus_{n \in \mathbb{Z}} V^1 \cap V_{n+\frac{1}{2}}$ and $a(n)V_n \subset V_{m+s-n-1}$ for any $a \in V_m$, then $V$ is said to be a graded vertex superalgebra. For a graded vertex superalgebra $V$, we define the weight of a homogeneous element $a \in V_n$ by $\text{wt}(a) := m$. This completes the definition.

Remark 2.2. We may consider that the vertex operator map $Y(\cdot, z)$ defines a linear map from $V$ to $\text{End}(V)[[z, z^{-1}]]$, the generating series of linear endomorphisms on $V$, and we will write $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$, $a(n) \in \text{End}(V)$. The generating series $Y(a, z)$ is called the vertex operator of $a$.

Remark 2.3. For a vertex superalgebra $(V, Y(\cdot, z), 1, \partial)$, the underlying vector space $V$ is sometimes called the Fock space of the structure.

Remark 2.4. Let $(V, Y(\cdot, z), 1, \partial)$ be a vertex superalgebra. Then the derivation operator $\partial$ is just the operator which take each $a \in V$ to $a(-2)1 \in V$.

The following are direct consequences of the definition. (cf. [FLM][FHL][Li1]).

**Proposition 2.5.** Let $a, b \in V$ be $\mathbb{Z}_2$-homogeneous.

1. **Skew-symmetry:** $Y(a, z)b = (-1)^{\epsilon(a,b)}e^{z\partial}Y(b, z)a$. In particular, $Y(a, z)1 = e^{z\partial}a$.

2. **Associativity:** For any $n \in \mathbb{Z}$, we have

$$Y(a(n), z)b = \text{Res}_w\{(w - z)^nY(a, w)Y(b, z) - (-1)^{\epsilon(a,b)}(-z + w)^nY(b, z)Y(a, w)\}.$$ 

By (1) of the proposition above, we see that a one-sided ideal of a vertex superalgebra is also a 1-sided ideal.

**Proposition 2.6.** ([Li1, Proposition 2.2.4][MN]) Let $V$ be a vertex superalgebra, $a, b \in V$ $\mathbb{Z}_2$-homogeneous and $p, q, r \in \mathbb{Z}$. Then the following Borcherds identity holds in $\text{End}(V)$:

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)} = \sum_{j=0}^{\infty} (-1)^j \binom{r}{j} \left\{a_{(p+r-j)}v_{(q+j)} - (-1)^{\epsilon(a,b)+r}b_{(q+r-j)}a_{(p+j)}\right\}.$$ 

The following are consequences of the Borcherds identity:

**Commutator:**
\[ a_{(m)} b_{(n)} - (-1)^{e(a,b)} b_{(n)} a_{(m)} = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)} b)_{(m+n-i)}, \]

**Iterate:**
\[ (a_{(m)} b)_{(n)} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} \left\{ a_{(m-i)} b_{(n+i)} - (-1)^{m+e(a,b)} a_{(m+n-i)} b_{(i)} \right\}. \]

The following property is referred to as the *state-field correspondence* and will be used frequently:

**Proposition 2.7.** The vertex operator map \( Y(\cdot, z) \) is injective.

**Proof:** Assume that \( Y(a, z) = 0 \). Then applying \( Y(a, z) \) to the vacuum element, we obtain \( 0 = Y(a, z) 1 \in a + V[z]z \). Therefore, \( a = 0 \). \( \Box \)

### 2.2 Vertex operator superalgebras

Later, we will present another approach to the definition of vertex superalgebras using a theory of local systems [Li1].

**Definition 2.8.** A vertex operator superalgebra \((V, Y(\cdot, z), 1, \omega)\) is a graded vertex superalgebra \((V, Y(\cdot, z), 1, \partial)\), where \( \partial \) is a linear endomorphism on \( V \) taking each \( a \in V \) to \( a_{(-2)} \in V \), with an additional element \( \omega \in V \) called the conformal vector of \( V \) such that

- the vertex operator \( Y(\omega, z) \) of the conformal vector \( \omega \) defines a representation of the Virasoro algebra on \( V \):
  \[ [L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0} \frac{m^3 - m}{12} c, \quad m, n \in \mathbb{Z}, \]  
  (2.1)

  where we have set \( Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \), i.e., \( L(n) = \omega(n+1) \), and \( c \in \mathbb{C} \) is called the central charge of \( \omega \);

- \( L(-1) = \partial \), i.e., \( Y(L(-1)a, z) = [L(-1), Y(a, z)] = \frac{d}{dz} Y(a, z) \) for any \( a \in V \);

- The \( \frac{1}{2} \mathbb{Z} \)-graded decomposition \( V = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}} V_n \) coincides with \( L(0) \)-eigenspace decomposition, that is, \( L(0)|_{V_n} = n \cdot 1_{V_n} \), and also we have \( \dim \mathbb{C} V_n < \infty \) and \( V_n = 0 \) for sufficiently small \( n \). *In this note, we only treat the SVOAs such that \( V_0 = \mathbb{C} 1 \) and \( V_n = 0 \) for \( n < 0 \). This completes the definition.*

A vertex operator superalgebra \( V = V^0 \oplus V^1 \) with \( V^1 = 0 \) is called a vertex operator algebra and we call it a VOA for short. Similarly, a vertex operator superalgebra is shortly referred to as a super VOA or an SVOA.
3 Fields and vertex algebras

In this section we present another formulation of concept of a vertex superalgebra.

3.1 Fields

Let $M = M^0 \oplus M^1$ be a $\mathbb{Z}_2$-graded $\mathbb{C}$-linear space. Consider the following two subspaces of $\text{End}(M)$:

$$\text{End}(M)^0 := \text{Hom}_\mathbb{C}(M^0, M^0) \oplus \text{Hom}_\mathbb{C}(M^1, M^1) \subset \text{End}(M),$$

$$\text{End}(M)^1 := \text{Hom}_\mathbb{C}(M^0, M^1) \oplus \text{Hom}_\mathbb{C}(M^1, M^0) \subset \text{End}(M).$$

Then the decomposition $\text{End}(M) = \text{End}(M)^0 \oplus \text{End}(M)^1$ defines a $\mathbb{Z}_2$-grading on the associative algebra $\text{End}(M)$. A formal power series $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \text{End}(M)[z, z^{-1}]$ is called a field on $M$ if it satisfies $a(z)v \in M((z))$ for any $v \in M$. The space of fields on $M$ forms a $\mathbb{C}$-linear space which we will denote by $\mathcal{F}(M)$. It is easy to see that $\mathcal{F}(M)$ has a $\mathbb{Z}_2$-graded decomposition $\mathcal{F}(M) = \mathcal{F}(M)^0 \oplus \mathcal{F}(M)^1$ where we have set

$$\mathcal{F}(M)^i := \{a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \mathcal{F}(M) \mid a(n) \in \text{End}(M)^i\}.$$

We call a field $a(z) \in \mathcal{F}(M)$ $\mathbb{Z}_2$-homogeneous if $a(z)$ is homogeneous with respect to the $\mathbb{Z}_2$-grading on $\mathcal{F}(M)$, i.e., $a(z) \in \mathcal{F}(M)^i$. It is clear that $\mathbb{1}_M(z) := \text{id}_M$ is a field on $M$. We call $\mathbb{1}_M(z)$ the vacuum field on $M$.

3.2 Normal product

Let $a(z), b(z) \in \mathcal{F}(M)$ be $\mathbb{Z}_2$-homogeneous. Its composition $a(z)b(z)$ does not be a field on $M$ in general. However, we can define the following product on $\mathcal{F}(M)$. For $n \in \mathbb{Z}$, we define the $n$-th normal product $a(z) \circ_n b(z)$ of $a(z)$ and $b(z)$ by means of

$$a(z) \circ_n b(z) := \text{Res}_{z_1} \{ (z_1 - z)^n a(z_1) b(z) - (-1)^{\epsilon(a,b)} (-z + z_1)^n b(z) a(z_1) \} \quad (3.1)$$

and extend linearly on $\mathcal{F}(M)$, where $\epsilon$ is the standard parity function. One can check that $a(z) \circ_n b(z)$ is a field on $M$. Therefore, $\mathcal{F}(M)$ has infinitely many products $\circ_n : \mathcal{F}(M) \otimes \mathcal{F}(M) \to \mathcal{F}(M)$ for $n \in \mathbb{Z}$.

**Exercise 1.** Show that $a(z) \circ_n b(z) \in \mathcal{F}(M)$.

**Exercise 2.** Show that (i) $a(z) \circ_{-1} \mathbb{1}_M(z) = \mathbb{1}_M(z) \circ_{-1} a(z) = a(z)$, (ii) $a(z) \circ_{-2} \mathbb{1}_M(z) = \partial_z a(z)$, where $\partial_z f(z)$ denotes the formal differential of $f(z)$. 

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3.3 Locality

Let \( a(z), b(z) \in \mathcal{F}(M) \) \( \mathbb{Z}_2 \)-homogeneous. We say \( a(z) \) and \( b(z) \) are local if there is \( N \in \mathbb{Z} \) such that

\[
(z_1 - z_2)^Na(z_1)b(z_2) = (-1)^{(a,b)}(-z_2 + z_1)^Nb(z_2)a(z_1)
\]

(3.2) holds in \( \text{End}(M)[z_1^{\pm 1}, z_2^{\pm 1}] \), and we write \( a(z) \sim b(z) \). Among the integers \( N \) satisfying (3.2), the minimum one is called the order of locality between \( a(z) \) and \( b(z) \) and we denote it by \( N(a,b) \). Clearly \( N(a,b) = N(b,a) \) and we note that \( a(z) \circ_{N(a,b)+i} b(z) = 0 \) for any \( i \geq 0 \).

Remark 3.1. It is not true that \( a(z) \sim a(z) \) for any \( a(z) \in \mathcal{F}(M) \). It is also not true that \( a(z) \sim b(z) \) and \( b(z) \sim c(z) \) implies \( a(z) \sim c(z) \).

3.4 Dong’s lemma

Lemma 3.2. ([Li1][K1]) If \( a(z), b(z), c(z) \in \mathcal{F}(M) \) are pair-wise local, then so are \( c(z) \) and \( a(z) \circ_n b(z) \) for any \( n \in \mathbb{Z} \). In particular, \( a(z) \circ_n b(z) \sim a \circ_n b(z) \) if \( a(z) \sim a(z) \), \( b(z) \sim b(z) \) and \( a(z) \sim b(z) \).

The lemma above is known as Dong’s lemma. By this lemma, mutually local fields forms a subspace under the normal products. This fact characterizes the notion of a vertex algebra.

3.5 Another formulation of a Vertex superalgebra

Definition 3.3. A vertex superalgebra is a \( \mathbb{Z}_2 \)-graded subspace \( \mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1 \) of the \( \mathbb{Z}_2 \)-graded space \( \mathcal{F}(V) = \mathcal{F}(V)^0 \oplus \mathcal{F}(V)^1 \) of fields on a \( \mathbb{Z}_2 \)-graded linear space \( V = V^0 \oplus V^1 \) satisfying the following:

(i) any two fields in \( \mathfrak{A} \) are local.

(ii) \( \mathfrak{A} \) is closed under the normal products, i.e., \( \mathfrak{A} \circ_n \mathfrak{A} \subset \mathfrak{A} \) for any \( n \in \mathbb{Z} \).

(iii) \( \mathfrak{A} \) contains the vacuum element, i.e., \( 1_{\mathfrak{A}}(z) \in \mathfrak{A}. \)

The element \( 1_{\mathfrak{A}}(z) \) is called the vacuum element of \( \mathfrak{A} \) and denoted by \( 1_{\mathfrak{A}} \).

The above definition of a vertex algebra seems to differ from the previous one. But it is known that these two definitions are equivalent (cf. [Li1][K1]). For, if \( (V,Y(\cdot, z), 1, \partial) \) is a vertex superalgebra in the previous sense, then by the vertex operator map \( Y(\cdot, z) \) we have the subspace \( \{ Y(a, z) \mid a \in V \} \) of fields on \( V \) which satisfies all the conditions above. Moreover, the linear map \( V \ni a \mapsto Y(a, z) \in \mathcal{F}(V) \) is injective by the creation property \( Y(a, z)1 \in a + V[z]z \). Therefore, we can identify the structure \( (V,Y(\cdot, z), 1, \partial) \) as a vertex superalgebra defined as above. Conversely, if \( \mathfrak{A} \subset \mathcal{F}(V) \) is a vertex superalgebra in the above sense, set \( Y_{\mathfrak{A}}(a, z) := \sum_{n \in \mathbb{Z}} a \circ_n z^{-n-1} \in \text{End}(\mathfrak{A})[z, z^{-1}] \) for \( a \in \mathfrak{A} \), i.e., we define
\( a_{(n)}b := a \circ_n b \) for \( a, b \in \mathfrak{A} \). Clearly \( d/dz \) acts on \( \mathfrak{A} \) and the quadruple \((\mathfrak{A}, Y_\mathfrak{A}(\cdot, z), \mathbf{1}_\mathfrak{A}, d/dz)\) forms a vertex superalgebra in the previous sense. Therefore, these two definitions are equivalent.

### 3.6 Modules over vertex operator superalgebras

One of advantages of fields formulation of vertex superalgebras is that we can easily define a concept of modules over vertex superalgebras.

**Definition 3.4.** Let \( \mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1 \) be a vertex algebra. A representation of \( \mathfrak{A} \) is a linear map \( \phi \) from \( \mathfrak{A} \) to the space \( \mathcal{F}(M) \) of fields on a \( \mathbb{Z}_2 \)-graded vector space \( M = M^0 \oplus M^1 \) such that \( \phi(1_{\mathfrak{A}}) = 1_V(z) \) and \( \phi(a \circ_n b) = \phi(a) \circ_n \phi(b) \) for any \( a, b \in \mathfrak{A} \) and \( n \in \mathbb{Z} \). We call \( (M, \phi) \) an \( \mathfrak{A} \)-module and the field \( \phi(a) \) on \( M \) is referred to as the vertex operator of \( a \) on \( M \) for \( a \in \mathfrak{A} \).

For the definition of modules over vertex operator superalgebras, there are some choices.

**Definition 3.5.** Let \((V, Y(\cdot, z), 1, \omega)\) be a vertex operator superalgebra.

1. A **weak** \( V \)-module is just a module over a vertex superalgebra \( V \).
2. A \( \frac{1}{2}\mathbb{N} \)-graded \( V \)-module is a module \( M = M^0 \oplus M^1 \) over a vertex superalgebra \( V \) which has a \( \frac{1}{2}\mathbb{N} \)-grading \( M = \bigoplus_{n \in \frac{1}{2}\mathbb{N}} M(n) \) such that \( M(0) \neq 0 \), \( a_{(p)}M(n) \subset M(n + \text{wt}(a) - p - 1) \) for any \( a \in V, p \in \mathbb{Z} \) and \( M^i = \bigoplus_{n \in \mathbb{N}} M(n + i/2) \) for \( i = 0, 1 \).
3. A **strong** \( V \)-module is a weak \( V \)-module \( M \) with \( \mathbb{C} \)-graded decomposition \( M = \bigoplus_{r \in \mathbb{C}} M_r \) such that each homogeneous subspace \( M_r \) is finite dimensional and is a \( L(0) \)-eigenspace with eigenvalue \( r \), and \( M_r = 0 \) if \( \text{Re}(r) \ll 0 \).

**Remark 3.6.** It is known that a strong \( V \)-module is a \( \frac{1}{2}\mathbb{N} \)-graded \( V \)-module. And by the definition, every \( \frac{1}{2}\mathbb{N} \)-graded \( V \)-module is a weak \( V \)-module.

A VOA \( V \) itself is a strong \( V \)-module which is called the adjoint module. A VOA \( V \) is called simple if \( V \) as a \( V \)-module is irreducible.

**Definition 3.7.** A vertex operator superalgebra \( V \) is called **rational** if every \( \frac{1}{2}\mathbb{N} \)-graded \( V \)-module is a direct sum of irreducible submodules. Also, \( V \) is called **regular** if every weak \( V \)-module is a direct sum of irreducible strong \( V \)-submodules.

**Remark 3.8.** It is shown in [DLM1] that an irreducible \( \frac{1}{2}\mathbb{N} \)-graded \( V \)-module is an irreducible strong \( V \)-module. Therefore, by the remark above, the regularity implies the rationality.
4 Construction of vertex algebras

In order to construct examples of vertex algebras, the uniqueness theorem and the existence theorem will be quite useful.

4.1 Uniqueness Theorem

The following theorem is extremely useful in identifying a field with one of the fields of a vertex operator algebra.

**Theorem 4.1. ([Go][K1])** Let $V$ be a vertex superalgebra and let $t(z)$ be a field on $V$ which is mutually local with all the fields $Y(a, z), a \in V$. Suppose that $t(z) \mathbb{1} = e^{\varepsilon \partial_V} b$ for some $b \in V$. Then $t(z) = Y(b, z)$.

**Proof:** Since the derivation $\partial_V$ does not change the $\mathbb{Z}_2$-parity, we note that $\varepsilon(t, a) = \varepsilon(b, a)$ for any $a \in V$. By the assumption of locality we have:

$$(z_1 - z_2)N t(z_1) Y(a, z_2) \mathbb{1} = (-1)^{\varepsilon(t, a)} (-z_2 + z_1) Y(a, z_2) t(z_1) \mathbb{1}$$

for sufficient large $N > 0$. Since $Y(a, z) \mathbb{1} = e^{\varepsilon \partial_V} a$, we obtain:

$$(z_1 - z_2)N t(z_1)e^{z_2 \partial_V} a = (-1)^{\varepsilon(t, a)} Y(a, z_2) e^{z_1 \partial_V} b = (-1)^{\varepsilon(t, a)} Y(a, z_2) Y(b, z_1) \mathbb{1}.$$  

By taking sufficiently large $N$, we get

$$(z_1 - z_2)^N t(z_1) Y(a, z_2) \mathbb{1} = (-1)^{\varepsilon(t, a)} Y(a, z_2) Y(b, z_1) \mathbb{1} = (-1)^{\varepsilon(t, a)} Y(b, z_1) Y(a, z_2) \mathbb{1}.$$  

Letting $z_2 = 0$ and dividing by $z_1^N$, we get $t(z)a = Y(b, z)a$ for any $a \in V$. 

4.2 Existence Theorem

The following theorem allows one to construct vertex superalgebras.

**Theorem 4.2. (Theorem 4.5 of [K1])** Let $V$ be a vector superspace, let $\mathbb{1}$ be an even vector of $V$ and $\partial_V$ an even endomorphism of $V$. Let $\{a^\alpha(z) = \sum_{n \in \mathbb{Z}} a^\alpha_n z^{-n-1}\}_{\alpha \in I}$ be a collection of $\mathbb{Z}_2$-homogeneous fields on $V$ such that

(i) $[\partial_V, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all $\alpha \in I$,

(ii) $\partial_V \mathbb{1} = 0$, $a^\alpha(z) \mathbb{1} \in a^\alpha_{(-1)} \mathbb{1} + V[z] \mathbb{1}$, and $a^\alpha_{(-1)} \mathbb{1}$, $\alpha \in I$, are linearly independent in $V$,

(iii) $a^\alpha(z)$ and $a^\beta(z)$, $\alpha, \beta \in I$, are mutually local,

(iv) the vectors $a_{(j_1)}^{\alpha_1} \cdots a_{(j_n)}^{\alpha_n} \mathbb{1}$, $j_k \in \mathbb{Z}$, $n \geq 0$, linearly span $V$.

Then the formula

$$Y_V(a_{(j_1)}^{\alpha_1} \cdots a_{(j_n)}^{\alpha_n} \mathbb{1}, z) := a^{\alpha_1} \circ_{j_1} Y_V(a_{(j_2)}^{\alpha_2} \cdots a_{(j_n)}^{\alpha_n} \mathbb{1}, z)$$  

(4.1)

defines a unique structure of a vertex superalgebra on $V$ such that $\mathbb{1}$ is the vacuum vector, $\partial_V$ is the derivation and $Y(a_{(-1)}^\alpha \mathbb{1}, z) = a^\alpha(z)$ for all $\alpha \in I$.
Proof: Choose a basis among the vectors of the form (iv) and define the vertex operator $Y(a, z)$ by formula (4.1). By (iii) and Dong’s lemma, the locality axioms hold. Therefore, $V$ has a structure of a vertex superalgebra which depends on a choice of linear basis of $V$ at now. If we choose another basis among the monomials (iv) we get possibly different structure of a vertex superalgebra on $V$, which we denote by $Y'(\cdot, z)$. But all the fields of this new structure are mutually local with those of the old structure and satisfy $Y'(a, z)1 = e^{z\partial V}a$. Then by Theorem 4.1 it follows that these vertex superalgebra structure coincide. Thus (4.1) is well-defined and we obtain the uniqueness of the structure.

4.3 Free bosonic Vertex algebra

Let $\mathfrak{h}$ be a $\mathbb{C}$-linear space with non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. The affinization of $\mathfrak{h}$ is given by $\hat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ and we introduce the Lie bracket on $\hat{\mathfrak{h}}$ as follows:

$$[a \otimes t^m, b \otimes t^n] := \delta_{m+n,0}m(a, b)K, \quad [K, \hat{\mathfrak{h}}] = 0. \quad (4.2)$$

Then $\hat{\mathfrak{h}}$ forms an infinite dimensional Lie algebra so called the Heisenberg algebra.

For $a \in \mathfrak{h}$, set $a(z) := \sum_{n \in \mathbb{Z}} a \otimes t^n z^{-n-1} \in \hat{\mathfrak{h}}[z, z^{-1}]$. Assume that we are given a non-trivial $\hat{\mathfrak{h}}$-module $V$ with the following properties:

1. For any $a \in \mathfrak{h}$ and $v \in V$, $a(z)v \in V((z))$.
2. $a \otimes t^0, a \in \mathfrak{h}$, acts on $V$ semisimply.
3. The center $K$ of $\hat{\mathfrak{h}}$ acts on $V$ as $\text{id}_V$.

Then by the condition (1), all $a(z), a \in \mathfrak{h}$, are fields on $V$, and one can check that

$$(z_1 - z_2)^2a(z_1)b(z_2) = (z_1 - z_2)^2b(z_2)a(z_1) \quad (4.3)$$

holds on $V$ for any $a, b \in \mathfrak{h}$. Therefore, fields $1_V(z)$ and $a(z), a \in \mathfrak{h}$, generate a vertex algebra in $\mathcal{F}(V)$. This vertex algebra is called the free bosonic vertex (operator) algebra associated to a metric space $\mathfrak{h}$. One can show that the free bosonic vertex algebra is a simple vertex algebra.

Exercise 3. Prove (4.3).

Let us give an explicit construction of a free bosonic vertex algebra. Set $\hat{\mathfrak{h}}^\pm := \bigoplus_{n>0} \mathfrak{h} \otimes t^\pm n$ and $\hat{\mathfrak{h}}^0 := \mathfrak{h} \otimes 1 \oplus \mathbb{C}K$. Then we have a triangular decomposition $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^- \oplus \hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+$ and hence we can consider highest weight representations (cf. [K2][MP]). We identify $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ through the inner product. Let $\lambda \in \mathfrak{h}$ and define a one-dimensional $\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+$-module $\mathbb{C}e^\lambda$ by the relations

$$a \otimes t^n \cdot e^\lambda = 0 \text{ for } n < 0, \quad a \otimes 1 \cdot e^\lambda = \langle a, \lambda \rangle \cdot e^\lambda \quad \text{and} \quad K \cdot e^\lambda = e^\lambda$$

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and then define the induced $\hat{h}$-module $M_\hat{h}(1, \lambda)$ by
\[
M_\hat{h}(1, \lambda) := \mathfrak{u}(\hat{h}) \otimes_{\mathfrak{u}(\hat{h} \oplus \hat{m} \oplus \hat{c})} e^\lambda,
\]
where $\mathfrak{u}(X)$ denotes the universal enveloping algebra for a Lie algebra $X$. Then it is easy to see that $M_\hat{h}(1, \lambda)$ is a $\hat{h}$-module satisfying the conditions (1)-(3) above. We can introduce a vertex algebra structure on $M_\hat{h}(1, 0)$. Note that $a(z)e^0 \in a(-1)e^0 + M_\hat{h}(1, 0)[[z]]z$ and $M_\hat{h}(1, 0)$ is spanned by the vectors $a^1(-n_1) \cdots a^k(-n_k)e^0$, $a_i \in \hat{h}$, $n_1 \geq \cdots \geq n_k > 0$, $k \geq 0$.

Therefore, by the existence theorem and the uniqueness theorem, $M_\hat{h}(1, 0)$ has a unique vertex algebra structure with vacuum element $1_l = e^0$ such that $Y(a(-1)e^0, z) = a(z)$ for $a \in \hat{h}$. This is another definition of a free bosonic vertex algebra.

We can find the conformal vector in $M_\hat{h}(1, 0)$. Let $\{a_1, \ldots, a_r\}$ be a linear basis of $\hat{h}$ and $\{a_1, \ldots, a_r\}$ its dual basis. Then the element
\[
\omega := \frac{1}{2} \sum_{i=1}^r a^i(-1)a_i(-1)e^0 \in M_\hat{h}(1, 0)
\]
satisfies all the conditions to be a conformal vector of $M_\hat{h}(1, 0)$. Therefore, the quadruple $(M_\hat{h}(1, 0), Y(\cdot, z), e^0, \omega)$ forms a vertex operator algebra.

**Exercise 4.** Prove that $\omega$ satisfies all the conditions to be a conformal vector of $M_\hat{h}(1, 0)$.

### 4.4 Virasoro vertex algebra

Let $Vir = \oplus_{n \in \mathbb{Z}} \mathbb{C} L(n) \oplus \mathbb{C} \vec{c}$ be the Virasoro algebra, the Lie algebra defined by the following Lie brackets:
\[
[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} \vec{c}, \quad [\vec{c}, Vir] = 0. \tag{4.4}
\]

Assume that we are given a Vir-module $M$ such that for any $v \in M$ there exists $N \in \mathbb{Z}$ such that $L(i)v = 0$ for any $i \geq N$ and the center $\vec{c}$ acts on it as a scalar. Then the generating series $\omega(z) := \sum_{n \in \mathbb{Z}} L(n)z^{-n-2} \in \text{End}(M)[[z, z^{-1}]]$ defines a field on $M$. By the Virasoro commutator (4.4), one can show that
\[
(z_1 - z_2)^4 \omega(z_1)\omega(z_2) - (z_1 - z_2)^4 \omega(z_2)\omega(z_1) = 0. \tag{4.5}
\]

Therefore, $\omega(z)$ together with $1_M(z)$ generates a vertex algebra inside $\mathcal{F}(M)$. This vertex algebra is called the Virasoro vertex algebra.

**Exercise 5.** Prove (4.5).
Let us present an explicit construction of the Virasoro vertex algebra. Set Vir$^\pm := \oplus_{n>0} \mathbb{C}L(\pm n)$ and Vir$^0 := \mathbb{C}L(0) \oplus \mathbb{C}c$. Then we have a triangular decomposition Vir = Vir$^- \oplus$ Vir$^0 \oplus$ Vir$^+$ of the Virasoro algebra. For $c, h \in \mathbb{C}$, define a one-dimensional Vir$^- \oplus$ Vir$^0$-module $\mathbb{C}e^{c,h}$ by the relations
\[ \text{Vir}^- \cdot e^{c,h} = 0, \quad L(0)e^{c,h} = he^{c,h} \quad \text{and} \quad \bar{c} \cdot e^{c,h} = c e^{c,h}, \]
and then define the induced Vir-module $M_{\text{Vir}}(c, h)$ by
\[ M_{\text{Vir}}(c, h) := \mathfrak{M}(\text{Vir}) \otimes_{\mathfrak{M}((\text{Vir}^- \oplus \text{Vir}^0))} e^{c,h}. \]
It is easy to see that the generating series $\omega(z)$ defines a field on $M_{\text{Vir}}(c, h)$. The value $c \in \mathbb{C}$ is called the central charge.

Now consider $M_{\text{Vir}}(1, 0)$. A linear basis of $M_{\text{Vir}}(c, 0)$ is given by the following:
\[ \{ L(-n_1) \cdots L(-n_k)e^{c,0} \mid n_1 \geq \cdots \geq n_k > 0, \ k \geq 0. \} \]
It is not difficult to see that $L(-1)e^{c,0}$ is a singular vector, that is, Vir$^- L(-1)e^{c,0} = 0$. We also have $L(0)L(-1)e^{c,0} = L(-1)e^{c,0}$. Thus, there is a unique Vir-homomorphism $\phi : M_{\text{Vir}}(c, 1) \to M_{\text{Vir}}(c, 0)$ such that $\phi(e^{c,1}) = L(-1)e^{c,0}$. One can actually show that the linear map $\phi$ is injective (cf. see [K2][MP]). Therefore, the Vir-submodule generated by $L(-1)e^{c,0}$ is isomorphic to $M_{\text{Vir}}(c, 1)$ and so we can consider the quotient module $\bar{M}_{\text{Vir}}(c, 0) := M_{\text{Vir}}(c, 0)/M_{\text{Vir}}(c, 1)$. For $x \in M_{\text{Vir}}(c, 0)$, denote by $x$ its image in $\bar{M}_{\text{Vir}}(c, 0)$. Inside $\bar{M}_{\text{Vir}}(c, 0)$, we have a relation $\omega(z)\bar{e}^{c,0} \in L(-2)e^{c,0} + \bar{M}_{\text{Vir}}(c, 0)[z]z$. Therefore, by the uniqueness theorem and the existence theorem, we can introduce the unique VOA structure on $\bar{M}_{\text{Vir}}(c, 0)$ with the vacuum vector $1 = \bar{e}^{c,0}$ such that $Y(L(-2)e^{c,0}, z) = \omega(z)$.

It is clear from our universal construction that any Virasoro vertex algebra is isomorphic to a homomorphic image of $\bar{M}_{\text{Vir}}(c, 0)$. It is shown, for examples, in [K1][MP] that there is a unique maximal proper ideal $J_{\text{Vir}}(c, h)$ of $M_{\text{Vir}}(c, h)$. The quotient $L_{\text{Vir}}(c, h) := M_{\text{Vir}}(c, h)/J_{\text{Vir}}(c, h)$ is an irreducible Vir-module. Hence there is a unique simple quotient VOA of $\bar{M}_{\text{Vir}}(c, 0)$ which we continue to denote by $L_{\text{Vir}}(c, 0)$.

5 Intertwining operators and fusion algebras

In this section we give the definition of intertwining operators among modules.

5.1 Intertwining operators

Let $V$ be a regular VOA and let $(M^i, Y_{M^i}(\cdot, z))$, $i = 1, 2, 3$, be irreducible $V$-modules. By definition, we have decompositions $M^i = \oplus_{n \in \mathbb{N}} M^i_{h_i + n}$ such that $M^i_{h_i} \neq 0$. The minimal eigenvalue $h_i$ is called the top weight of $M^i$. 

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**Definition 5.1.** An $V$-intertwining operator of type $M^1 \times M^2 \to M^3$ is a linear map

$$I(\cdot, z) : M^1 \to \text{Hom}_\mathbb{C}(M^2, M^3)\{z\}$$

$$u \mapsto I(u, z) = \sum_{s \in \mathbb{C}} u_{(s)}z^{-s-1}$$

satisfying the following:

1. For any $u \in M^1$, $v \in M^2$, $u_{(s)}v = 0$ if $\text{Re}(s) \gg 0$.
2. $[L(-1), I(u, z)]v = I(L(-1)u, z)v = \frac{d}{dz}I(u, z)v$ holds for any $u \in M^1$, $v \in M^2$.
3. For any $a \in V$ and $u \in M^1$, there exists $N \in \mathbb{Z}$ such that

$$(z_1 - z_2)^NY_{M^3}(a, z_1)I(u, z_2)v = (z_1 - z_2)^NI(u, z_2)Y_{M^2}(a, z_1)v$$

holds for any $v \in M^2$.
4. For any $a \in V$, $u \in M^1$ and $n \in \mathbb{Z}$, the following holds for any $v \in M^2$:

$$I(a_{(n)}u, z)v = \text{Res}_w\{(w - z)^nY_{M^3}(a, w)I(u, z)v - (-z + w)^nI(u, z)Y_{M^2}(a, w)v\}.$$

The space of $V$-intertwining operators of type $M^1 \times M^2 \to M^3$ is denoted by $(M^1 M^2)_V$ and its dimension is called the fusion rule.

**Remark 5.2.** Let $(M, Y_M(\cdot, z))$ be a $V$-module. Then the vertex operator map $Y_M(\cdot, z)$ on $M$ defines a $V$-intertwining operator of type $V \times M \to M$. Therefore, the notion of intertwining operators is a generalization of that of vertex operators.

By the definition, we can show the following formula for intertwining operators:

**Proposition 5.3.** Let $V$ be a VOA, $M^i$, $i = 1, 2, 3$, strong $V$-modules and $I(\cdot, z) \in (M^1 M^2)_V$. If we expand $I(u, z) = \sum_{r \in \mathbb{C}} u_{(r)}z^{-r-1}$ for $u \in M^1$, then the following commutativity formula and iterate formula hold for any $v \in M^2$:

Commutator: $a_{(m)}u_{(r)} - u_{(r)}a_{(m)} = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}u)_{(m+r-i)}$.

Iterate: $(a_{(m)}u)_{(r)} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} \{a_{(m-i)}u_{(r+i)} - (-1)^m u_{(m+r-i)}a_{(i)}\}$.

The following result is fundamental.

**Proposition 5.4.** ([FHL]) Let $V$ be a VOA and $M^i$, $i = 1, 2, 3$, $V$-modules. Then we have the following linear isomorphism:

$$\left( \begin{array}{c} M^3 \\ M^1 M^2 \end{array} \right)_V \simeq \left( \begin{array}{c} M^3 \\ M^2 M^1 \end{array} \right)_V.$$
5.2 Fusion algebras

To a regular vertex operator algebra $V$, we can associate a finite dimensional commutative associative algebra called the fusion algebra as follows. It is shown in [DLM1] that a regular vertex operator algebra $V$ has finitely many inequivalent irreducible representations. Let $\{M^1, \ldots, M^n\}$ be the set of isomorphism classes of inequivalent irreducible $V$-modules. Consider the linear space $\mathcal{V}(V) := \oplus_{i=1}^{n} \mathbb{C}[M^i]$ spanned by formal elements $[M^i]$ labeled by elements in $\mathcal{S}$, and define a bilinear map on $\mathcal{V}(V)$ by

$$[M^i] \times [M^j] := \sum_{k=1}^{n} \dim \left( \frac{M^k}{M^i \cdot M^j} \right) \cdot [M^k]. \quad (5.1)$$

The algebra $\mathcal{V}(V)$ equipped with the product (5.1) is called the fusion algebra or the Verlinde algebra associated to $V$. By Proposition 5.4, the fusion algebra is commutative algebra. The following result is obtained by many authors:

**Theorem 5.5.** (cf. [ABD][FHL][H1][H2]) Let $V$ be a simple and regular VOA. The fusion algebra $\mathcal{V}(V)$ associate to $V$ is a unital commutative associative algebra. The identity element is given by the isomorphism class of adjoint module $V$.

Given a VOA $V$, it is usually a difficult problem to determine structures of the fusion algebra $\mathcal{V}(V)$. However, once we determine it, it will provide us a powerful tool to study VOAs containing $V$. Later we will present applications of fusion rules and fusion algebras.

We introduce the notion of simple current modules.

**Definition 5.6.** Let $V$ be a simple regular VOA and $\{M^1, \ldots, M^n\}$ as above. A $V$-module $M^i$ is referred to as a simple current if $\sum_{k=1}^{n} N_{ij}^k = 1$ for all $1 \leq j \leq n$. In other words, $M^i$ is a simple current if and only if for each $M^j$ there exists a unique $M^k$ such that $[M^i] \times [M^j] = [M^k]$.

We have the following simple criterion to find simple current modules:

**Lemma 5.7.** ([Y1]) Let $V$ be a simple regular VOA and $M$ an irreducible $V$-module. If there is an irreducible $V$-module $N$ such that $[M] \times [N] = [V]$, then $M$ is a simple current module.

**Proof:** Let $M$ and $N$ as in the statement. Let $U$ be an irreducible $V$-module. We first show that $[M] \times [U] \neq 0$. Assume false. Then by the associativity for the fusion algebra we have $[N] \times ([M] \times [U]) = ([N] \times [M]) \times [U] = [V] \times [U] = [U]$, which is a contradiction. Therefore, $[M] \times [U] \neq 0$ for any irreducible $V$-module $[U]$. The same is true for $N$. Now write

$$[M] \times [U] = \sum_{W} N_{MW}^{MU}[W],$$

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where $W$ runs over all the inequivalent irreducible $V$-modules and $N_{MU}^W$ denotes the fusion rule of type $M \times U \rightarrow W$. Multiplying $[N]$, we obtain

$$[N] \times [M] \times [U] = [V] \times [U] = [U] = \sum_W N_{MU}^W [N] \times [W].$$

Since $[N] \times [W] \neq 0$, we see that $N_{MU}^W = 0$ except for only one irreducible $V$-module $W$. Therefore, $M$ is a simple current.

6 Commutant subalgebras and tensor products

In this section we present a notion of sub VOAs and consider tensor products of VOAs and SVOAs. In the SVOAs case, we have to introduce some 2-cocycle to define an SVOA structure on a tensor product.

6.1 Virasoro vectors and sub VOAs

Let $(V, Y_V(\cdot, z), 1, \omega)$ be a VOA.

Definition 6.1. A weight two vector $e \in V_2$ is called a Virasoro vector if the coefficients of the vertex operator $Y_V(e, z) = \sum_{n \in \mathbb{Z}} e^{(n)} z^{-n-1} = \sum_{m \in \mathbb{Z}} L^e(m) z^{-m-2}$ (i.e., $L^e(m) := e_{(m+1)}$) generate a representation of the Virasoro algebra on $V$, that is, the following commutator relation holds on $V$:

$$[L^e(m), L^e(n)] = (m - n)L^e(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c_e,$$

where the scalar $c_e \in \mathbb{C}$ is called the central charge of $e$.

Lemma 6.2. ([M1],[La1]) A vector $e \in V_2$ is a Virasoro vector with central charge $c_e$ if and only if it satisfies $e^{(1)}e = 2e$ and $e^{(3)}e = (c_e/2)1$.

Proof: The following proof is due to C.H. Lam [La1]. Recall Borcherds identity in Proposition 2.6), we have the following commutator formula for any $a, b \in V$ and $m, n \in \mathbb{Z}$:

$$[a(m), b(n)] = \sum_{i=0}^{\infty} \binom{m}{i} (a(i)b)(m+n-i). \quad (6.1)$$

Now let $e \in V_2$ be such that $e^{(1)}e = 2e$ and $e^{(3)}e = (c_e/2)1$. Since $e^{(n)}e \in V_{3-n}$ and $V_m = 0$ for $m < 0$, we have $e^{(n)}e = 0$ for $n \geq 4$. By setting $L^e(m) = e_{(m+1)}$, we have:

$$[L^e(m), L^e(n)] = (e^{(0)}e)_{(m+n+2)} + \binom{m+1}{1} (e^{(1)}e)_{(m+n+1)} + \binom{m}{2} (e^{(2)}e)_{(m+n)}$$

$$+ \binom{m}{3} (e^{(3)}e)_{(m+n-1)}.$$
By the skew-symmetry in Proposition 2.5, we have
\[ e_{(2)}e = -e_{(2)}e + L(-1)e_{(3)}e, \]
\[ e_{(0)}e = -e_{(0)}e + L(-1)e_{(1)}e - \frac{1}{2}L(-1)^2e_{(2)}e + \frac{1}{6}L(-1)^3e_{(3)}e. \]
Since \( a_{(n)}1 = 0 \) for any \( a \in V, n \geq 0 \), we have \( L(-1)1 = \omega_{(0)}1 = 0 \). Therefore, 
\[ L(-1)e_{(3)}e = 0 \] as \( e_{(3)}e \in \mathbb{C}1 \), and thus \( e_{(2)}e = 0 \) by the first equality above. Then by the second equality above we obtain \( 2e_{(0)}e = L(-1)e_{(1)}e \). Hence,
\[ [L^e(m), L^e(n)] = \frac{1}{2}(L(-1) \cdot 2e)_{(m+n+2)} + 2(m+1)e_{(m+n+1)} + \frac{m^3-m}{6} \cdot \frac{c_e}{2} \cdot 1_{(m+n-1)} \]
\[ = -(m+n+2)e_{(m+n+1)} + 2(m+1)e_{(m+n+1)} + \frac{m^3-m}{12} \delta_{m+n,0}c_e \]
\[ = (m-n)L^e(m+n) + \delta_{m+n,0} \frac{m^3-m}{12} c_e, \]
where in the above calculations we have used the derivation property \( Y_V(L(-1)a, z) = \frac{d}{dz}Y_V(a, z) \) which is equivalent to \( (L(-1)a)_{(n)} = -na_{(n-1)} \) for \( n \in \mathbb{Z} \), and the property of the vacuum that \( Y_V(1, z) = \text{id}_V \) which is equivalent to \( 1_{(n)} = \delta_{n,-1} \).

Conversely, if \( e \in V_2 \) is a Virasoro vector, then \( L^e(m) = e_{(m+1)}, m \in \mathbb{Z}, \) satisfy the Virasoro commutator relation (2.1) with the central charge \( c = c_e \). Since \( e = e_{(-1)}e \) by the property of the vacuum element, we have
\[ e_{(1)}e = e_{(1)}e_{(-1)}1 = L^e(0)L^e(-2)1 = L^e(-2)L^e(0)1 + [L^e(0), L^e(-2)]1 \]
\[ = e_{(-1)}e_{(1)}1 + 2L^e(-2)1 = 0 + 2e_{(-1)}1 \]
\[ = 2e \]
and
\[ e_{(3)}e = e_{(3)}e_{(-1)}1 = L^e(2)L^e(-2)1 = L^e(-2)L^e(2)1 + [L^e(2), L^e(-2)]1 \]
\[ = 0 + (c_e/2)1 = (c_e/2)1. \]
This completes the proof.

If \( V \) contains a Virasoro vector \( e \) with central charge \( c_e \), then the vertex operator \( Y_V(e, z) \) gives a Virasoro field on \( V \) and hence \( Y_V(e, z) \) together with \( Y_V(1, z) \) generates a Virasoro VOA inside \( \mathcal{F}(V) \) which is isomorphic to a homomorphic image of \( \bar{M}_{Vir}(c_e, 0) \) (cf. Section 4.4). Since a state \( a \in V \) and its field \( Y_V(a, z) \in \mathcal{F}(V) \) are in one-to-one correspondence (cf. Proposition 2.7), by the remark above \( e \) generates a subalgebra of \( V \) isomorphic to a Virasoro VOA. To be precise, we give the definition of a sub VOA as follows:

**Definition 6.3.** A vertex operator subalgebra of \( (V, Y_V(\cdot, z), 1_V, \omega) \) is a pair \((U, e)\) of a subspace \( U \) of \( V \) containing \( 1_V \) and a Virasoro vector \( e \in U \cap V_2 \) such that the structure \( (U, Y_V(\cdot|U, z)|_U, 1_V, e) \) forms a vertex operator algebra whose grade decomposition is given by \( U = \oplus_{n \geq 0} U \cap V_n \).
By the definition above, any Virasoro vector $e$ of $V$ together with the vacuum vector $1_V$ generates a sub VOA. We denote this sub VOA by $\text{Vir}(e)$ as it is generated by $e$ and is a homomorphic image of the Virasoro VOA $\tilde{M}_{\text{Vir}}(e,0)$.

### 6.2 Tensor product of VOAs

Let us consider a tensor product of VOAs.

**Lemma 6.4.** Let $(V^i, Y^i(\cdot, z), 1^i, \omega^i)$, $i = 1, 2$, be vertex operator algebras.

1. The structure $(V^1 \otimes V^1, Y^1(\cdot, z) \otimes Y^2(\cdot, z), 1^1 \otimes 1^2, \omega^1 \otimes \omega^2 + 1^1 \otimes 1^2)$ forms a vertex operator algebra.

2. Let $(M^i, Y_{M^i}(\cdot, z))$, $i = 1, 2$, be respectively $V^i$-modules. Then the tensor product $(M^1 \otimes M^2, Y_{M^1}(\cdot, z) \otimes Y_{M^2}(\cdot, z))$ is a $V^1 \otimes V^2$-module.

**Proof:** Since we have a characterization of vertex algebras by local fields given in Section 3.5, it is easy to see that the structure $(V^1 \otimes V^2, Y^1(\cdot, z) \otimes Y^2(\cdot, z), 1^1 \otimes 1^2)$ is a vertex algebra. Since $V^i$, $i = 1, 2$, has a homogeneous decomposition $V^i = \bigoplus_{n \geq 0} V^i_n$, their tensor product $V^1 \otimes V^2$ also carries a homogeneous decomposition

$$V^1 \otimes V^2 = \bigoplus_{n \geq 0} (V^1 \otimes V^2)_n, \quad (V^1 \otimes V^2)_n = \bigoplus_{s,t \geq 0, s+t=n} V^1_s \otimes V^2_t,$$

and this decomposition is given by the conformal $\omega^1 \otimes 1^2 + 1^1 \otimes \omega^2$. Therefore, the structure $(V^1 \otimes V^1, Y^1(\cdot, z) \otimes Y^2(\cdot, z), 1^1 \otimes 1^2, \omega^1 \otimes 1^2 + 1^1 \otimes \omega^2)$ is a vertex operator algebra. The assertion (2) follows similarly.

It is obvious to see that $(V^1 \otimes 1^2, \omega^1 \otimes 1^2)$ and $(1^1 \otimes V^2, 1^1 \otimes \omega^2)$ are sub VOAs of $V^1 \otimes V^2$ and canonically isomorphic to $V^1$ and $V^2$, respectively. Thus, we can identify that $V^1$ and $V^2$ are sub VOAs of $V^1 \otimes V^2$. By the commutator formula (6.1), one can easily verify that

$$[Y_{V^1 \otimes V^2}(a \otimes 1^2, z_1), Y_{V^1 \otimes V^2}(1^1 \otimes b, z_2)] = 0$$

for $a \in V^1, b \in V^2$, where $Y_{V^1 \otimes V^2}(\cdot, z)$ denotes the vertex operator map on $V^1 \otimes V^2$. Therefore, $V^1$ and $V^2$ are mutually commutative subalgebras of $V^1 \otimes V^2$. A tensor product of VOAs has the following universal property.

**Proposition 6.5.** Let $V$ be a VOA and let $U^i$, $i = 1, 2$, be sub VOAs of $V$. If $U^1$ and $U^2$ are mutually commutative, that is, $[Y(a, z_1), Y(b, z_2)] = 0$ for any $a \in U^1$ and $b \in U^2$, then the subalgebra of $V$ generated by $U^1$ and $U^2$ is a homomorphic image of the tensor product VOA $U^1 \otimes U^2$.

**Proof:** Let $W$ be the subalgebra generated by $U^1$ and $U^2$. First we observe that $W$ is spanned by $a_{(-1)}b$, $a \in U^1$, $b \in U^2$. Since $[a_{(m)}, b_{(n)}] = 0$ for any $a \in U^1$, $b \in U^2$, $m, n \in \mathbb{Z}$,
Lemma 6.6. 

we see that \(a_{(m)} b = a_{(m)} b_{(-1)} = b_{(-1)} a_{(m)} I\) and \(b_{(n)} a = b_{(n)} a_{(-1)} = a_{(-1)} b_{(n)} I\) and hence \(a_{(m)} b = b_{(n)} a = 0\) if \(m, n \geq 0\). By definition the set \(\{a \otimes I^2, I^1 \otimes b \mid a \in U^1, b \in U^2\}\) generates \(W\). So by using the Borcherds identity in Proposition 2.6 we see that \(W\) is linearly spanned by elements of the form

\[a_{(m_1)}^{1} \cdots a_{(m_s)}^{1} \cdot b_{(n_1)}^{1} \cdots b_{(n_t)}^{1}, \quad a^{i} \in U^1, \quad b^{i} \in U^2,\]

as \(U^1\) and \(U^2\) are mutually commutative. Then by \(b_{(n_1)}^{1} \cdots b_{(n_t)}^{1} I \in U^2\), any element of \(W\) is of the form

\[a_{(m_1)}^{1} \cdots a_{(m_s)}^{1} \cdot b, \quad a^{i} \in U^1, \quad b \in U^2.\]

Now by using the Borcherds identity and the property \(a_{(n)}^{i} b = 0\) for \(n \geq 0\) we can inductively show that

\[(a_{(m_1)}^{1} \cdots a_{(m_s)}^{1} I)_{(-1)} b = a_{(m_1)}^{1} \cdots a_{(m_s)}^{1} b.\]

Thus \(W\) is linearly spanned by \(a_{(-1)} b, \quad a \in U^1, \quad b \in U^2\).

We define a linear map \(\phi : U^1 \otimes U^2 \rightarrow W\) by \(\phi(a \otimes b) := a_{(-1)} b = b_{(-1)} a\) for \(a \in U^1, b \in U^2\). We have seen that \(\phi\) is surjective. It remains to show that \(\phi\) defines a VOA-homomorphism. For any \(a, a' \in U^1, b, b' \in U^2\), we have

\[
\phi \left((a \otimes I^2)_{(n)} (a' \otimes b)\right) = \phi ((a_{(n)} a' \otimes b) = (a_{(n)} a')_{(-1)} b = b_{(-1)} (a_{(n)} a')
\]

\[= a_{(n)} b_{(-1)} a' = a_{(n)} a'_{(-1)} b
\]

\[= \phi (a \otimes I^2)_{(n)} \phi (a' \otimes b),\]

where \(n \in \mathbb{Z}\) is arbitrary. Similarly, we have \(\phi \left((I^1 \otimes b)_{(n)} (a \otimes b')\right) = \phi (I^1 \otimes b)_{(n)} \phi (a \otimes b').\) Since \(a \otimes b = (a \otimes I^2)_{(-1)} (I^1 \otimes b), V^1 \otimes I^2\) and \(I^1 \otimes V^2\) generate \(V^1 \otimes V^2\). Thus \(\phi\) is a VOA-homomorphism.

\[\]

6.3 Commutant subalgebras

In this subsection we consider commutant subalgebras in VOA theory. Let \((V, \omega)\) be a VOA and \((U, e)\) its sub VOA. First, we observe some lemmas concerning the mutually commutativity.

Lemma 6.6. For \(a, b \in V, [Y(a, z_1), Y(b, z_2)] = 0\) if and only if \(a_{(i)} b = 0\) for all \(i \geq 0\).

Proof: Assume that \([Y(a, z_1), Y(b, z_2)] = 0\). Then for any \(i \geq 0, a_{(i)} b = a_{(i)} b_{(-1)} = b_{(-1)} a_{(i)} I = a_{(i)} I = 0\) as \(a_{(i)} I = 0\). Conversely, assume that \(a_{(i)} b = 0\) for any \(i \geq 0\). Then by the commutator formula (6.1), we have \([a_{(m)}, b_{(n)}] = 0\) for any \(m, n \in \mathbb{Z}\).

Lemma 6.7. For a subset \(S\) of \(V\), the subspace \(S^c := \{a \in V \mid a_{(i)} v = 0 \text{ for any } v \in S, i \geq 0\}\) forms a subalgebra in \(V\).
that which is a special case of the Borcherds identity in Proposition 2.6.

By the lemmas above, we define the commutant subalgebra of a sub VOA \((U, e)\) by

\[ \text{Com}_V(U) := \{ a \in V \mid a(v) = 0 \text{ for any } v \in U, \ i \geq 0 \}. \]

In the definition above, the notion of a commutant subalgebra seems to depend on \(U\). But it is completely determined only by the Virasoro vector \(e\) of \(U\) as we will see below.

**Proposition 6.8.** Let \((U, e)\) be a sub VOA of \(V\). Then \(\text{Com}_V(U) = \ker V e(0)\).

**Proof:** Let \(a \in \text{Com}_V(U)\). Then \(e(i0)a = e(i0)a(i-1)1 = a(i-1)e(0)1 = 0\). Thus \(a \in \ker V e(0)\). Conversely, let \(a \in \ker V e(0)\) and \(v \in U\). Then we can find an integer \(N\) such that \(v(n)a = 0\) for all \(n \geq N\) and \(v(N-1)a \neq 0\). If \(N > 0\), then \(e(0)v(N)a = v(N)e(0)a + [e(0), v(N)]a = -Nv(N-1)a = 0\), which is a contradiction. So \(Y_V(v, z)a \in V[z]\). Then by the skew-symmetry in Proposition 2.5 we have \(Y_V(b, z)v = e^{zL(-1)}Y_V(v, -z)b \in V[z]\). Thus \(b(i)v = 0\) for all \(i \geq 0\), \(b \in \text{Com}_V(U)\).

Let us introduce the notion of orthogonality of Virasoro vectors.

**Definition 6.9.** A decomposition \(\omega = \omega^1 + \cdots + \omega^n\) of the conformal vector \(\omega\) is called orthogonal if each \(\omega^i\) is a Virasoro vector and \([Y(\omega^i, z_1), Y(\omega^j, z_2)] = 0\) for \(i \neq j\).

Let \(\omega = \omega^1 + \cdots + \omega^n\) be an orthogonal decomposition. Since \(\text{Vir}(\omega^i)\) is generated by \(\omega^i\), the orthogonality implies that if \(i \neq j\) then \([Y(a, z_1), Y(b, z_2)] = 0\) for any \(a \in \text{Vir}(\omega^j)\) and \(b \in \text{Vir}(\omega^j)\). Namely, subalgebras \(\text{Vir}(\omega^i), i = 1, \ldots, n,\) are mutually commutative. Therefore, by Proposition 6.5 sub VOA generated by \(\text{Vir}(\omega^i), i = 1, \ldots, n,\) is isomorphic to a homomorphic image of a tensor product VOA \(\text{Vir}(\omega^1) \otimes \cdots \otimes \text{Vir}(\omega^n)\).

**Lemma 6.10.** ([FZ, Theorem 5.1]) For a Virasoro vector \(e \in V, \ \omega = e + (\omega - e)\) is orthogonal if and only if \(\omega(2)e = 0\).

**Proof:** By the \(L(-1)\)-derivation property we have \(\omega(0)e = e(-2)1\). On the other hand, \(e(0)e = e(0)e(-1)1 = L^e(-1)L^e(-2)1 = L^e(-2)L^e(-1)1 + L^e(-3)1 = e(-1)e(0)1 + e(-2)1 = e(-2)1\). Since \(\omega(1)e = e(1)e = 2e, \ (\omega - e)(1)e = 0\). By assumption, \((\omega - e)(2)e = 0\). As \(V\) has a grading such that \(V_0 = \mathbb{C}1\) and \(V_n = 0\) for \(n < 0\), we have \((\omega - e)(3)e \in \mathbb{C}1\) and \((\omega - e)(n)e = 0\) for \(n \geq 4\). Then by the skew-symmetry formula

\[
(a(n)b)_{(i)} = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a(n-j)b(i+j) - (-1)^n b(n+i-j)a(j),
\]

which is a special case of the Borcherds identity in Proposition 2.6.
we have
\[ e_{(0)}(\omega - e) = \sum_{0 \leq i \leq 3} \frac{(-1)^{i+1}}{i!} L(-1)^i(\omega - e)_{(i)} e = 0. \]

Thus \( \omega - e \in \text{Com}_V(e) \).

**Proposition 6.11.** Let \( \omega = e + (\omega - e) \) is an orthogonal decomposition. Then
(i) \( (\ker e_{(0)}, \omega - e) \) is a sub VOA of \( V \).
(ii) \( \ker e_{(0)} \) is the unique maximal sub VOA of \( V \) whose conformal vector is \( \omega - e \).

**Proof:**
(i): We have shown that \( \ker e_{(0)} \) is a subalgebra of \( V \). It is clear that \( 1 \in \ker e_{(0)} \) and \( \omega - e \in \ker e_{(0)} \). So we have to prove that \( \omega - e \) is a conformal vector of \( \ker e_{(0)} \). Let \( a \in \ker e_{(0)} \). Then \( e_{(0)}a = 0 \) so that we have the \( L(-1) \)-derivation property \( Y((\omega - e)_{(0)}a, z) = Y(\omega(a), z) = \frac{d}{dz} Y(a, z) \). Since \( e_{(1)}a = 0 \), we have \( (\omega - e)_{(1)}a = \omega_{(1)}a \) so that \( (\omega - e)_{(1)} \) acts on \( \ker e_{(0)} \) semisimply with a graded decomposition \( \ker e_{(0)} = \oplus_{n \in \mathbb{Z}} (\ker e_{(0)} \cap V_n) \). Thus, \( \omega - e \) is a conformal vector of \( \ker e_{(0)} \).

(ii): Let \( (W, \omega - e) \) be a sub VOA of \( V \). For each \( x \in W \), we have \( Y(e_{(0)}x, z) = Y_V(\omega(a), z) + Y_V((\omega - e)_{(0)}x, z) = \frac{d}{dz} Y(x, z) - \frac{d}{dz} Y(x, z) = 0 \). In particular, \( e_{(0)}x = \text{Res}_z Y(e_{(0)}x, z)1 = 0 \). Thus \( x \in \ker e_{(0)} \) and \( M \) is a subalgebra of \( \ker e_{(0)} \).

By this proposition, if we have an orthogonal decomposition \( \omega = \omega^1 + \omega^2 \), then we have a two mutually commuting sub VOAs \( U^1 = (\ker e_{(0)}, \omega^1) \) and \( U^2 = (\ker e_{(0)}, \omega^2) \). Denote by \( T \) the subalgebra generated by \( U^1 \) and \( U^2 \). It is shown in Proposition 6.5 that \( T \) is a homomorphic image of a tensor product VOA \( U^1 \otimes U^2 \). If one of \( U^i \), \( i = 1, 2 \), is simple, then \( T \) is actually isomorphic to \( U^1 \otimes U^2 \).

**Lemma 6.12.** Let \( V \) be a VOA and let \( V^1, V^2 \) be two sub VOAs of \( V \) such that \( V^1 \) and \( V^2 \) generate \( V \) and \( [Y_V(a^i, z_1), Y_V(a^j, z_2)] = 0 \) for any \( a^i \in V^i, i = 1, 2 \). Suppose that \( V^1 \) is simple. Then \( V \) is isomorphic to \( V^1 \otimes V^2 \).

**Proof:** We may assume that \( V^1 \) is simple. As we have seen, a linear map \( V^1 \otimes V^2 \ni a \otimes b \mapsto a_{(-1)} b \in V \) is an epimorphism. Assume that this is not injective. In this case, since \( V^1 \otimes V^2 \) as a \( V^1 \)-module is isomorphic to a direct sum of copies of \( V^1 \) and \( V^1 \) is an irreducible \( V^1 \)-module, the kernel of the above epimorphism is also a direct sum of copies of \( V^1 \) and thus contains an element of the form \( \mathbf{1} \otimes a \in V^1 \otimes V^2 \), \( 0 \neq a \in V^2 \). However, we have \( \mathbf{1} \otimes a \mapsto \mathbf{1}_{(-1)} a = a \neq 0 \), a contradiction. Thus \( V \) is isomorphic to \( V^1 \otimes V^2 \).

By this lemma, if one of \( U^i \), \( i = 1, 2 \), is simple, then \( V \) has a sub VOA isomorphic to \( U^1 \otimes U^2 \) whose conformal vector is the same as that of \( V \). In this case, the pair \((U^1, U^2)\) is often called a *commutant pair*. For a sub VOA \( V^1 \) of \( V \), the construction \( V^1 \hookrightarrow \text{Com}_V(V^1) \) is referred to as the *commutant construction* or *coset construction*. Many important VOAs are constructed by the coset construction (cf. [GKO][LLY]).
6.4 Representation of tensor products

A tensor product of VOAs inherits some properties of VOAs.

Proposition 6.13. ([FHL]) Let $V^1, V^2$ be VOAs.

1. A tensor product $V^1 \otimes V^2$ of VOAs $V^1$ and $V^2$ is simple if and only if both $V^1$ and $V^2$ are simple VOAs. Similarly, $V^1 \otimes V^2$ is rational (resp. regular) if and only if both $V^1$ and $V^2$ are rational (resp. regular).

2. A tensor product $M^1 \otimes M^2$ of $V^i$-module $M^i$, $i = 1, 2$, is an irreducible $V^1 \otimes V^2$-module if and only if each $M^i$ is an irreducible $V^i$-module for $i = 1, 2$.

A tensor product of intertwining operators defines an intertwining operators for tensor product of VOAs.

Proposition 6.14. ([DMZ][ABD][ADL]) Let $V^1, V^2$ be VOAs, $M^i$, $i = 1, 2, 3$, strong $V^1$-modules and $N^1$, $i = 1, 2, 3$, strong $V^2$-modules.

1. For $I(\cdot, z) \in \left( \frac{M^1}{M^1 \otimes M^2} \right)_{V^1}$ and $J(\cdot, z) \in \left( \frac{N^1}{M^1 \otimes M^2} \right)_{V^2}$, its tensor product $I(\cdot, z) \otimes J(\cdot, z)$ is in $\left( \frac{M^3 \otimes N^3}{M^3 \otimes N^3} \right)_{V^1 \otimes V^2}$. This defines an injective linear map from $\left( \frac{M^3 \otimes N^3}{M^3 \otimes N^3} \right)_{V^1 \otimes V^2}$ to $\left( \frac{M^1}{M^1 \otimes M^2} \right)_{V^1} \otimes \left( \frac{N^1}{M^1 \otimes M^2} \right)_{V^2}$.

2. If one of $V^1$ or $V^2$ is regular, then we have the following linear isomorphism:

$$\left( \frac{M^3 \otimes N^3}{M^1 \otimes N^1 \otimes M^2 \otimes N^2} \right)_{V^1 \otimes V^2} \simeq \left( \frac{M^3}{M^1 \otimes M^2} \right)_{V^1} \otimes \left( \frac{N^3}{N^1 \otimes N^2} \right)_{V^2}.$$

6.5 Tensor product of SVOAs

Contrary to the case of VOAs, a usual tensor product of SVOAs does not form an SVOA as there is a parity condition due to odd parts. However, by introducing suitable cocycles, we can define a tensor product of SVOAs. For later purpose, here we consider the most general case.

Let $(V^i, Y^i(\cdot, z))$, $i = 1, \ldots, r$, be SVOAs and consider a tensor product $W := V^1 \otimes \cdots \otimes V^r$. For each code word $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_2^r$, set $W^\alpha := (V^1)^{\alpha_1} \otimes \cdots \otimes (V^r)^{\alpha_r}$, where we denote the $\mathbb{Z}_2$-grading on $V^i$ by $V^i = (V^i)^0 \oplus (V^i)^1$. Then we have a natural $\mathbb{Z}_2$-grading $W = \oplus_{\alpha \in \mathbb{Z}_2^r} W^\alpha$. For $u = u^1 \otimes \cdots \otimes u^r \in V^1 \otimes \cdots \otimes V^r = W$, set

$$Y_W(u, z) := Y^{\alpha}(u^1, z) \otimes \cdots \otimes Y^{\alpha}(u^r, z)$$

and extend linearly on $W$. Then $Y_W(u, z)$, $u \in W$, are fields on $W$ and we have the following locality for $u^\alpha \in W^\alpha$, $u^\beta \in W^\beta$, $\alpha, \beta \in \mathbb{Z}_2^r$:

$$Y_W(u^\alpha, z_1)Y_W(u^\beta, z_2) \simeq (-1)^{[\alpha, \beta]}Y_W(u^\beta, z_2)Y_W(u^\alpha, z_1), \quad (6.2)$$

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Lemma 6.15. By direct computation we obtain: a linear basis of \( \text{an SVOA} \). In order to introduce an SVOA structure on \( W \) therefore, \( W \) has a natural \( \mathbb{Z}_2 \)-grading induced from those of \( V^i \)'s:

\[
W = \bigoplus_{m \in \frac{1}{2} \mathbb{N}} W_m, \quad \text{where} \quad W_m := \bigoplus_{n_1, \ldots, n_r \in \frac{1}{2} \mathbb{N}} V_{n_1} \otimes \cdots \otimes V_{n_r}.
\]

Therefore, \( W \) has a natural \( \mathbb{Z}_2 \)-grading \( W = W(0) \oplus W(1) \), \( W(j) = \oplus_{m \in \mathbb{N}} W_{m+j/2}, j = 0, 1 \). If we set \( Z^r_{2}(j) := \{ \alpha \in Z^r_2 \mid \langle \alpha, \alpha \rangle = j \} \) for \( j = 0, 1 \), then the \( \mathbb{Z}_2 \)-homogeneous parts of \( W \) are given by \( W(j) = \oplus_{\alpha \in Z^r_2(j)} W^\alpha, j = 0, 1 \). However, by (6.2), the parity property of \( Y_W(\cdot, z) \) is not consistent with this \( \mathbb{Z}_2 \)-grading on \( W \) and so \( (W, Y_W(\cdot, z)) \) does not form an SVOA. In order to introduce an SVOA structure on \( W \), we have to modify the vertex operator map \( Y_W(\cdot, z) \) by a suitable 2-cocycle.

Proposition 6.16. The structure \( (W, \tilde{Y}_W(\cdot, z)) \) forms an SVOA.

Remark 6.18. By the locality (6.2), if an even subcode \( D \) of \( Z^r_2 \) is self-orthogonal, that is, \( \langle \alpha, \beta \rangle = 0 \) for any \( \alpha, \beta \in D \), then we do not have to introduce the 2-cocycle \( \lambda \) to define a VOA structure on \( U_D \); namely \( (U_D, \tilde{Y}_W(\cdot, z)) \) forms a VOA.

By direct computation we obtain:

Lemma 6.15. We have \( \lambda(\alpha, \beta) + \lambda(\beta, \alpha) = \langle \alpha, \beta \rangle + \langle \alpha, \beta \rangle \langle \beta, \beta \rangle \) for any \( \alpha, \beta \in \mathbb{Z}_2^r \).

Now set \( \hat{Y}_W(u^\alpha, z)u^\beta := (-1)^{\lambda(\alpha, \beta)} Y_W(u^\alpha, z)u^\beta \) for \( u^\alpha \in W^\alpha, u^\beta \in W^\beta, \alpha, \beta \in \mathbb{Z}_2^r \) and extend linearly on \( W \). Then by (6.2) and Lemma 6.15, we obtain the following locality:

\[
\hat{Y}_W(u, z_1)\hat{Y}_W(v, z_2) \simeq (-1)^{\epsilon(u, v)} \hat{Y}_W(v, z_2)\hat{Y}_W(u, z_1),
\]

where \( u, v \) are \( \mathbb{Z}_2 \)-homogeneous elements in \( W \) and the \( \mathbb{Z}_2 \)-grading on \( W \) is given by \( W = W(0) \oplus W(1) \). So we have constructed a tensor product of SVOAs.

Proposition 6.17. The structure \( (U_D, \tilde{Y}_W(\cdot, z)) \) is a subalgebra of \( W \). It is a VOA if \( D(1) = \phi \) and otherwise it is an SVOA.
7 Simple VOAs

In this section we observe that a simple vertex operator algebra behaves like a commutative associative algebra.

7.1 Irreducible modules

Let $V$ be a vertex operator algebra and $M$ a $V$-module. For subsets $A$ of $V$ and $B$ of $M$, we define

$$A \cdot B := \text{Span}_{\mathbb{C}}\{a_{(n)}v \mid a \in V, \ v \in M, \ n \in \mathbb{Z}\}.$$

Lemma 7.1. (Lemma 3.12 of [Li2]) Let $a, b \in V$ and $v \in M$. Let $k \in \mathbb{Z}$ be such that $z^kY(a, z)v \in M[z]$. Then for $p, q \in \mathbb{Z},$

$$a_{(p)}b_{(q)}v = \sum_{i=0}^{n} \sum_{j=0}^{\infty} \binom{p-k}{i} \binom{k}{j} (a_{(p-k-i+j)}b)_{(q+k+i+j)}v,$$

where $n$ is any non-negative integer such that $z^{n+q+1}Y(b, z)v \in M[z]$.

As a corollary, we have $A^1 \cdot (A^2 \cdot B) \subset (A^1 \cdot A^2) \cdot B$ for subsets $A^1, A^2 \subset V$ and $B \subset M$. In particular, $V \cdot w$ is a submodule of $M$ for any $w \in M$.

Proposition 7.2. ([DL]) Let $M^i, i = 1, 2, 3$, be $V$-modules and $I(\cdot, z)$ a $V$-intertwining operator of type $M^1 \times M^2 \rightarrow M^3$. Assume that there are subsets $S^i \subset M^i, i = 1, 2, 3$, such that $I(s^i, z)s^2 = 0$ for any $s^i \in S^i$ and $s^2 \in S^2$. If $M^1$ has no proper submodule containing $S^i$ for $i = 1, 2$, then $I(\cdot, z) = 0$. In particular, if both $M^1$ and $M^2$ are irreducible, then $I(u, z)v = 0$ for some non-zero $u \in M^1, v \in M^2$ implies $I(\cdot, z) = 0$.

Proof: Let $s^1 \in S^1$ and $s^2 \in S^2$. By assumption, we have $Y(a, z_1)I(s^1, z_2)s^2 = 0$ for any $a \in V$. By the definition of intertwining operators, there is an $N > 0$ such that $(z_1 - z_2)^N I(s^1, z_2)Y(a, z_1) = (z_1 - z_2)^N Y(a, z_1)I(s^1, z_2)s^2 = 0$. Since there are finite negative powers of $z_1$ in $I(s^1, z_2)Y(a, z_1)s^2$, we have $I(s^1, z_2)Y(a, z_1)s^2 = 0$ for any $a \in V$. Then we have

$$I(a_{(n)}s^1, z)s^2 = \text{Res}_{z_0} \{(z_0 - z)^nY(a, z_0)I(s^1, z) - (-z + z_0)^nI(s^1, z)Y(a, z_0)\}s^2 = 0,$$

and hence $I(M^1, z)S^2 = 0$ as $V \cdot S^1 = M^1$. By the commutator formula in Proposition 5.3,

$$I(u, z)a_{(n)}s^2 = a_{(n)}I(u, z)s^2 - [a_{(n)}, I(u, z)]s^2$$

$$= -\sum_{i=0}^{\infty} \binom{n}{i} I(a_{(i)}u, z)s^2$$

$$= 0$$

for any $a \in V$ and $n \in \mathbb{Z}$. Therefore, $I(M^1, z)M^2 = 0$ and hence $I(\cdot, z) = 0$. 

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Corollary 7.3. Let $V$ be a simple VOA.

(1) For any $a, b \in V$, $Y(a, z)b = 0$ implies $a = 0$ or $b = 0$.

(2) Let $W$ be an irreducible $V$-module. Then for non-zero $a_1, \ldots, a_k \in V$ and linearly independent $w_1, \ldots, w_k \in W$, $\sum_{i=1}^k Y(a_i, z)w_i \neq 0$.

(3) Let $W$ be a $V$-module. Then $Y(a, z)w = 0$ implies $a = 0$ or $w = 0$.

Proof: (1) is clear from Proposition 7.2 since the vertex operator map $Y(\cdot, z)$ is a $V$-intertwining operator of type $V \times V \rightarrow V$. Consider (2). If $\sum_i Y(a_i, z)w^i = 0$, then we can show that $\sum_i Y(a_i, z)Y(b^i, z_1)w^i = 0$ for any $b^1 \in V$ by a similar argument used in the proof of Proposition 7.2. Thus $\sum_i Y(a_i, z)Y(b^n, z_n) \cdots Y(b^1, z_1)w^i = 0$ for $b^1, \ldots, b^n \in V$.

Since $w^i$ are linearly independent in $W$ and $\mathbb{C}$ is algebraically closed, we are reduced to the case $i = 1$ by the density theorem. Hence the assertion follows from the previous proposition.

Consider (3). Assume that $a \neq 0$ and $w \neq 0$. Replacing $W$ by $V \cdot w$ if necessary, we may assume that $W = V \cdot w$. By Zorn’s lemma, we can take a maximal submodule $U$ of $W$ which does not contain $w$. Then the quotient $W/U$ is an irreducible $V$-module and a relation $Y(a, z)w = 0$ in $W$ yields a contradiction.

7.2 Quantum Galois theory

Let $(V, Y(\cdot, z), 1, \omega)$ be a simple VOA.

Definition 7.4. An automorphism $\sigma$ of $V$ is a linear isomorphism on $V$ such that $\sigma 1 = 1$, $\sigma \omega = \omega$ and $\sigma Y(a, z)b = Y(\sigma a, z)\sigma b$ for any $a, b \in V$. We denote the group of automorphisms of $V$ by $\text{Aut}(V)$.

Remark 7.5. In the definition above, we may remove the condition $\sigma 1 = 1$, because of the state-field correspondence (cf. Proposition 2.7).

Let $G$ be a finite subgroup of $\text{Aut}(V)$. Then the fixed point subalgebra $V^G := \{a \in V \mid \sigma a = a \text{ for any } \sigma \in G\}$ is called the $G$-orbifold subalgebra or simply orbifold of $V$. The study of $V^G$ was initiated in [DVVV] in physical point of view and was begun by [DM1] in mathematical point of view. Let $\text{Irr}(G)$ be the set of all inequivalent irreducible characters of $G$. For each $\chi \in \text{Irr}(G)$, we fix an irreducible $\mathbb{C}[G]$-module $M_\chi$ affording the character $\chi$. Then we have a decomposition

$$V = \bigoplus_{\chi \in \text{Irr}(G)} M_\chi \otimes \text{Hom}_G(M_\chi, V).$$

Set $V_\chi := \text{Hom}_G(M_\chi, V)$. Then $V^G = V_{1_G}$ where $1_G \in \text{Irr}(G)$ is the principal character of $G$. Each $V_\chi$ is a $V^G$-module since the action of $\mathbb{C}[G]$ on $V$ commutes with that of $V^G$.

Thus $V$ is a module over $\mathbb{C}[G] \otimes \mathbb{C} V^G$. The following Schur-Weyl type duality theorem is shown in [DM1] and [DLM3].
Theorem 7.6. With reference to the setup above, we have:

(1) For each $\chi \in \text{Irr}(G)$, $V_\chi \neq 0$.

(2) All $V_\chi$, $\chi \in \text{Irr}(G)$, are irreducible $V^G$-modules. In particular, $V^G$ is a simple VOA.

(3) $V_\lambda \simeq V_\mu$ as $V^G$-modules if and only if $M_\lambda \simeq M_\mu$ as $\mathbb{C}[G]$-modules.

By this theorem, $V^G$ is a simple sub VOA of $V$ and the pair $(\mathbb{C}[G], V^G)$ forms a dual pair on $V$. We will later consider a simple VOA $V$ graded by an abelian group $A$, that is, $V$ has a decomposition $V = \bigoplus_{\alpha \in A} V_\alpha$ such that $V_\alpha \neq 0$ and $V_\alpha \cdot V_\beta \subset V_{\alpha + \beta}$ for $\alpha, \beta \in A$.

In this case the dual group $A^*$, the group of all group homomorphisms from $A$ to $\mathbb{C}^*$, naturally acts on $V$. By the theorem above, each $V_\alpha$ is an irreducible module over the $A^*$-fixed point subalgebra $V_{A^*}$ of $V$.

The following Galois correspondence is established in [HMT] by using Theorem 7.6.

Theorem 7.7. Assume that $V$ is a simple VOA. Given a finite subgroup $G$ of $\text{Aut}(V)$, define the map $\Phi : H \mapsto V^H$ which associates to a subgroup $H$ of $G$ the sub VOA $V^H$ of $V$. Then $\Phi$ induces a bijective correspondence between subgroups of $G$ and sub VOAs of $V$ containing $V^G$.

Remark 7.8. By theorem 7.6 and 7.7, the representation theory of $G$ helps us to study $V$ as a $V^G$-module.

Let $\sigma \in G$. For a $V$-module $(M, Y_M(\cdot, z))$, we can define another module structure on it as follows. Define the $\sigma$-conjugate vertex operator $Y^\sigma_M(\cdot, z)$ by

$$Y^\sigma_M(a, z)v := Y_M(\sigma a, z)v$$

for $a \in V$ and $v \in M$. Then one can easily check that $(M, Y^\sigma_M(\cdot, z))$ is also a $V$-module.

We usually denote $(M, Y^\sigma_M(\cdot, z))$ by $M^\sigma$ and call it the $\sigma$-conjugate of $M$. It is obvious that $M^\sigma$ is irreducible if and only if $M$ is. If $M^\sigma$ is isomorphic to $M$ as a $V$-module, then $M$ is referred to as $\sigma$-stable. The $\sigma$-stability is equivalent to saying that there is a $V$-isomorphism $\phi(\sigma) : M \rightarrow M$ such that $\phi(\sigma)Y_M(a, z)v = Y_M(\sigma a, z)\phi(\sigma)v$ for $a \in V$ and $v \in M$. The linear isomorphism $\phi(\sigma)$ is called a $\sigma$-stabilizing automorphism. If $M$ is irreducible, then the stabilizing automorphism is unique up to linearity.

Similarly, we can define the conjugation of intertwining operators.

Proposition 7.9. Let $V$ be a VOA, $\rho \in \text{Aut}(V)$ and $(M^i, Y_M^i(\cdot, z))$, $i = 1, 2, 3$, $V$-modules. Then we have the following natural isomorphism:

$$\begin{pmatrix} M^3 \\ M^1 & M^2 \end{pmatrix}_V \simeq \begin{pmatrix} (M^3)^\rho \\ (M^1)^\rho & (M^2)^\rho \end{pmatrix}_V.$$

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Proof: Let $I(\cdot, z)$ be a $V$-intertwining operator of type $M^1 \times M^2 \rightarrow M^3$. Let $\psi_i : M^i \rightarrow (M^i)^\rho$, $i = 1, 2, 3$, be canonical linear isomorphisms such that $Y_{(M^i)^\rho}(a, z)\psi_i = \psi_i Y_{M^i}(\rho a, z)$ for all $a \in V$. For $u \in (M^1)^\rho$ define

$$I^\rho(u, z) := \psi_3 I(\psi_1^{-1}u, z)\psi_2^{-1} \in \text{Hom}_C \left( (M^2)^\rho, (M^3)^\rho \right) \{z\}.$$  

Then it is easy to check that $I^\rho(\cdot, z)$ is a $V$-intertwining operator of type $(M^1)^\rho \times (M^2)^\rho \rightarrow (M^3)^\rho$. Thus the map

$$\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix} \ni I(\cdot, z) \mapsto I^\rho(\cdot, z) \in \begin{pmatrix} (M^3)^\rho \\ (M^1)^\rho \ (M^2)^\rho \end{pmatrix}_V$$

gives the desired linear isomorphism.

As a corollary, we have:

Corollary 7.10. Let $V$ be a simple regular VOA. For $\rho \in \text{Aut}(V)$, define its action on the fusion algebra $\mathcal{V}(V)$ associated to $V$ by the conjugation $[M]^\rho := [M^\rho]$, where $M$ runs over irreducible $V$-modules. Then the conjugation by an automorphism preserves the fusion rules and thus $\text{Aut}(V)$ acts on the fusion algebra $\mathcal{V}(V)$.

8 Ising model

In this section we present an explicit realization of the Ising model SVOA $L_{\text{Vir}}(1/2, 0) \oplus L_{\text{Vir}}(1/2, 1/2)$.

8.1 Construction

Let $A_\psi$ be the $\mathbb{R}$-algebra generated by $\{\psi_k \mid k \in \mathbb{Z} + 1/2\}$ subject to the defining relation

$$[\psi_m, \psi_n]_+ := \psi_m \psi_n + \psi_n \psi_m = \delta_{m+n,0}, \quad m, n \in \mathbb{Z} + 1/2,$$

and denote a subalgebra of $A_\psi$ generated by $\{\psi_k \mid k \in \mathbb{Z} + \frac{1}{2}, k > 0\}$ by $A_\psi^+$. Let $\mathbb{R}1$ be a trivial $A_\psi^+$-module. Define the induced $A_\psi$-module $X_\mathbb{R}$ by

$$X_\mathbb{R} := \text{Ind}_{A_\psi^+}^{A_\psi} \mathbb{R}1 = A_\psi \otimes A_\psi^+ \mathbb{R}1.$$  

Then a linear basis of $X_\mathbb{R}$ is given by the set

$$\{\psi_{-r_1} \psi_{-r_2} \cdots \psi_{-r_k} 1 \mid r_1 > r_2 > \cdots > r_k > 0, \ k \geq 0\}. \quad (8.1)$$

Therefore, there is a natural $\mathbb{Z}_2$-grading $X_\mathbb{R} = X_\mathbb{R}^0 \oplus X_\mathbb{R}^1$ defined by

$$X_\mathbb{R}^i := \text{Span}_\mathbb{R}\{\psi_{-r_1} \cdots \psi_{-r_k} 1 \mid r_1 > \cdots > r_k > 0, \ k \geq 0, \ r_1 + \cdots + r_k \in \mathbb{N} + i/2\}.$$
We can introduce an $\mathcal{A}$-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $X_{\mathbb{R}}$. Define a bilinear form $\langle \cdot, \cdot \rangle$ on $X$ so that the basis (8.1) forms an orthonormal basis. Then $\langle \cdot, \cdot \rangle$ satisfies the $\mathcal{A}$-invariance property $\langle \psi_r u, v \rangle = \langle u, \psi_r v \rangle$, $r \in \mathbb{Z} + 1/2$, for $u, v \in X_{\mathbb{R}}$.

Consider the generating series $\psi(z) := \sum_{m \in \mathbb{Z}} \psi_{m+1/2} z^{-m-1}$ on $X_{\mathbb{R}}$. It is obvious that $\psi(z)$ is an odd field on the $\mathbb{Z}_2$-graded space $X_{\mathbb{R}} = X_{\mathbb{R}}^0 \oplus X_{\mathbb{R}}^1$. By a computation, we can show the following locality:

$$(z_1 - z_2) \psi(z_1) \psi(z_2) = -(z_1 - z_2) \psi(z_2) \psi(z_1).$$

Therefore, $\psi(z)$ together with the vacuum field $1_X(z)$ generates a vertex superalgebra inside $\mathcal{F}(X_{\mathbb{R}})$, which is explicitly as follows. Since $\psi(z)1 = \psi_{-1/2}1 + X_{\mathbb{R}}^1[z]z$, there exists the unique vertex superalgebra structure on $X_{\mathbb{R}} = X_{\mathbb{R}}^0 \oplus X_{\mathbb{R}}^1$ with the vacuum $1$ such that $Y(\psi_{-1/2}1, z) = \psi(z)$ by the uniqueness theorem and the existence theorem given in Section 4. Now consider the element $\omega := \frac{1}{2} \psi_{-3/2} \psi_{-1/2}1 \in X_{\mathbb{R}}^0$. By a direct computation, one can check that the vertex operator $Y(\omega, z)$ defines a representation of the Virasoro algebra on $X_{\mathbb{R}}$ with central charge $1/2$. Since $X_{\mathbb{R}}$ possesses a positive definite $\mathcal{A}$-invariant bilinear form, we can show that $X_{\mathbb{R}}$ as a Vir-module is semisimple and each $X_{\mathbb{R}}^i$, $i = 0, 1$, is an irreducible Vir-module over $\mathbb{R}$. Therefore, we have the following isomorphisms as Vir-modules over $\mathbb{C}$ (cf. [KR]; see Section 4.4 for notation):

$$\mathbb{C} \otimes X_{\mathbb{R}}^0 \simeq L_{Vir}(1/2, 0), \quad \mathbb{C} \otimes X_{\mathbb{R}}^1 \simeq L_{Vir}(1/2, 1/2).$$

Since $\omega \in X_{\mathbb{R}}^0$, the vertex superalgebra structure on $X_{\mathbb{R}}$ implies that $X_{\mathbb{R}}^0$ is isomorphic to the simple Virasoro VOA $L_{Vir}(1/2, 0)$ over $\mathbb{R}$. For later purpose, we set $X := \mathbb{C} \otimes_{\mathbb{R}} X_{\mathbb{R}}$, $X^0 := \mathbb{C} \otimes_{\mathbb{R}} X_{\mathbb{R}}^0$, and $X^1 := \mathbb{C} \otimes_{\mathbb{R}} X_{\mathbb{R}}^1$. Since $X_{\mathbb{R}}$ is a real-form of $X$ with positive definite bilinear form, the complexification $X^0$ is isomorphic to the simple Virasoro VOA $L_{Vir}(1/2, 0)$ over $\mathbb{C}$.

**Theorem 8.1.** There is a structure of a simple SVOA on $L_{Vir}(1/2, 0) \oplus L_{Vir}(1/2, 1/2)$ which has a positive definite bilinear form. Its real form is isomorphic to the vertex superalgebra $X_{\mathbb{R}}$ constructed as above.

**Remark 8.2.** The field $\psi(z)$ is called a free fermionic field in physics and it appears as a symmetry of 2-dimensional Ising model, a famous solvable model of statistical mechanics. Therefore, we often call the VOA $L_{Vir}(1/2, 0)$ the *Ising model*.

We can also realize another representation of the Virasoro algebra with central charge $1/2$ as follows. Let $\mathcal{A}_0$ be a $\mathbb{C}$-algebra generated by $\{\phi_m \mid m \in \mathbb{Z}\}$ with the relation $[\phi_m, \phi_n]_+ = \delta_{m+n, 0}$, $m, n \in \mathbb{Z}$. Let $\mathcal{A}_0^+$ be a subalgebra of $\mathcal{A}_0$ generated by $\{\phi_m \mid m > 0\}$ and let $\mathbb{C}e^{1/16}$ be a trivial $\mathcal{A}_0^+$-module. Consider the induced module

$$\tilde{X} := \text{Ind}_{\mathcal{A}_0^+}^{\mathcal{A}_0} e^{1/16} = \mathcal{A}_0 \otimes_{\mathcal{A}_0^+} e^{1/16}.$$
where $\phi, \phi_s$: is defined to be $\phi_r \phi_s$ if $r \leq s$ and $\phi_s \phi_r$ otherwise. Then $\{\tilde{L}(n) \mid n \in \mathbb{Z}\}$ is a set of well-defined linear endomorphisms on $\tilde{X}$ and they defines a representation of the Virasoro algebra on $\tilde{X}$ with central charge $1/2$. It is shown in [KR] that $\tilde{X} \simeq L_{\text{Vir}}(1/2, 1/16) \oplus L_{\text{Vir}}(1/2, 1/16)$ as a Vir-module (see Section 4.4 for notation). The Virasoro field $\tilde{\omega}(z) := \sum_{n \in \mathbb{Z}} \tilde{L}(n) z^{-n-2}$ defines a representation of the Virasoro VOA $\tilde{M}_{\text{Vir}}(1/2, 0)$ and it is known that this action factor out the maximal ideal $J_{\text{Vir}}(1/2, 0)$, that is, the field $\tilde{\omega}(z)$ actually defines a representation of the Ising model VOA $L_{\text{Vir}}(1/2, 0)$.

We present the following fact without proof. Set $|\frac{1}{16}|^z := (\sqrt{2}\phi \pm 1)e^{\frac{1}{16}}$ and denote by $\hat{\mathfrak{X}}^\pm$ the $\mathfrak{A}_\varphi$-modules generated by $|\frac{1}{16}|^z$. Then $\hat{\mathfrak{X}} \simeq \hat{\mathfrak{X}}^+ \oplus \hat{\mathfrak{X}}^-$ and each $\hat{\mathfrak{X}}^\pm$ is isomorphic to $L_{\text{Vir}}(1/2, 1/16)$ as a Vir-module. As modules over the Virasoro algebra, $X^0 \simeq L_{\text{Vir}}(1/2, 0)$ and $X^1 \simeq L_{\text{Vir}}(1/2, 1/2)$ are generated by $1 \in X^0$ and $\psi_{1/2}1 \in X^1$, respectively. Therefore, $X^0$ is linearly spanned by the vectors of the form

$$L(-n_1) \cdots L(-n_k)1, \quad n_1 \geq \cdots \geq n_k \geq -2, \quad k \geq 0,$$

and $X^1$ is linearly spanned by the vectors of the form

$$L(-n_1) \cdots L(-n_k)\psi_{-1/2}1, \quad n_1 \geq \cdots \geq n_k \geq -1, \quad k \geq 0.$$

Consider the generating series

$$\phi(z) := \sum_{n \in \mathbb{Z}} \phi_n z^{-n-1/2}. \quad (8.2)$$

Even though the powers of $z$ in $\phi(z)$ are contained in $\mathbb{Z} + 1/2$, one can check the following locality:

$$(z_1 - z_2)\phi(z_1)\phi(z_2) = -(z_1 - z_2)\phi(z_2)\phi(z_1). \quad (8.3)$$

Using $\phi(z)$, define linear maps $I^i, (\cdot, z) : X^i \times \hat{\mathfrak{X}}^\pm \to \hat{\mathfrak{X}}^\pm \{z\}, i = 0, 1$, as follows:

$$I^0, (1, z) := \text{id}_{\hat{\mathfrak{X}}^\pm}, \quad I^1, (\psi_{1/2}1, z) := \phi(z),$$

and inductively we set

$$I^i, (L(-n)u, z) := \text{Res}_z \{ (x - z)^{-n+1}\tilde{\omega}(x)I^i, (u, z) - (1 + x)^{-n+1}I^i, (u, z)\tilde{\omega}(x) \}.$$

Then
Proposition 8.3. (1) The linear maps $I^{0,\pm}(\cdot, z)$ defined as above are well-defined and the pairs $(X^{\pm}, I^{0,\pm}(\cdot, z))$ form isomorphic irreducible $X^0$-modules.

(2) The linear maps $I^{1,\pm}(\cdot, z)$ defined as above are well-defined and give $X^0$-intertwining operators of type $X^1 \times X^1 \rightarrow X^\pm$ which are equivalent to $L_{Vir}(1/2, 0)$-intertwining operators the type

$$L_{Vir}(1/2, 1/2) \times L_{Vir}(1/2, 1/16) \rightarrow L_{Vir}(1/2, 1/16).$$

Moreover, each of $I^{1,\pm}(\cdot, z)$ is local to itself, that is, for $u, v \in X^1$, there exists $N \in \mathbb{N}$ such that

$$(z_1 - z_2)^N I^{1,\pm}(u, z_1) I^{1,\pm}(v, z_2) = -(z_1 - z_2)^N I^{1,\pm}(v, z_2) I^{1,\pm}(u, z_1).$$

8.2 Fusion rules of Ising model

We have explicitly constructed the simple vertex operator algebra $L_{Vir}(1/2, 0)$ and its irreducible modules $L_{Vir}(1/2, 1/2)$ and $L_{Vir}(1/2, 1/16)$. We will study a class of VOAs by using certain symmetry in the fusion algebra associated to $L_{Vir}(1/2, 0)$ which we will present in this subsection.

The representation theory of $L_{Vir}(1/2, 0)$ is well-known.

Theorem 8.4. ([DMZ][DLM2])

(1) The VOA $L_{Vir}(1/2, 0)$ is regular. There are exactly three irreducible representations, $L_{Vir}(1/2, h)$, $h = 0, 1/2, 1/16$.

(2) The fusion algebra associated to $L_{Vir}(1/2, 0)$ has the following structure:

$$[0] \times [h] = [h], \quad h = 0, 1/2, 1/16, \quad [1/2] \times [1/2] = [0],$$

$$[1/2] \times [1/16] = [1/16], \quad [1/16] \times [1/16] = [0] + [1/2],$$

where we have set $[h] := L_{Vir}(1/2, h)$ for simplicity.

Remark 8.5. Since the fusion algebra is commutative by Proposition 5.4, we only present fusion rules for partial pairs in the theorem above.

It follows from the theorem above that $L_{Vir}(1/2, 0)$ and $L_{Vir}(1/2, 1/2)$ are simple current $L_{Vir}(1/2, 0)$-modules (cf. Definition 5.6). The important point is that we can find $\mathbb{Z}_2$-symmetries in the fusion algebra associated to $L_{Vir}(1/2, 0)$ given as below.

Proposition 8.6. On the fusion algebra $\mathcal{V}(L_{Vir}(1/2, 0)) = \mathbb{C}[0] \oplus \mathbb{C}[1/2] \oplus \mathbb{C}[1/16]$, where $[L_{Vir}(1/2, h)]$ denotes $[h]$, has an involution $\tau$ defined as identity on $\mathbb{C}[0] \oplus \mathbb{C}[1/2]$ and acts as $-1$ on $\mathbb{C}[1/16]$. On the $\langle \tau \rangle$-fixed-point subalgebra $\mathcal{V}(L_{Vir}(1/2, 0))^{(\tau)} = \mathbb{C}[0] \oplus \mathbb{C}[1/2]$, we have an involution $\sigma$ defined as identity on $\mathbb{C}[0]$ and acts as $-1$ on $\mathbb{C}[1/2]$.

Remark 8.7. Note that the involution $\tau$ above belongs to $\text{Aut}(\mathcal{V}(L_{Vir}(1/2, 0)))$, but the involution $\sigma$ above belongs to $\text{Aut}(\mathcal{V}(L_{Vir}(1/2, 0))^{(\tau)})$, not to $\text{Aut}(\mathcal{V}(L_{Vir}(1/2, 0)))$. 

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8.3 Miyamato involution

We consider a VOA \( V \) which contains the Ising model \( L_{\text{Vir}}(1/2,0) \) as a sub VOA.

**Definition 8.8.** A Virasoro vector \( e \in V \) is called an *Ising vector* if the Virasoro sub VOA \( \text{Vir}(e) \) generated by \( e \) is isomorphic to the Ising model VOA \( L_{\text{Vir}}(1/2,0) \).

Given a Virasoro vector \( e \) with central charge 1/2, it always generates a Virasoro VOA with central charge 1/2. The following lemma gives us a criterion to check whether it generates a simple Virasoro VOA \( L_{\text{Vir}}(1/2,0) \) or not.

**Lemma 8.9.** ([DMZ]) A vector \( e \in V \) is an Ising vector if and only if \( e \) is a Virasoro vector with central charge 1/2 and it satisfies the following singular vector condition:

\[
\left( e^3_{(-3)} + 93e^2_{(-2)} - 264e_{(-3)}e_{(-1)} - 108e_{(-5)} \right) \cdot 1 = 0.
\]

Now assume that a VOA \( V \) contains an Ising vector \( e \). Since \( \text{Vir}(e) \cong L_{\text{Vir}}(1/2,0) \) is regular, \( V \) is a completely reducible \( \text{Vir}(e) \)-module. By (1) of Theorem 8.4, there are three irreducible \( L_{\text{Vir}}(1/2,0) \)-modules, \( L_{\text{Vir}}(1/2,h), h = 0, 1/2, 1/16 \). Therefore, we can decompose \( V \) as follows:

\[
V_e(0) \oplus V_e(1/2) \oplus V_e(1/16),
\]

where \( V_e(h) \) denotes the sum of all irreducible \( \text{Vir}(e) \)-modules isomorphic to \( L_{\text{Vir}}(1/2,h) \). The \( \mathbb{Z}_2 \)-symmetry of the fusion algebra associated to \( L_{\text{Vir}}(1/2,0) \) given in Proposition 8.6 gives rise to an involutive automorphism of VOAs containing \( L_{\text{Vir}}(1/2,0) \).

**Theorem 8.10.** ([M1]) Let \( V \) be a VOA and \( e \in V \) an Ising vector.

1. A linear isomorphism \( \tau(e) \) on \( V \) defined as identity on \( V_e(0) \oplus V_e(1/2) \) and \(-1 \) on \( V_e(1/16) \) is an involutive automorphism of a VOA \( V \).

2. A linear isomorphism \( \sigma(e) \) on the \( \langle \tau(e) \rangle \)-fixed point subalgebra \( V^{\langle \tau(e) \rangle} = V_e(0) \oplus V_e(1/2) \) of \( V \) defined as identity on \( V_e(0) \) and \(-1 \) on \( V_e(1/2) \) is an involutive automorphism of a VOA \( V^{\langle \tau(e) \rangle} \).

**Proof:** By the fusion rules given in (2) of Theorem 8.4, \( V \) must have the following structure:

\[
V_e(0) \cdot V_e(h) \subset V_e(h), \quad h = 0, 1/2, 1/16, \quad V_e(1/2) \cdot V_e(1/2) \subset V_e(0),
\]

\[
V_e(1/2) \cdot V_e(1/16) \subset V_e(1/16), \quad V_e(1/16) \cdot V_e(1/16) \subset V_e(0) \oplus V_e(1/2).
\]

Therefore, the assertion follows.

The involutions \( \tau(e) \in \text{Aut}(V) \) and \( \sigma(e) \in \text{Aut}(V^{\langle \tau(e) \rangle}) \) defined as above are often called the *Miyamato involutions* of \( \tau \)-type and \( \sigma \)-type, respectively.
Remark 8.11. The famous moonshine VOA $V^\natural$ constructed Frenkel-Lepowsky-Meurman [FLM] has the Monster sporadic finite simple group as its full-automorphism group. In the Monster, there are two conjugacy classes of involutions, the 2A- and 2B-conjugacy classes (cf. [ATLAS]). It is shown in [C] and [M1] that there is a one-to-one correspondence between 2A-involutions of the Monster and Ising vectors of $V^\natural$ by associating the Miyamoto involution $\tau(e) \in \text{Aut}(V^\natural)$ to each Ising vector $e \in V^\natural$.

9 Framed VOAs

In this section we consider an important class of VOAs called framed VOAs which includes the famous moonshine VOA constructed in [FLM].

9.1 Frames and automorphisms

We give the definition of framed VOAs.

Definition 9.1. A simple VOA $(V, \omega)$ is referred to as a framed VOA if the conformal vector $\omega$ admits an orthogonal decomposition $\omega = e_1 + \cdots + e_n$ such that each $e_i$ is an Ising vector. The orthogonal decomposition $\omega = e_1 + \cdots + e_n$ is called an Ising frame.

Let $V$ be a framed VOA with an Ising frame $\omega = e_1 + \cdots + e_n$. As we have seen in Section 6.3, the sub VOA generated by $e_1, \ldots, e_n$ is isomorphic to a tensor product $\text{Vir}(e_1) \otimes \cdots \otimes \text{Vir}(e_n) \cong L_{\text{Vir}}(1/2, 0) \otimes n$. Set $F := \text{Vir}(e_1) \otimes \cdots \otimes \text{Vir}(e_n)$. Since $L_{\text{Vir}}(1/2, 0)$ is simple and regular, so is $F$ by Proposition 6.13. Therefore, $V$ as an $F$-module is completely reducible and is a direct sum of irreducible $F$-submodules

$$L_{\text{Vir}}(1/2, h_1) \otimes \cdots \otimes L_{\text{Vir}}(1/2, h_n), \quad h_i \in \{0, 1/2, 1/16\}.$$ 

Assign to an irreducible $F$-submodule $\otimes_{i=1}^n L_{\text{Vir}}(1/2, h_i)$ its 1/16-word $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_2^n$ by $\alpha_i = 1$ if and only if $h_i = 1/16$. For each $\alpha \in \mathbb{Z}_2^n$, set $V^\alpha$ to be the sum of all irreducible $F$-submodules whose 1/16-words are equal to $\alpha$ and define

$$S := \{ \alpha \in \mathbb{Z}_2^n \mid V^\alpha \neq 0 \}.$$ 

Then we obtain the 1/16-word decomposition $V = \bigoplus_{\alpha \in S} V^\alpha$. Since the 1/16-word decomposition is based on the isotypical decomposition, it is well-defined. By the fusion rule in Theorem 8.4, the 1/16-word decomposition provides an $S$-graded structure for $V$, that is, $V^\alpha \cdot V^\beta = V^{\alpha+\beta}$. In particular, $S$ is a linear code.

Set $\nu^1 = (1, 0, \ldots, 0), \nu^2 = (0, 1, \ldots, 0), \ldots, \nu^n = (0, 0, \ldots, 1)$. Then the Miyamoto involution $\tau(e')$ preserves each subspace $V^\alpha$ and acts on it as $(-1)^{\langle \alpha', \alpha \rangle}$, where $\langle \alpha, \beta \rangle = \alpha_1 \beta_1 + \cdots + \alpha_n \beta_n \in \mathbb{Z}_2$ for $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n$. Therefore, Miyamoto
involutions $\tau(e^i), 1 \leq i \leq n$, generate an elementary abelian 2-subgroup of Aut($V$).

Denote by $P$ the group generated by $\tau(e^i), 1 \leq i \leq n$. Then $P$ is isomorphic to the dual group $S^*$ of $S$. Since $V^{(0^*)} = V^P$, we see from Theorem 7.6 that

**Lemma 9.2.** All $V^\alpha, \alpha \in S$, are inequivalent irreducible $V^{(0^*)}$-modules.

Consider the $P$-orbifold subalgebra $V^P$ of $V$. Since $V^P = V^{(0^*)}$, the $P$-orbifold $V^P$ as an $F$-module has the following form:

$$V^P = V^{(0^*)} = \bigoplus_{h_i \in \{0, 1/2\}} m_{h_1, \ldots, h_n} L_{\text{Vir}}(1/2, h_i) \otimes \cdots \otimes L_{\text{Vir}}(1/2, h_n),$$

where $m_{h_1, \ldots, h_n} \in \mathbb{N}$ denotes the multiplicity. On $V^{(0^*)}$, Ising vectors $e^i, 1 \leq i \leq n$, define Miyamoto involutions of $\sigma$-type. Denote by $Q$ the elementary abelian 2-group generated by $\sigma(e^i), 1 \leq i \leq n$. Then $(V^{(0^*)})^Q = m_{0, \ldots, 0} \text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^n)$ is simple so that $m_{0, \ldots, 0} = 1$ as $\dim V_0 = 1$. Moreover, each isotypical component

$$m_{h_1, \ldots, h_n} L_{\text{Vir}}(1/2, h_i) \otimes \cdots \otimes L_{\text{Vir}}(1/2, h_n), \quad h_i \in \{0, 1/2\},$$

is an irreducible $(V^{(0^*)})^Q$-module by Theorem 7.6. Therefore,

**Lemma 9.3.** We have $m_{h_1, \ldots, h_n} \in \{0, 1\}$.

Now set

$$D := \{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n \mid m_{\beta_1/2, \ldots, \beta_n/2} \neq 0\}.$$

Then from (9.1) we see that

$$V^{(0^*)} = \bigoplus_{\beta = (\beta_1, \ldots, \beta_n) \in D} L_{\text{Vir}}(1/2, \beta_1/2) \otimes \cdots \otimes L_{\text{Vir}}(1/2, \beta_n/2).$$

By the fusion rule in Theorem 8.4, it is easy to see that $D$ is an even linear code.

To summarize, given a framed VOA $V$ with an Ising frame $\omega = e^1 + \cdots + e^n$, we have obtained a pair $(D, S)$ of binary codes of length $n$. These codes are called the structure codes of the framed VOA $V$. The structure codes satisfy the following.

**Proposition 9.4.** Let $V$ be a framed VOA and $(D, S)$ its structure codes.

1. For each $\alpha \in S$, its weight $|\alpha|$ is divisible by 8.
2. $D$ is an even linear code.
3. $S \subset D^\perp$.

**Proof:** (1) and (2) are obvious as $V$ has an $\mathbb{N}$-grading $V = \bigoplus_{n \in \mathbb{N}} V_n$. For $\alpha \in S$, the vertex operator map $Y(\cdot, z)$ restricted on $V^{(0^*)} \otimes V^\alpha$ is a $V^{(0^*)}$-intertwining operator.
of type $V^{(0^n)} \times V^\alpha \to V^\alpha$. In Section 8.1 we have constructed an $L_{Vir}(1/2, 0)$-intertwining operator $I^{1,+}(\cdot, z)$ of the type

$$L_{Vir}(1/2, 1/2) \times L_{Vir}(1/2, 1/16) \to L_{Vir}(1/2, 1/16)$$

(cf. Proposition 8.3). By the fusion rule in Theorem 8.4, any $L_{Vir}(1/2, 0)$-intertwining operator of this type is a linear multiple of $I^{1,+}(\cdot, z)$. Since the powers of $z$ in $I^{1,+}(\cdot, z)$ are contained in $1/2 + Z$ and those of $z$ in the vertex operator map $Y(\cdot, z)$ on $V$ are contained in $Z$, we have $\langle \alpha, D \rangle = 0$ by (2) of Proposition 6.14.

Given a framed VOA with a frame $\omega = e^1 + \cdots + e^n$, we have seen that we can associate its structure codes $(D, S)$. It is also shown in the previous subsection that $V$ is an $S$-graded extension of $V^{(0^n)}$ and $V^{(0^n)}$ is a $D$-graded extension of the Ising frame $Vir(e^1) \otimes \cdots \otimes Vir(e^n)$ of $V$. Here a natural question arises: “Can we recover the VOA structure on $V$ from the structure codes $(D, S)$?” It is not difficult to see that the answer is not true in general. However, if $S = 0$, then the answer is always true as we will see in the next subsection.

### 9.2 Code VOAs

Recall the simple SVOA $X = X^0 \oplus X^1 \simeq L_{Vir}(1/2, 0) \oplus L_{Vir}(1/2, 1/2)$ we constructed in Section 8.1 (cf. Theorem 8.1). Consider the tensor product SVOA $X^\otimes n$ of $n$ copies of SVOA $X$ as we constructed in Section 6.5. It is clear that the even part of $X^\otimes n$ is a framed VOA. Let $\omega = e^1 + \cdots + e^n$ be the Ising frame of $X^\otimes n$. Then $(X^0)^\otimes n = Vir(e^1) \otimes \cdots \otimes Vir(e^n)$ and as an $(X^0)^\otimes n$-module, $X^\otimes n$ is isomorphic to

$$X^\otimes n \simeq \bigoplus_{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_2} L_{Vir}(1/2, \alpha_1/2) \otimes \cdots \otimes L_{Vir}(1/2, \alpha_n/2).$$

For a subset $D$ of $\mathbb{Z}^n_2$, set

$$V_D := \bigoplus_{\alpha = (\alpha_1, \ldots, \alpha_n) \in D} L_{Vir}(1/2, \alpha_1/2) \otimes \cdots \otimes L_{Vir}(1/2, \alpha_n/2) \subset X^\otimes n.$$

If $D$ is a code, that is, a subspace of $\mathbb{Z}^n_2$, then as a subalgebra of $X^\otimes n$ the space $V_D$ forms a simple SVOA. Moreover, if $D$ is even, then $V_D$ is a simple VOA. Therefore, for any even linear code $D$, we can construct a framed VOA $V_D$ with structure codes $(D, 0)$.

**Proposition 9.5.** For any even linear code $D$, there exists a framed VOA with structure codes $(D, 0)$. It is unique up to isomorphism.

**Proof:** The existence is already proved as above. The uniqueness is a general consequence of simple current extensions (cf. [DM2][Y1]). Let $(V, Y(\cdot, z))$ be a framed VOA
with an Ising frame $\omega = e^1 + \cdots + e^n$ such that its structure codes are $(D, 0)$. Then there is a $D$-graded decomposition

$$V = \bigoplus_{a=(a_1, \ldots, a_n) \in D} V^a, \quad V^a = L_{\Vir}(l/2, \alpha_1/2) \otimes \cdots \otimes L_{\Vir}(l/2, \alpha_n/2),$$

as an $F := \Vir(e^1) \otimes \cdots \otimes \Vir(e^n)$-module. Assume that we have another vertex operator map $\tilde{Y}(\cdot, z)$ on $V$ such that $(V, \tilde{Y}(\cdot, z))$ is a framed VOA with structure codes $(D, 0)$ and its Ising frame is the same as that of $(V, Y(\cdot, z))$. By (2) of Proposition 6.14 and Theorem 8.4, the space of $F$-intertwining operators of type $V^\alpha \times V^\beta \to V^{\alpha + \beta}$ is one-dimensional for any $\alpha, \beta \in D$. Thus there are scalars $c(\alpha, \beta) \in \mathbb{C}$ such that

$$\tilde{Y}(x^\alpha, z)x^\beta = c(\alpha, \beta)Y(x^\alpha, z)x^\beta$$

for any $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$. Clearly $c(0_D, \alpha) = 1$ for all $\alpha \in D$ as $Y(1, z) = \tilde{Y}(1, z) = \id_{V_D}$. Since both $(V, Y(\cdot, z))$ and $(V, \tilde{Y}(\cdot, z))$ are simple, all $c(\alpha, \beta)$, $\alpha, \beta \in D$, are non-zero by (1) of Corollary 7.3. In the following, $x^\alpha$ denotes an arbitrary element of $V^\alpha$ for $\alpha \in D$. By the skew-symmetry property in Proposition 2.5, we have

$$\tilde{Y}(x^\alpha, z)x^\beta = e^{L(-1)}z\tilde{Y}(x^\beta, z)x^\alpha = c(\beta, \alpha)e^{L(-1)}zY(x^\beta, z)x^\alpha = c(\beta, \alpha)Y(x^\alpha, z)x^\beta$$

and we deduce

$$c(\alpha, \beta) = c(\beta, \alpha). \quad (9.2)$$

And by the locality there is a positive integer $N \in \mathbb{N}$ such that

$$(z_1 - z_2)^N Y(x^\alpha, z_1)Y(x^\beta, z_2)x^\gamma = (z_1 - z_2)^N Y(x^\beta, z_2)Y(x^\alpha, z_1)x^\gamma$$

and

$$(z_1 - z_2)^N \tilde{Y}(x^\alpha, z_1)\tilde{Y}(x^\beta, z_2)x^\gamma = (z_1 - z_2)^N \tilde{Y}(x^\beta, z_2)\tilde{Y}(x^\alpha, z_1)x^\gamma.$$ 

Comparing the equalities above we obtain

$$c(\alpha, \beta + \gamma)c(\beta, \gamma) = c(\beta, \alpha + \gamma)c(\alpha, \gamma). \quad (9.3)$$

By (9.2) and (9.3), we deduce the following 2-cocycle condition:

$$c(\alpha, \beta + \gamma)c(\beta, \gamma) = c(\alpha, \beta)c(\alpha + \beta, \gamma). \quad (9.4)$$

Therefore, $c(\cdot, \cdot)$ defines an abelian central extension of $D$ by $\mathbb{C}^*$. Since $\mathbb{C}$ is algebraically closed, such an extension splits by Theorem 3.2 of [Kar]. This is equivalent to saying that $c(\cdot, \cdot)$ is a 2-coboundary, that is, there is a map $t(\cdot) : D \to \mathbb{C}^*$ such that

$$c(\alpha, \beta) = \frac{t(\alpha)t(\beta)}{t(\alpha + \beta)}.$$
Since \( c(0_D, \alpha) = 1 \), we have \( t(0_D) = 1 \). Now define a linear isomorphism \( \psi \) from \((V, \tilde{Y}(\cdot, z))\) to \((V, Y(\cdot, z))\) by \( \psi(x^\alpha) = t(\alpha)x^\alpha \). Then

\[
\psi \left( \tilde{Y}(x^\alpha, z)x^\beta \right) = t(\alpha + \beta)\tilde{Y}(x^\alpha, z)x^\beta = t(\alpha + \beta)c(\alpha, \beta)Y(x^\alpha, z)x^\beta
\]

\[
= t(\alpha)t(\beta)Y(x^\alpha, z)x^\beta = Y(t(\alpha)x^\alpha, z)t(\beta)x^\beta
\]

\[
= Y(\psi(x^\alpha), z)\psi(x^\beta).
\]

It is obvious \( \psi(1) = 1 \) and \( \psi(\omega) = \omega \) as \( t(0_D) = 1 \). Therefore, \( \psi \) defines a VOA-isomorphism between \((V, Y(\cdot, z))\) and \((V, \tilde{Y}(\cdot, z))\).  

By the proposition above, the VOA \( V_D \) is uniquely determined as the framed VOA with structure codes \((D, 0)\). So \( V_D \) is referred to as the code VOA associated to code \( D \) (cf. \[M2\]).

Let \( D < \mathbb{Z}_2^n \) be an even linear code and \( V_D \) the associated code VOA. Take an arbitrary \( \gamma \in \mathbb{Z}_2^2 \). As we have constructed \( V_D \) as a subalgebra of \( V_{\mathbb{Z}_2^n} = X^\otimes n \), it is easy to see that

**Lemma 9.6.** The subspace

\[
V_{D+\gamma} := \bigoplus_{\beta=(\beta_1, \ldots, \beta_n) \in D+\gamma} L_{V_D}(\ell/2, \beta_1/2) \otimes \cdots \otimes L_{V_D}(\ell/2, \beta_n/2)
\]

is an irreducible \( V_D \)-submodule of \( V_{\mathbb{Z}_2^n} \).

We will call \( V_{D+\gamma} \) a coset type module. Let \( \{\gamma^1, \ldots, \gamma^r\} \) be the representatives of \( \mathbb{Z}_2^2/D \). As a \( V_D \)-module, we have a decomposition

\[
V_{\mathbb{Z}_2^n} = \bigoplus_{i=1}^r V_{D+\gamma^i}.
\]

Denote by \( Y_{V_{\mathbb{Z}_2^n}}(\cdot, z) \) the vertex operator map on the SVOA \( V_{\mathbb{Z}_2^n} \). Then the restriction of \( Y_{V_{\mathbb{Z}_2^n}}(\cdot, z) \) on \( V_{D+\gamma^i} \otimes V_{D+\gamma^j} \) realizes a \( V_D \)-intertwining operator of type \( V_{D+\gamma^i} \times V_{D+\gamma^j} \to V_{D+\gamma^i+\gamma^j} \).

**Lemma 9.7.** The space of \( V_D \)-intertwining operators of type \( V_{D+\gamma^i} \times V_{D+\gamma^j} \to V_{D+\gamma^i+\gamma^j} \) is one-dimensional, that is,

\[
\left( \begin{array}{cc}
V_{D+\gamma^i+\gamma^j} \\
V_{D+\gamma^i} & V_{D+\gamma^j}
\end{array} \right)_{V_D} = CY_{V_{\mathbb{Z}_2^n}}(\cdot, z)|_{V_{D+\gamma^i} \otimes V_{D+\gamma^j}}.
\]

**Proof:** It is clear that \( Y_{V_{\mathbb{Z}_2^n}}(\cdot, z)|_{V_{D+\gamma^i} \otimes V_{D+\gamma^j}} \) is a non-zero \( V_D \)-intertwining operator of the type above. As an \( F \)-module, we have a decomposition

\[
V_{D+\gamma} = \bigoplus_{\beta \in D+\gamma} V^\beta
\]

\[\text{This notation has nothing to do with the coset construction in Section 6.3.}\]
with \( V^\beta = L_{Vir}(\frac{1}{2}, \beta_1/2) \otimes \cdots \otimes L_{Vir}(\frac{1}{2}, \beta_n/2) \) for \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n \). Therefore, by a restriction, we have a natural linear map
\[
\Phi : \left( \begin{array}{cc} V_{D+\gamma_i+\gamma_j} & V_{D+\gamma_i} \\ V_{D+\gamma_i} & V_{D+\gamma_j} \end{array} \right)_{V_D} \rightarrow \left( \begin{array}{cc} V_{\gamma_i+\gamma_j} & V_{\gamma_i} \\ V_{\gamma_i} & V_{\gamma_j} \end{array} \right)_F.
\]

Since all the coset type modules above are irreducible \( V_D \)-modules, \( \Phi \) is injective by Proposition 7.2. By (2) of Proposition 6.14, the space of \( F \)-intertwining operators of type \( V_{\gamma_i} \times V_{\gamma_j} \rightarrow V_{\gamma_i+\gamma_j} \) is one-dimensional so that the assertion follows.

The representation theory of code VOAs is completed by Miyamoto [M3]. We state the following fact without proof (cf. [DGH][La1][M3][Y2]).

**Theorem 9.8.** A code VOA \( V_D \) associated to an even linear code \( D \) is regular.

Now consider irreducible modules over code VOAs. Let \( D \) be an even code of length \( n \) and \( M \) an irreducible \( V_D \)-module. Since \( F \) is regular, \( M \) is a direct sum of irreducible \( F \)-submodules. Take an irreducible \( F \)-submodule \( N \) of \( M \). By Proposition 6.13, \( N \) is isomorphic to
\[ L_{Vir}(\frac{1}{2}, h_1) \otimes \cdots \otimes L_{Vir}(\frac{1}{2}, h_n), \quad h_i \in \{0, 1/2, 1/16\}. \]

Now define the 1/16-word \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_2^n \) of \( N \) by \( \alpha_i = 1 \) if and only if \( h_i = 1/16 \). By the fusion rules in Theorem 8.4, the 1/16-word \( \alpha \) is independent of choice of an irreducible \( F \)-submodule \( N \) of \( M \). Therefore, we can define the 1/16-word of an irreducible \( V_D \)-module \( M \) by \( \alpha \).

**Proposition 9.9.** ([M3]) Let \( M \) be an irreducible \( V_D \)-module and \( \alpha \in \mathbb{Z}_2^n \) its 1/16-word. 
(1) \( \alpha \in D^\perp \).
(2) If \( \alpha = (0^n) \), then \( M \) is isomorphic to the coset type module \( V_{D+\gamma} \) for some \( \gamma \in \mathbb{Z}_2^n \).

**Proof:** Since the powers of \( z \) in the \( L_{Vir}(\frac{1}{2}, 0) \)-intertwining operator of type
\[ L_{Vir}(\frac{1}{2}, \frac{1}{2}) \times L_{Vir}(\frac{1}{2}, \frac{1}{16}) \rightarrow L_{Vir}(\frac{1}{2}, \frac{1}{16}) \]
are in \( \mathbb{Z} + 1/2 \) and those of \( z \) in the vertex operator map on \( M \) is in \( \mathbb{Z} \), the assertion (1) follows from (2) of Proposition 6.14. For the proof of the assertion (2), see [M3].

**Remark 9.10.** In [M3], Miyamoto classified all the irreducible \( V_D \)-modules. He also gave a construction of all irreducible \( V_D \)-modules having 1/16-word \( \alpha \) for all \( \alpha \in D^\perp \).

### 9.3 Hamming code VOA

In this subsection we consider the Hamming code VOA.
define

For simplicity, set

\[ \text{Consider a tensor product} \] X \[\text{or} \quad Z \]

Therefore, there are exactly 16 coset type

subsection, we assume that the VOA structure on \( X \) -intertwining operators of types

By the listed above, we note that all the coset type

Since \( \text{dim}_{\mathbb{Z}_2} H_8 = 4 \), there are \( 2^8/2^4 = 16 \) cosets in \( \mathbb{Z}_2^8/H_8 \) and we may choose the following coset representatives of \( \mathbb{Z}_2^8/H_8 \):

\[ \mathbb{Z}_2^8/H_8 = \{ \nu^j + H_8, \nu^i + \nu^j + H_8 \mid 1 \leq i, j \leq 8 \}. \quad (9.5) \]

Therefore, there are exactly 16 coset type \( V_{H_8} \)-modules:

\[ V_{H_8 + \nu^i + \nu^j}, \quad 1 \leq i, j \leq 8. \]

By the listed above, we note that all the coset type \( V_{H_8} \)-modules are graded by either \( Z \) or \( Z + 1/2 \).

Next consider irreducible \( V_{H_8} \)-modules having 1/16-word equal to \( (1^8) \). Recall the construction of \( X^0 \simeq L_{\text{Vir}}(1/2, 0) \), \( X^1 \simeq L_{\text{Vir}}(1/2, 1/2) \) and an irreducible \( X^0 = L_{\text{Vir}}(1/2, 0) \)-module \( \hat{X}^+ \simeq L_{\text{Vir}}(1/2, 1/16) \) in Section 8.1. In Proposition 8.3 we have also constructed \( X^0 \)-intertwining operators \( I^{i+} (r, z) \) of type \( X^i \times \hat{X}^+ \to \hat{X}^+, i \in \mathbb{Z}_2 \), which give \( L_{\text{Vir}}(1/2, 0) \)-intertwining operators of types

\[ L_{\text{Vir}}(1/2, i/2) \times L_{\text{Vir}}(1/2, 1/16) \to L_{\text{Vir}}(1/2, 1/16). \]

Consider a tensor product \( R := (\hat{X}^+) \otimes^8 \simeq L_{\text{Vir}}(1/2, 1/16)^{\otimes 8} \). By our construction, \( V_{H_8} \) has the \( H_8 \)-grading

\[ V_{H_8} = \bigoplus_{\alpha = (\alpha_1, \ldots, \alpha_8) \in H_8} X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_8}. \quad (9.6) \]

For simplicity, set

\[ V_{H_8}^\alpha := X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_8} \]

for \( \alpha = (\alpha_1, \ldots, \alpha_8) \in H_8 \). And for \( x = x^1 \otimes \cdots \otimes x^8 \in X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_8} = V_{H_8}^\alpha, \alpha \in H_8, \)

define

\[ J(x, z) := I^{\alpha_1+} (x^1, z) \otimes \cdots \otimes I^{\alpha_8+}(x^8, z) \]

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and extend linearly on $V_{H_8}$. Then by Theorem (1) of Proposition 6.14 and (2) of Proposition 8.3, $J(\cdot, z)|_{V_{H_8}^\alpha \otimes R}$ is an $(X^0)^{\otimes 8}$-intertwining operator of type $V_{H_8}^\alpha \times R \to R$. Moreover, $J(x, z), x \in V_{H_8}$, are mutually local fields on $R$ again by (2) of Proposition 8.3. Then we can show

**Lemma 9.12.** ([M3]) The pair $(R, J(\cdot, z))$ is an irreducible $V_{H_8}$-module.

Let $\omega = e^1 + \cdots + e^8$ be the canonical Ising frame of the code VOA $V_{H_8}$. Since $V_{H_8}$ is graded by integers, there is no $L_{Vir}(1/2, 1/16)$-component inside $V_{H_8}$. Therefore, all of $e^i$, $1 \leq i \leq 8$, define Miyamoto involutions of $\sigma$-type on $V_{H_8}$ by Theorem 8.10. It is easy to see that $\sigma(e^i)$ preserves the decomposition (9.6) and acts on each component $V_{H_8}^\alpha$ as $(-1)^{\alpha \cdot \omega \alpha}$. Therefore, the group $Q$ generated by $\{\sigma(e^i) \mid 1 \leq i \leq 8\}$ is isomorphic to an elementary abelian 2-group $\Z_2^8$. By Schur’s lemma, we see that

**Lemma 9.13.** All the conjugate modules $R^\mu, \mu \in Q$, are inequivalent irreducible $V_{H_8}$-modules.

Now we have constructed $16 + 16 = 32$ inequivalent irreducible $V_{H_8}$-modules whose top weights are in $\Z + 1/2$. For later purpose, we introduce the following notation:

$$H(0, \lambda) := V_{H_8 + \lambda}, \quad H(1, \mu) := R^\mu, \quad \lambda = \lambda + H_8 \in \Z_2^8 \otimes H_8, \quad \mu \in \Z_2^8 / H_8 \simeq Q, \quad (9.7)$$

where we have used the canonical isomorphism $Q \simeq \Z_2^8 / H_8$. Now we classify irreducible $V_{H_8}$-modules whose top weight are in $\Z + 1/2$.

**Theorem 9.14.** ([M3]) Every irreducible $V_{H_8}$-module whose top weight is in $\Z + 1/2$ is isomorphic to one of $H(i, \lambda), i \in \Z_2, \lambda \in \Z_2^8 / H_8$.

**Proof:** Let $(M, Y_M(\cdot, z))$ be an irreducible $V_{H_8}$-module whose top weight is in $\Z + 1/2$. Then the 1/16-word $\alpha_M \in \Z_2^8$ of $M$ is either $(0^8)$ or $(1^8)$. If $\alpha_M = (0^8)$, then by the fusion rule in Theorem 8.4 there exists $\beta = (\beta_1, \ldots, \beta_8) \in \Z_2^8$ such that $M$ contains a $V_{H_8}^{(0^8)} \simeq \otimes L_{Vir}(1/2, 0)^{\otimes 8}$-submodule $N$ of the form

$$L_{Vir}(1/2, \beta_1/2) \otimes \cdots \otimes L_{Vir}(1/2, \beta_8/2).$$

Then by Lemma 7.1 and Corollary 7.3, all $V_{H_8}^\alpha \cdot N, \alpha \in H_8$, are non-zero irreducible $V_{H_8}^{(0^8)}$-submodules and $M = V \cdot N = \sum_{\alpha \in H_8} V_{H_8}^\alpha \cdot N$ as $M$ is irreducible. Since $V_{H_8}^\alpha \cdot N, \alpha \in H_8$, are inequivalent irreducible $V_{H_8}^{(0^8)}$-submodules, $M = \oplus_{\alpha \in D} V_{H_8}^\alpha \cdot N$ so that $M$ as a $V_{H_8}^{(0^8)}$-module is isomorphic to $V_{H_8 + \beta}$. By adding suitable modification to the proof of Proposition 9.5, one can show the uniqueness of a $V_{H_8}$-module structure on $V_{H_8 + \beta}$. Thus, $M$ is isomorphic a coset type $V_{H_8}$-module $H(0, \beta + H_8)$ in this case.
If \( \alpha_M = (1^8) \), then \( M \) contains \( L_{\text{Vir}}(1/2, 1/16)^\otimes 8 \) as a \( V_{H_8}^{(0^8)} \)-module so that \( M \) is of the form

\[
M \simeq L_{\text{Vir}}(1/2, 1/16)^\otimes 8 \otimes P,
\]

where \( P \) denotes the space of multiplicity:

\[
P = \text{Hom}_{V_{H_8}^{(0^8)}}(L_{\text{Vir}}(1/2, 1/16)^\otimes 8, M).
\]

For each \( \alpha \in H_8 \), the space of \( V_{H_8}^{(0^8)} \)-intertwining operators of type

\[
V_{H_8}^{\alpha} \times L_{\text{Vir}}(1/2, 1/16)^\otimes 8 \to L_{\text{Vir}}(1/2, 1/16)^\otimes 8
\]

is one-dimensional by (2) of Proposition 6.14. Therefore, under the identification \( M \simeq R \otimes P \) as \( V_{H_8}^{(0^8)} \)-modules, there exist \( f(\alpha) \in \text{End}\mathbb{C}(P) \), \( \alpha \in H_8 \), such that for any \( x^\alpha \in V_{H_8}^{\alpha} \), the vertex operator map \( Y_M(\cdot, z) \) is of the form

\[
Y_M(x^\alpha, z) = Y_R(x^\alpha, z) \otimes f(\alpha).
\]

By Lemma 7.1 and Corollary 7.3, each \( f(\alpha) \) is invertible as \( P \) is of finite dimension. Since both \( Y_M(\cdot, z) \) and \( Y_R(\cdot, z) \) satisfy the Borcherds identity in Proposition 2.6, we can verify the following relation:

\[
f(\alpha)f(\beta) = f(\alpha + \beta).
\]

Therefore, \( P \) affords an action of the group algebra \( \mathbb{C}[H_8] \) via \( f : H_8 \ni \alpha \mapsto f(\alpha) \in \text{End}(P) \). Therefore, the irreducibility of \( M \) implies that \( P \) is one-dimensional and \( f \) is a linear character of \( H_8 \). Since dual group \( H_8^* \) of \( H_8 \) is canonically isomorphic to \( \mathbb{Z}_2^8/H_8 \), \( M \) is isomorphic to one of \( H(1, \delta + H_8) \), \( \delta \in \mathbb{Z}_2^8 \).

In the Hamming code VOA \( V_{H_8} \), we can find three Ising frames. Set \( u^0 = 1 \in X^0 \), \( u^1 = \psi_{-1/2} \mathbf{1} \in X^1 \) and for \( \alpha = (\alpha_1, \ldots, \alpha_8) \in \mathbb{Z}_2^8 \), set

\[
u^\alpha := u^{\alpha_1} \otimes \cdots \otimes u^{\alpha_8} \in V_{Z_2^8}.
\]

For \( \alpha \in \mathbb{Z}_2^8 \), set

\[
s(\alpha) := \frac{1}{8} \sum_{i=1}^8 e^i + \frac{1}{8} \sum_{|\beta| = 4 \in H_8} (-1)^{(\alpha, \beta)} w^\beta = \frac{1}{8} \omega + \frac{1}{8} \sum_{|\beta| = 4 \in H_8} (-1)^{(\alpha, \beta)} w^\beta \in V_{H_8}.
\]

**Lemma 9.15.** ([M2]/[M4]) For each \( \alpha \in \mathbb{Z}_2^8 \), \( s(\alpha) \) defines an Ising vector of \( V_{H_8} \).

**Proof:** By explicit calculations one can verify \( s(\alpha)_{(1)}s(\alpha) = 2s(\alpha) \) and \( s(\alpha)_{(3)}s(\alpha) = (1/4) \mathbf{1} \) so that \( s(\alpha) \) is a Virasoro vector with central charge 1/2 by Lemma 6.2. It remains to show that the Virasoro sub VOA \( \text{Vir}(s(\alpha)) \) is simple. Since the SVOA \( X \) has a real-form \( X_{\mathbb{R}} \) with positive definite bilinear form (cf. Section 8.1), a real-form of \( V_{H_8} \) inherits
a positive definite bilinear form. Since \( s(\alpha) \) is contained the real-form of \( \text{Vir}(\alpha) \), \( \text{Vir}(s(\alpha)) \) also possesses a positive definite bilinear form. Since an ideal of \( \text{Vir}(s(\alpha)) \) is contained in the radical of the bilinear form on \( \text{Vir}(s(\alpha)) \), the positivity implies that \( \text{Vir}(s(\alpha)) \) is a simple VOA. 

Since \( H_8 \) is self-dual, it is easy to see that \( s(\alpha) = s(\beta) \) if and only if \( \alpha - \beta \in H_8 \). Thus, by (9.5), there are 16 distinct Ising vectors of the form \( s(\alpha) \), \( \alpha \in \mathbb{Z}_2^8 \). We give the following fact without proof.

**Proposition 9.16.** (\([La1][MM][M4]\))

1. There are exactly 24 Ising vectors inside \( \text{Vir}(H_8) \) which are listed as follows:

   \[ \{ e^i, s(\nu^j), s(\nu^j + \nu^k) \mid 1 \leq i, j, k \leq 8 \}. \]

2. There are exactly three Ising frames inside \( \text{Vir}(H_8) \), given as follows:

   \[ \omega = \sum_{i=1}^{8} e^i = \sum_{j=1}^{8} s(\nu^j) = \sum_{k=1}^{8} s(\nu^j + \nu^k). \]

By the proposition above, we set:

\[ I_0 := \{ e^1, \ldots, e^8 \}, \quad F_0 := \text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^8), \]

\[ I_1 := \{ s(\nu^1), \ldots, s(\nu^8) \}, \quad F_1 := \text{Vir}(s(\nu^1)) \otimes \cdots \otimes \text{Vir}(s(\nu^8)), \]

\[ I_2 := \{ s(\nu^1 + \nu^1), \ldots, s(\nu^1 + \nu^8) \}, \quad F_2 := \text{Vir}(s(\nu^1 + \nu^1)) \otimes \cdots \otimes \text{Vir}(s(\nu^1 + \nu^8)). \]

The Ising frames inside \( \text{Vir}(H_8) \) are conjugate under \( \sigma \)-involutions.

**Lemma 9.17.** (\([M4][MM]\)) For \( u \in I_a \), \( \sigma(u)I_a = I_a \) and \( \sigma(u)I_b = I_c \), where \( a, b, c \) are such that \( \{a, b, c\} = \{0, 1, 2\} \). Moreover, if \( v \in I_b \), \( a \neq b \), then \( \sigma(u)v = \sigma(v)u \).

**Proof:** The proof is given by direct verification. Let \( u = e^i \in I_0 \). Then it is easy to show that \( \sigma(e^i)s(\alpha) = s(\alpha + \nu^i) \) for \( \alpha \in \mathbb{Z}_2^8 \). So \( \sigma(e^i)I_1 = I_2 \). Other cases can also be verified directly, but this is not a smart way; it needs long computations. There is a general identity on a VOA \( V \) with \( V_1 = 0 \) which will reduce the computations. For details, see \([M1]\) and \([MM]\). 

Let \( V \) be a VOA and \( e \in V \) an Ising vector. For any automorphism \( \rho \in \text{Aut}(V) \) we have \( \rho(\sigma(e)) = \sigma(\rho e) \rho \) as one can easily show. Then by the lemma above, we can deduce from \( \sigma(s(\nu^i))e^j = \sigma(e^j)s(\nu^i) = s(\nu^j + \nu^i) \) that \( \sigma(s(\nu^i))s(\nu^j + \nu^i) = \sigma(s(\nu^j + \nu^i))s(\nu^i) = e^{j(i,j)}. \)

Let us consider \( H(1, \lambda), \lambda \in \mathbb{Z}_2^8/H_8 \), as an \( F_i \)-module for \( i = 1, 2 \). It suffices to compute eigenvalues of \( u_{(1)}, u \in I_i \), on the top level of \( R = H(1, H_8) \), which is one-dimensional and spanned by the vector \( (\frac{1}{16})^++8 \) (cf. Section 8.1).
Lemma 9.18. (1) On the top level of $R$, the grade keeping operator $s(\alpha)_{(1)}$ of $s(\alpha)$ acts as 0 if $s(\alpha) = s((0)^8)$, as $1/2$ if $s(\alpha) = s(\nu^i + \nu^j)$, $2 \leq i \leq 8$, and as $1/16$ if $s(\alpha) = s(\nu^i)$, $1 \leq j \leq 8$.

(2) On the top level of $R$, the grade keeping operator $(\sigma(s(\alpha))w^\beta)_{(1)}$ of $\sigma(s(\alpha))w^\beta$, $\alpha \in \mathbb{Z}_2^8$, $\beta \in H_8 \setminus \{(0)^8, (1)^8\}$, acts as $1/4$ if $\langle \alpha, \alpha \rangle = 0$ and as $0$ if $\langle \alpha, \alpha \rangle = 1$.

Proof: Let $\beta \in H_8$ with $|\beta| = 4$. By the definition of the vertex operator map $J(\cdot, z)$ on $R$, the grade keeping operator $w^\beta_{(1)}$ of $w^\beta$ acts on $(\frac{1}{16})^+ \otimes \mathbb{Z}_2^8$ as $1/4$ since $\phi_0(\frac{1}{16})^+ = (1/\sqrt{2})|\frac{1}{16})^+$. Therefore, by (9.8), $s(\alpha)_{(1)}$ acts on $(\frac{1}{16})^+ \otimes \mathbb{Z}_2^8$ as

$$\frac{1}{8} \cdot \frac{1}{2} + \frac{1}{8} \sum_{\beta \in H_8 \atop |\beta| = 4} (-1)^{\langle \alpha, \beta \rangle} \cdot \frac{1}{4} = \frac{1}{16} + \frac{1}{32} \sum_{\beta \in H_8 \atop |\beta| = 4} (-1)^{\langle \alpha, \beta \rangle}.$$

The right hand side is equal to $1/2$ if $\alpha \equiv (0)^8 \mod H_8$, $0$ if $\alpha \equiv \nu^i + \nu^j \mod H_8$, $2 \leq i \leq 8$, and $1/16$ if $\alpha \equiv \nu^j \mod H_8$, $1 \leq j \leq 8$. Thus the assertion (1) holds. By a direct (but long) computation, we obtain

$$\sigma(s(\alpha))w^\beta = \frac{1}{2}w^\beta + \frac{1}{2}(-1)^{\langle \alpha, \alpha \rangle}w^{\beta + (1^8)} - \frac{1}{2}(-1)^{\langle \alpha, \beta \rangle} \left( \sum_{i \in \text{Supp}(\beta)} e^i - \sum_{j \notin \text{Supp}(\beta)} e^j \right).$$

Therefore, the assertion (2) also holds.

As a corollary, we can show the following amazing fact. For $i, j \in \{1, \ldots, 8\}$ we define $p(i, j)$ to be the unique $k \in \{1, \ldots, 8\}$ such that $\nu^k + \nu^i + \nu^j \in H_8$. Then

Proposition 9.19. ([M4]) We have the following conjugate isomorphisms:

(i) $H(0, \alpha + H_8)^{\sigma(e^i)} \simeq H(0, \alpha + H_8)$ for any $\alpha \in \mathbb{Z}_2^8$,

(ii) $H(0, \nu^i + \nu^j + H_8)^{\sigma(s(\alpha))} \simeq H(0, \nu^i + \nu^j + H_8)$ for any $\alpha \in \mathbb{Z}_2^8$,

(iii) $H(1, \alpha + H_8)^{\sigma(e^i)} \simeq H(1, \alpha + \nu^i + H_8)$ for any $\alpha \in \mathbb{Z}_2^8$,

(iv) $H(1, \nu^i + H_8)^{\sigma(s(\nu^i + \nu^j))} \simeq H(0, \nu^i + \nu^j + \nu^i + H_8) = H(0, \nu^{p(i,j)} + H_8)$,

(v) $H(1, \nu^i + \nu^j + H_8)^{\sigma(s(\nu^j))} \simeq H(0, \nu^i + \nu^j + \nu^i + H_8) = H(0, \nu^{p(i,j)} + H_8)$,

(vi) $H(1, \nu^i + H_8)^{\sigma(s(\nu^i))} \simeq H(1, \nu^i + H_8)$,

(vii) $H(1, \nu^i + \nu^j + H_8)^{\sigma(s(\nu^i + \nu^j))} \simeq H(1, \nu^i + \nu^j + H_8)$.

Proof: For (i), we know that for each $0 \neq \alpha \in \mathbb{Z}_2^8$, $V_{H_8} \oplus H(0, \alpha + H_8) = V_{H_8} \oplus V_{H_8 + \alpha}$ is a subalgebra of the SVOA $V_{\mathbb{Z}_2^8}$ and $\sigma(e^i)$ is a well-defined automorphism on $V_{H_8} \oplus V_{H_8 + \alpha}$ which keep the decomposition. Thus the isomorphism in (i) is manifest. Moreover, $\sigma(s(\beta))$, $\beta \in \mathbb{Z}_2^8$, are also well-defined on $V_{H_8} \oplus V_{H_8 + \nu^i + \nu^j}$, $2 \leq j \leq 8$, and keep the decomposition so that (ii) also follows. The assertion (iii) is just the definition of $H(1, \alpha + H_8)$. 

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Consider (iv). The proof is given by direct verification of computation of eigenvalues of grade keeping operators of $e^j$, $1 \leq j \leq 8$, on the top levels. The action of $e^j$ on $H(1, \nu^i + H_8)^{s(\alpha(i))}$ is by definition given by $\sigma(s(\alpha))e^j$ on $H(1, \nu^i + H_8)$. By Lemma 9.17, $\sigma(s(\alpha))e^j = \sigma(e^j)s(\alpha) = s(\alpha + \nu^j)$, so that the action of $e^j$ on $H(1, \nu^i + H_8)^{s(\alpha(i))}$ is equal to that of $s(\alpha + \nu^j)$ on $H(1, \nu^i + H_8)$. Since $H(1, \nu^i + H_8) = H(1, H_8)^{\sigma(e^j)}$, this action can be identified with that of $s(\alpha + \nu^j) = s(\alpha + \nu^j + \nu^j)$ on $H(1, H_8) = R$. Therefore, by (1) of Lemma 9.18, $H(1, \nu^i + H_8)^{s(\alpha(i))}$ is isomorphic to $H(0, \nu^j + H_8)$ if $\nu^i + \nu^j + \alpha \in H_8$. The proof of (v) is similar.

Consider (vi). By (1) of Lemma 9.18, $H(1, \nu^j + H_8)^{s(\nu^j)} \simeq H(1, \lambda + H_8)$ for some $\lambda \in \mathbb{Z}_2$ so that we have to determine eigenvalues of grade keeping operators of $w^\beta$, $\beta \in H_8 \setminus \{(0^8), (1^8)\}$, on the top levels. Since $H(1, \nu^j + H_8) = H(1, H_8)^{\sigma(e^j)}$ and $\sigma(e^j)^\prime(s(\nu^j)) = \sigma(s(\nu^j + \nu^j))\sigma(e^j)$, we have the following isomorphism:

$$H(1, \nu^j + H_8)^{s(\nu^j)} \simeq H(1, H_8)^{\sigma(s(\nu^j + \nu^j))\sigma(e^j)}.$$ 

Therefore, the action of $w^\beta_{(1)}$ on $H(1, \nu^j + H_8)^{s(\nu^j)}$ is equivalent to that of the grade keeping operator of $(-1)^{[\nu^j,\beta]}\sigma(s(\nu^j + \nu^j))w^\beta$ on $H(1, H_8)$. By (2) of Lemma 9.18, this is equal to $1/4$ if $i \not\in \text{Supp}(\beta)$ and to $-1/4$ if $i \in \text{Supp}(\beta)$. Therefore, the assertion follows. The proof of (vii) is similar.

We need the following fact to determine certain fusion rules of $V_{H_8}$-modules.

**Proposition 9.20.** ([M3]) On $H(1, H_8) \oplus H(1, \nu^i + H_8)$, there exists a unique irreducible $V_{H_8}(H_8 + \nu^i)$ module structure. In particular, there exists a non-zero $V_{H_8}$-intertwining operator of type $H(0, \nu + H_8) \times H(1, H_8) \rightarrow H(1, \nu^i + H_8)$.

Now we can determine the following fusion rules:

**Theorem 9.21.** ([M4]) The fusion rules of irreducible $V_{H_8}$-modules with integral or half-integral top weights are as follows:

$$[H(i, \alpha + H_8)] \times [H(j, \beta + H_8)] = [H(i + j, \alpha + \beta + H_8)] \quad i, j \in \mathbb{Z}_2, \alpha, \beta \in \mathbb{Z}_2^8.$$ 

**Proof:** First, consider the fusion rule $[H(0, \alpha + H_8)] \times [H(0, \beta + H_8)]$. Let $M$ be an irreducible $V_{H_8}$-module such that there is a non-zero $V_{H_8}$-intertwining operator of type $H(0, \alpha + H_8) \times H(0, \beta + H_8) \rightarrow M$.

Then by the fusion rules in Theorem 8.4, $M$ as a $V_{H_8}(\nu^i)$-module contains an irreducible $V_{H_8}(\nu^i)$-submodule of the form

$$L_{V_{H_8}}(t/2, (\alpha + \beta)_{1/2}) \otimes \cdots \otimes L_{V_{H_8}}(t/2, (\alpha + \beta)_8/2).$$
Then, as we showed in the proof of Theorem 9.14, $M$ is isomorphic to $H(0, \alpha + \beta + H_8)$. Since the space of $V_{H_8}$-intertwining operators of type

$$H(0, \alpha + H_8) \times H(0, \beta + H_8) \rightarrow H(0, \alpha + \beta + H_8)$$

is one-dimensional by Lemma 9.7, we obtain the following fusion rules:

$$[H(0, \alpha + H_8)] \times [H(0, \beta + H_8)] = [H(0, \alpha + \beta + H_8)].$$

The remaining cases are easily verified by using Corollary 7.10. Consider $[H(1, \nu^j + H_8)] \times [H(1, \nu^i + H_8)]$. By Proposition 9.19, we compute

$$[H(1, \nu^i + H_8) ] \times [H(1, \nu^i + H_8)]$$

$$= [H(0, \nu^i + H_8) \sigma(s(\nu^i+\nu^i)) \times [H(1, \nu^j + H_8)]$$

$$= \left\{ [H(0, \nu^i + H_8)] \times [H(1, \nu^j + H_8)] \sigma(s(\nu^i+\nu^j)) \right\}$$

$$= \left\{ [H(0, \nu^i + H_8)] \times [H(0, \nu^{p(i,j)} + H_8)] \sigma(s(\nu^i+\nu^j)) \right\}$$

$$= [H(0, \nu^i + \nu^{p(i,j)} + H_8)]$$

$$= [H(0, \nu^i + \nu^j + H_8)].$$

Similarly, we can show that $[H(1, \nu^i + \nu^i + H_8)] \times [H(1, \nu^i + \nu^j + H_8)] = [H(0, \nu^i + \nu^j + H_8)].$

Now, by Lemma 5.7, we know that all $H(i, \alpha + H_8)$, $i \in \mathbb{Z}_2$, $\alpha \in \mathbb{Z}_8^*$, are simple current $V_{H_8}$-modules as $[H(i, \alpha + H_8)] \times [H(i, \alpha + H_8)] = [H(0, H_8)] = [V_{H_8}].$

Consider the fusion rules of type $[H(0, \nu^i + \nu^j + H_8)] \times [H(1, H_8)]$. By considering the 1/16-word, we see that there exists $\gamma \in \mathbb{Z}_8^*$ such that

$$[H(0, \nu^i + \nu^j + H_8)] \times [H(1, H_8)] = [H(1, \gamma + H_8)].$$

If $\gamma \equiv \nu^j \mod H_8$ for some $1 \leq j \leq 8$, then

$$\{ [H(0, \nu^i + \nu^i H_8)] \times [H(1, H_8)] \} \sigma(s(\nu^j + H_8)) = [H(1, \nu^j + H_8)] \sigma(s(\nu^j + H_8)) = [H(0, \nu^j + H_8)].$$

But the left hand side is equal to

$$[H(0, \nu^i + \nu^i + H_8)] \sigma(s(\nu^i + H_8)) \times [H(1, H_8)] \sigma(s(\nu^i + H_8))$$

$$= [H(0, \nu^i + \nu^j + H_8)] \times [H(1, H_8)]$$

$$= [H(1, \nu^j + H_8)],$$

which yields a contradiction. Therefore, $\gamma \equiv \nu^i \equiv \nu^j$ for some $1 \leq k \leq 8$. Since the fusion algebra is associative by Theorem 5.5, by considering

$$[H(0, \nu^i + \nu^j + H_8)] \times [H(1, H_8)] \times [H(1, H_8)] = [H(1, \gamma + H_8)] \times [H(1, H_8)],$$

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we see that \([H(0, \nu^1 + \nu^i + H_8)] = [H(0, \gamma + H_8)]\) and hence \(\gamma \equiv \nu^1 + \nu^i\). So we have obtained
\[
[H(0, \nu^1 + \nu^i + H_8)] \times [H(1, H_8)] = [H(1, \nu^1 + \nu^i + H_8)].
\]
By Proposition 9.20, we also know that
\[
[H(0, \nu^i + H_8)] \times [H(1, H_8)] = [H(1, \nu^i + H_8)].
\]
Now by using (i) and (iii) of Proposition 9.19, we get the following fusion rules:
\[
[H(0, \alpha + H_8)] \times [H(1, \beta + H_8)] = [H(1, \alpha + \beta + H_8)], \quad \alpha, \beta \in \mathbb{Z}_8^2.
\]
The associativity of the fusion algebra implies the remaining case
\[
[H(1, \nu^i + H_8)] \times [H(1, \nu^1 + \nu^j + H_8)] = [H(0, \nu^1 + \nu^i + \nu^j + H_8)]
\]
\[
= [H(0, \nu^i + H_8)] \times [H(1, H_8)] \times [H(1, \nu^1 + \nu^j + H_8)]
\]
\[
= [H(0, \nu^i + H_8)] \times [H(0, \nu^1 + \nu^j + H_8)]
\]
\[
= [H(0, \nu^i + \nu^j + H_8)].
\]
This completes the proof of Theorem 9.21.

### 9.4 Construction of framed VOAs

In this section we present a method of constructing framed VOAs. As we have seen in Section 9.2, given an even linear code \(D\) of \(\mathbb{Z}_8^2\), we can always construct a framed VOA whose structure codes are \((D, \{(0^n)\})\). Given a pair \((D, S)\) of even linear codes satisfying the conditions in Proposition 9.4, we do not have a method to construct a framed VOA with structure codes \((D, S)\) in general. However, if \(D\) contains “many” subcodes isomorphic to the Hamming code \(H_8\), then we can construct framed VOAs.

Before we state a general result, we consider simple cases.

First, consider the Hamming code VOA \(V_{H_8}\) and its irreducible module \(H(1, \alpha + H_8)\), \(\alpha \in \mathbb{Z}_8^2\). It is shown in Proposition 9.19 that there exists an automorphism \(\sigma \in \text{Aut}(V_{H_8})\) such that \(H(1, \alpha + H_8)\sigma \simeq V_{H_8 + \nu^i}\). We know that \(V_{H_8} \oplus V_{H_8 + \nu^i}\) is a sub SVOA of the SVOA \(V_{
abla 8}^2\) so that by conjugation by \(\sigma\) we can introduce the SVOA structure on \(V_{H_8} \oplus H(1, \alpha + H_8)\) which is isomorphic to \(V_{H_8} \oplus V_{H_8 + \nu^i}\).

Let \(D\) be a direct sum \(H_8^\oplus r\) of copies of the Hamming code and \(V_{D}\) the associated code VOA. Then \(V_{D}\) is isomorphic to a tensor product \(V_{H_8}^\oplus r\). Let \(U\) be an irreducible \(V_{D}\)-module which is isomorphic to
\[
H(0, \alpha_1 + H_8) \otimes \cdots \otimes H(0, \alpha_s + H_8) \otimes H(1, \alpha_{s+1} + H_8) \otimes \cdots \otimes H(1, \alpha_r + H_8), \quad \alpha_i \in \mathbb{Z}_8^2.
\]
as a $V_{H_8}^\otimes r$-module. We also assume that the top weight of $U$ is integral. By Proposition 9.19, there exists an automorphism $\rho \in \text{Aut}(V_D)$ such that the conjugate module $U^\rho$ is isomorphic to

$$H(0, \alpha_1 + H_8) \otimes \cdots \otimes H(0, \alpha_s + H_8) \otimes H(0, \beta_{s+1} + H_8) \otimes \cdots \otimes H(1, \beta_r + H_8)$$

as a $V_{H_8}^\otimes r$-module. Then $U^\rho$ is isomorphic to the coset type module $V_{H_8}^{\otimes r + (\alpha_1, \ldots, \alpha_s, \beta_{s+1}, \ldots, \beta_r)}$ so that there is a unique VOA structure on $V_D \oplus U^\rho$ isomorphic to the code VOA $V_{H_8}^{\oplus (H_8 \oplus (\alpha_1, \ldots, \alpha_s, \beta_{s+1}, \ldots, \beta_r))}$. Therefore, by conjugation by $\rho \in \text{Aut}(V_D)$, we can introduce a structure of a framed VOA on $V_D \oplus U$ with structure codes $(D, \{(0^{br}), (0^{b_s}1^{8(r-s)})\})$.

In this way, by using the symmetries of the fusion algebra associated to the Hamming code VOA $V_{H_8}^{\oplus r}$, we can construct framed VOAs.

**Assumption 1.** The data $\{(D, S), \{V_\alpha \mid \alpha \in S\}\}$ are as follows:

1. $(D, S)$ is a pair of even linear codes of $\mathbb{Z}_2^n$ such that
   
   (1-i) $D \subset S^1$,
   
   (1-ii) For each $\alpha \in S$, there is a subcode $E_\alpha$ such that $E_\alpha$ is a direct sum of the Hamming code $H_s$ and $\text{Supp}(E_\alpha) = \text{Supp}(\alpha)$, where $\text{Supp}(A) = \bigcup_{\beta \in A} \text{Supp}(\beta)$ for a subset $A$ of $\mathbb{Z}_2^n$.

2. $V^{(0r)}$ is the code VOA associated to the code $D$.

3. $\{V_\alpha \mid \alpha \in S\}$ is a set of irreducible $V^{(0r)}$-modules such that
   
   (3-i) The 1/16-word of $V_\alpha$ is equal to $\alpha$,
   
   (3-ii) all $V_\alpha$, $\alpha \in S$, have integral top weights,
   
   (3-iii) $(V_\alpha \circ V_\beta)_{V^{(0r)}} \neq 0$ for $\alpha, \beta \in S$.

Under Assumption 1 we can prove that the space $\oplus_{\alpha \in S} V_\alpha$ forms a structure of a framed VOA with structure codes $(D, S)$. Since the proofs needs deep results on VOA theory, we give only results here and omit their proofs. For interested readers, refer to [M4] and [Y1].

The first step is to show that all $V_\alpha$, $\alpha \in S$, are simple current $V^{(0r)}$-modules under Assumption 1.

**Lemma 9.22.** ([M4]/[Y1]) Under Assumption 1, all $V_\alpha$, $\alpha \in S$, are inequivalent simple current $V^{(0r)}$-modules and we have the fusion rules $[V_\alpha] \times [V_\beta] = [V_{\alpha+\beta}]$ for $\alpha, \beta \in S$.

Using the lemma above, we can prove the following.

**Theorem 9.23.** ([M4]/[Y1]) Under Assumption 1, the space $V := \oplus_{\alpha \in S} V_\alpha$ has a unique structure of a framed VOA with structure codes $(D, S)$.  

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Remark 9.24. In [M4], Miyamoto assumed stronger conditions than those in Assumption 1. In particular, he assumed that the length of structure codes is a multiple of 8. A refinement in [Y1] removes this restriction and enables us to construct framed VOAs with structure codes of any length as long as Assumption 1 is satisfied. This refinement is actually used in [Y3]. In [Y3], the author considered a framed VOA with structure codes of length 47 which has the baby-monster sporadic finite simple group as its full-automorphism group.

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