Non left-orderable surgeries and generalized Baumslag-Solitar relators

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**$L$-space Conjecture**

**$L$-space Conjecture [Boyer-Gordon-Watson, 2011]**

$M$: an irreducible rational homology sphere

$M$ is an $L$-space if and only if $\pi_1(M)$ is not LO

**left-orderability**

A non-trivial group $G$ is called **left-orderable (LO)** if $\exists <$: a strict total order on $G$ which is left invariant:

$$g < h \quad \rightarrow \quad fg < fh \quad \text{for } \forall f, g, h \in G$$

**$L$-space**

A rational homology sphere $M$ is called an **$L$-space** if $\text{rk}\hat{HF}(M) = |H_1(M; \mathbb{Z})|$ holds for $\hat{HF}(M)$: Heegaard Floer homology.
Dehn surgery

Dehn surgery is one of the simple ways to construct $L$-spaces.

The following operation to obtain another 3-manifold from a given 3-manifold is called a **Dehn surgery**.

$K$: a knot in a 3-manifold $M$

**Dehn surgery on $K$**

1. remove an open regular neighborhood of $K$ from $M$ (drilling)
2. glue a solid torus $V$ back along a slope $p/q$ (Dehn filling)
**Left-orderable surgery and $L$-space surgery**

$K$: a knot in 3-sphere $S^3$

$K(p/q)$: a 3-manifold obtained by Dehn surgery on $K$ along the slope $p/q$

**left-orderable surgery**

A Dehn surgery on $K$ is called a **non left-orderable surgery** if it yields a closed 3-manifold with $\pi_1(K(p/q))$ is non left-orderable.

**$L$-space surgery**

A Dehn surgery on $K$ is called an **$L$-space surgery** if it yields a closed 3-manifold which is an $L$-space.

**Question**

Which knots in $S^3$ have non-LO and/or $L$-space surgery?
Known results - Pretzel knots -

Theorem [Lidman-Moore, preprint (arXiv:1306.6707v1)]

For $s \geq 3$, only $(-2, 3, 2s + 1)$-pretzel knots have $L$-space surgeries among hyperbolic pretzel knots.

Hence, if $L$-space Conjecture is true, among hyperbolic pretzel knots, only $(-2, 3, 2s + 1)$-pretzel knots would have non-LO surgeries.
Known results - Pretzel knots -

Theorem [Nakae, Clay-Watson, 2013]
For $s \geq 3$, $(-2, 3, 2s + 1)$-pretzel knots have non left-orderable surgeries.

Corollary
If a $(-2, 3, 2s + 1)$-pretzel knot has an $L$-space surgery, then it has a non left-orderable surgery.

Remark: It is still open whether the opposite statement holds.
Main Theorem

As an extension of Nakae’s result, we have:

**Theorem [Ichihara-Temma, 2014]**

Let $K$ be a knot in a 3-manifold $M$. Suppose that $\pi_1(M - K)$ has a presentation such as

$$\langle a, b | (w_1 a^m w_1^{-1})b^{-r}(w_2^{-1} a^n w_2)b^{r-k}\rangle$$

with $m, n \geq 0$, $r \in \mathbb{Z}$, $k \geq 0$, and $a$: a meridian of $K$. Suppose that the longitude of $K$ is represented as $a^{-s}w a^{-t}$

with $s, t \in \mathbb{Z}$ and $w$ is a word without $a^{-1}, b^{-1}$.

If $q \neq 0$ and $p/q \geq s + t$, then Dehn surgery on $K$ along the slope $p/q$ yields a closed 3-manifold with $\pi_1(K(p/q))$ is non left-orderable.
**Remark:**
The relator in the presentation in Theorem can be regarded as a generalization of the well-known **Baumslag-Solitar relator**.

The Baumslag-Solitar relator

is the relator $x^{-n}yx^my^{-1}$ with $m, n \neq 0$ in the group generated by $x, y$.

It plays an important role and is well-studied in combinatorial group theory and geometric group theory. For example;

**Theorem [Shalen, 2001]**
The Baumslag-Solitar relator cannot appear in the fundamental group of an orientable 3-manifold.
Known results - Twisted Torus knots -

Note:

$(-2, 3, 2s + 1)$-pretzel knots = twisted torus knots $K(3, 5; 2, s - 2)$.

Twisted torus knot $K(3, -4; 2, 2)$
**Corollary**

**Known results - Twisted Torus knots -**

**Theorem [Vafaee, 2014]**

For $p \geq 2$, $k \geq 1$, $r > 0$ and $0 < s < p$,

$K(p, kp \pm 1; s, r)$ has an $L$-space surgery if and only if either $s = p - 1$ or $s \in \{2, p - 2\}$ and $r = 1$.

**Corollary**

$K(3, q; 2, s)$ has an $L$-space surgery if $q > 0$ and $s \geq 1$.

**Theorem [Clay-Watson, 2013]**

$K(3, 3k + 2; 2, s)$ has a non left-orderable surgery if (1) $k \geq 0$ and $s = 1$, or (2) $k = 1$ and $s \geq 0$. 
Corollary [Ichihara-Temma, 2014]

For \( k, s \geq 0 \), \( K(3, 3k + 2; 2, s) \) has a non left-orderable surgery.

Precisely \( \pi_1(K(p/q)) \) is non left-orderable if \( p/q \geq 3(3k + 2) + 2s \).
Recent extensions

Our results have been extended as follows.

Theorem (Christianson-Goluboff-Hamann-Varadaraj)
For $p, k, s > 0$, $K(p, pk \pm 1; p - 1, s)$ and $K(p, pk \pm 1; p - 2, 1)$ have non left-orderable surgeries.

This is obtained in Columbia University math REU program by undergraduates.

Corollary
For $s > 0$, $K(3, q; 2, s)$ have non left-orderable surgeries.

Corollary
If $K(3, q; 2, s)$ has an $L$-space surgery, then it has a non left-orderable surgery.
Question

How about "negatively" twisted cases? i.e., the cases that $s < 0$?

Theorem [Motegi, 2014]

For $p > q \geq 2$ and $s \geq -1$, $K(p, q; p - q, s)$ has an $L$-space surgery.

Corollary [Ichihara-Temma, 2014]

For $k \geq 0$, $s \geq -1$, $K(3, 3k + 2; 2, s)$ has a non left-orderable surgery.
Left-orderability

The following is well-known for experts:

**Theorem**

A countable group $G$ is left-orderable if and only if $G$ is isomorphic with a subgroup of $Homeo^+(\mathbb{R})$.

Set $G := \pi_1(K(p/q))$.

Let us consider a homomorphism $G \to Homeo^+(\mathbb{R})$.

Abusing notations, we will confuse the image of $g \in G$ and $g$. 
Sample calculations

\[ w_1 a^m w_1^{-1} b^{-r} w_2^{-1} a^n w_2 b^{r-k} = 1 \]

\[ \Rightarrow a^n w_2 b^{r-k} w_1 a^m = w_2 b^r w_1 \]

Assume: \( x < ax \) for any \( x \in \mathbb{R} \)

\[ a^n w_2 b^{r-k} w_1 a^m x = w_2 b^r w_1 x \]

\[ < w_2 b^r w_1 a^m x \]

\[ < a^n w_2 b^r w_1 a^m x \]

\[ b^{r-k} x < b^r x \Rightarrow x < b^k x \Rightarrow \boxed{ x < bx } \quad (\forall x \in \mathbb{R}) \]