Uniform hyperbolicity for curve graphs of nonorientable surfaces

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Theorem 1.1 (Hensel-Przytycki-Webb 2013)

\(S\): an orientable surface

\(\mathcal{C}(S)\): the curve graph of \(S\)

If \(\mathcal{C}(S)\) is connected, then it is \(17\)-(Gromov) hyperbolic.
Theorem 1.1 (Hensel-Przytycki-Webb 2013)

$S$: an orientable surface

$C(S)$: the curve graph of $S$

If $C(S)$ is connected, then it is 17-(Gromov) hyperbolic.

Theorem 1.2 (Bestvina-Fujiwara 2007, Masur-Schleimer 2013)

$N$: a nonorientable surface

$C(N)$: the curve graph of $N$

Each $C(N)$ is Gromov hyperbolic.
Question 1.3

Is $\mathcal{C}(N)$ uniformly hyperbolic?
How large is the hyperbolicity constant?
Is $C(N)$ uniformly hyperbolic?
How large is the hyperbolicity constant?

We try applying Hensel-Przytycki-Webb’s argument to the case of nonorientable surfaces!
Question 1.3

Is $C(N)$ uniformly hyperbolic?
How large is the hyperbolicity constant?

We try applying Hensel-Przytycki-Webb’s argument to the case of nonorientable surfaces!

[Reference]
S. Hensel, P. Przytycki, and R. C. H. Webb,
Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs,
$N = N_{g,n}$: the **nonorientable surface** of genus $g$ with $n$ boundary components
Arcs on $N$: properly embedded and essential, i.e. is not homotopic into $\partial N$

curves on $N$: properly embedded and essential, i.e. does not bound a disk or a Möbius band, and is not homotopic into $\partial N$
Definition 2.1

**Arc curve graph** $\mathcal{AC}(N)$

The vertex set $\mathcal{AC}^{(0)}(N) = \{ \text{homotopy classes of arcs and curves on } N \}$

$(a, b): \text{edge in } \mathcal{AC}(N)$, $(a, b \in \mathcal{AC}^{(0)}(N))$

$\iff$ the corresponding arcs or curves can be realized disjointly.

**Arc graph** $\mathcal{A}(N)$

$\iff$ the subgraph of $\mathcal{AC}(N)$ consists of homotopy classes of arcs on $N$

**Curve graph** $\mathcal{C}(N)$

$\iff$ the subgraph of $\mathcal{AC}(N)$ consists of homotopy classes of curves on $N$.

We consider the graphs as geodesic spaces.
A graph is **hyperbolic** if there exists a constant $k \geq 0$ (we call this constant the **hyperbolicity constant**), such that for all geodesic triangles $T = abd$ ($a, b, d$: vertices) in the space, there exists a vertex $p$ such that distances between edges of the triangle and $p$ are at most $k$ ($p$ is called the $k$-center of $T$).
Definition 2.2

**Uniformly hyperbolic**

$\iff$ the hyperbolicity constant does not depend on topological types of surfaces.
Definition 2.2

**Uniformly hyperbolic**

\[ \Leftrightarrow \text{the hyperbolicity constant does not depend on topological types of surfaces.} \]

The important point of Main theorem

\( \mathcal{C}(N) \) is 17-hyperbolic

independently of the topological types of \( N \)!
**Definition of unicorn paths**

\[ \overline{\alpha \alpha'}_a: \text{the subarc of an arc } a \text{ whose endpoints are } \alpha \text{ and } \alpha' \]

**Definition 3.1 (Unicorn arc)**

- \(a, b\): arcs on \(N\) which are in minimal position
- \(\alpha, \beta\): one of the endpoints of \(a\) and \(b\)
- \(\pi \in a \cap b\)
- \(a' \subset a\): the arc whose endpoints are \(\alpha\) and \(\pi\)
- \(b' \subset b\): the arc whose endpoints are \(\beta\) and \(\pi\)

When \(a' \cup b'\) is an embedded arc in \(N\), we say that \(a' \cup b'\) is a **unicorn arc** obtained from \(a^\alpha, b^\beta\) and \(\pi\).
Definition of unicorn paths

Example
Definition 3.2 (Order)

\[ a' \cup b', \quad a'' \cup b'' : \text{two unicorn arcs obtained from } a^\alpha, b^\beta \]
\[ a' \cup b' \leq a'' \cup b'' \iff a'' \subseteq a' \text{ and } b' \subseteq b'' \]
Definition of unicorn paths

Definition 3.2 (Order)

\[ a', b', \quad a'', b'': \text{two unicorn arcs obtained from } a^\alpha, b^\beta \]
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\[ \leq \text{ is total order.} \]
**Definition 3.2 (Order)**

\[ a' \cup b', \ a'' \cup b'': \text{two unicorn arcs obtained from } a^\alpha, b^\beta \]
\[ a' \cup b' \leq a'' \cup b'' \iff a'' \subseteq a' \text{ and } b' \subseteq b'' \]

\( \leq \) is total order.

\( (c_1, c_2, \ldots, c_{n-1}) \): ordered set of unicorn arcs obtained from \( a^\alpha, b^\beta \)

**Definition 3.3 (Unicorn path)**

\[ \mathcal{P}(a^\alpha, b^\beta) := (a = c_0, c_1, \ldots, c_{n-1}, c_n = b) \]

is called the \textbf{unicorn path} between \( a^\alpha \) and \( b^\beta \).
Example of a unicorn path

\[ P(\alpha^a, \beta^b) := (a, \overline{\alpha \pi_3 a} \cup \pi_3 \beta_b, \overline{\alpha \pi_4 a} \cup \pi_4 \beta_b, \overline{\alpha \pi_5 a} \cup \pi_5 \beta_b, b). \]
Unicorn paths are paths in arc graphs

\[ a, b: \text{two vertices of } A(N) \]

**Proposition 3.4**

*Unicorn path \( P(a^\alpha, b^\beta) \) is a path connecting two vertices \( a \) and \( b \) in \( A(N) \).*
Unicorn paths are paths in arc graphs

\( a, b: \) two vertices of \( \mathcal{A}(N) \)

**Proposition 3.4**

Unicorn path \( \mathcal{P}(a^\alpha, b^\beta) \) is a path connecting two vertices \( a \) and \( b \) in \( \mathcal{A}(N) \).

**Proof**
Theorem 4.1 (K.)

If $C(N)$ is connected, then it is 17-hyperbolic.
Theorem 4.1 (K.)

If \( C(N) \) is connected, then it is 17-hyperbolic.

Hensel-Przytycki-Webb’s hyperbolicity constant does not depend on the topological types of surfaces!

Note

\( C(N) \) is connected if \( g = 1, 2 \) and \( g + n \geq 5 \), and \( g \geq 3 \).
Define a retraction \( r : \mathcal{A}C(N) \rightarrow \mathcal{C}(N) \) as follows.

If \( a \in \mathcal{C}^{(0)}(N) \), then \( r(a) = a \).

If \( a \in \mathcal{A}^{(0)}(N) \), we assign a boundary component of a regular neighborhood of its union with \( \partial N \) to \( r(a) \).
Define a retraction \( r : \mathcal{A}C(N) \to \mathcal{C}(N) \) as follows. If \( a \in \mathcal{C}^{(0)}(N) \), then \( r(a) = a \). If \( a \in \mathcal{A}^{(0)}(N) \), we assign a boundary component of a regular neighborhood of its union with \( \partial N \) to \( r(a) \).

Note

If there are two boundary components of the regular neighborhood, then we choose the essential one.
Construction of a retraction

Similar to the case of orientable surfaces, we can show the following claim.

Claim 4.3

The difference from the case of orientable surfaces

We have to consider "twisted" $r(a)$ and $r(b)$.
Similar to the case of orientable surfaces, we can show the following claim.

Claim 4.3

\( r \) is 2-Lipschitz, i.e. \( d_c(r(a), r(b)) \leq 2d_{AC}(a, b) \) for any \( a, b \in AC(N) \).
Similar to the case of orientable surfaces, we can show the following claim.

**Claim 4.3**

$r$ is 2-Lipschitz,

i.e. $d_C(r(a), r(b)) \leq 2d_{AC}(a, b)$ for any $a, b \in AC(N)$.

The difference from the case of orientable surfaces

We have to consider "twisted" $r(a)$ and $r(b)$.
Construction of a retraction

\[ \gamma_1 \quad r(a) \quad \gamma_2 \]

\[ \gamma_1 \quad a \quad r(a) \quad \gamma_2 \]
Outline of Proof of Claim 4.3

We prove only for the pairs \((a; b)\) \(\in AC(N)\) which fill \(\text{d} AC(a; b) = 1\).

The goal: to show \(\text{d}C(r(a); r(b)) = 1\).

(1) \(a; b \in C(0)(N)\) \(\text{d}C(r(a); r(b)) = \text{d}C(a; b) = \text{d}AC(a; b) = 1\).

(2) \(a \in C(0)(N)\) and \(b \in A(0)(N)\).

We can take the regular neighborhood of the union of \(b\) and boundary components which have endpoints of \(b\) without intersecting \(a\).
Outline of Proof of Claim 4.3

We prove only for the pairs \((a, b) \ (a, b \in \mathcal{AC}(N))\) which fill \(d_{\mathcal{AC}}(a, b) = 1\).
Outline of Proof of Claim 4.3

We prove only for the pairs \((a, b) \ (a, b \in \mathcal{AC}(N))\) which fill \(d_{\mathcal{AC}}(a, b) = 1\).

The goal: to show \(d_C(r(a), r(b)) \leq 2\)
Construction of a retraction

Outline of Proof of Claim 4.3

We prove only for the pairs \((a, b)\) \((a, b \in AC(N))\) which fill \(d_{AC}(a, b) = 1\).

The goal: to show \(d_{C}(r(a), r(b)) \leq 2\)

\[(1)\] \(a, b \in C^{(0)}(N)\)
\[d_{C}(r(a), r(b)) = d_{C}(a, b) = d_{AC}(a, b) = 1 \leq 2.\]
Outline of Proof of Claim 4.3

We prove only for the pairs \((a, b) (a, b \in \mathcal{AC}(N))\) which fill \(d_{\mathcal{AC}}(a, b) = 1\).

The goal: to show \(d_{\mathcal{C}}(r(a), r(b)) \leq 2\)

(1) \(a, b \in \mathcal{C}^{(0)}(N)\)
\[ d_{\mathcal{C}}(r(a), r(b)) = d_{\mathcal{C}}(a, b) = d_{\mathcal{AC}}(a, b) = 1 \leq 2. \]

(2) \(a \in \mathcal{C}^{(0)}(N)\) and \(b \in \mathcal{A}^{(0)}(N)\)
We can take the regular neighborhood of the union of \(b\) and boundary components which have endpoints of \(b\) without intersecting \(a\).
\[ d_{\mathcal{C}}(r(a), r(b)) = d_{\mathcal{C}}(a, r(b)) \leq 1 \leq 2. \]
Construction of a retraction

(3) \( a, b \in A^{(0)}(N) \)

The following is all cases of the pairs.
If \( r(a) \) and \( r(b) \) are not disjoint, we can take a curve \( \alpha \) without intersect both \( r(a) \) and \( r(b) \) as follows.

**Example**
In all cases, there exists an essential curve \( \alpha \) which does not intersect both \( r(a) \) and \( r(b) \).

Hence, we can show that

\[
d_C(r(a), r(b)) \leq d_C(r(a), \alpha) + d_C(\alpha, r(b)) \leq 2.
\]
Outline of proof of Main theorem

Theorem 4.4 (K.)

If $C(N)$ is connected, then it is 17-hyperbolic.

Outline of Proof of Main theorem
**Theorem 4.4 (K.)**

*If \( C(N) \) is connected, then it is 17-hyperbolic.*

**Outline of Proof of Main theorem**

\( T = abd \): any geodesic triangle in \( C(N) \) \((a, b, d \in C^{(0)}(N))\)

\( \bar{a}, \bar{b}, \bar{d} \in A^{(0)}(N) \):

arcs which are adjacent to \( a, b, d \) in \( AC(N) \).
Outline of proof of Main theorem

Theorem 4.4 (K.)

*If $C(N)$ is connected, then it is 17-hyperbolic.*

Outline of Proof of Main theorem

T = abd: any geodesic triangle in $C(N)$ ($a, b, d \in C^{(0)}(N)$)

$a, b, d \in A^{(0)}(N)$:

arcs which are adjacent to $a, b, d$ in $AC(N)$.

Now we use the following claim.

Claim 4.5

$P$: any unicorn path between $\bar{a}$ and $\bar{b}$.

$G = ab$: the edge of $T$ connecting $a$ and $b$ in $C(N)$.

$c \in P$: at maximal distance $k$ from $G$.

Then $k \leq 8$. 

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Outline of proof of Main theorem

Uniform hyperbolicity for $C(N)$

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Outline of proof of Main theorem

\(\alpha, \beta, \delta\): one of the endpoints of \(\bar{a}, \bar{b}, \bar{d}\).

**Lemma 4.6**

There exist \(\bar{c}' \in \mathcal{P}(\bar{a}^\alpha, \bar{b}^\beta)\), \(\bar{c}'' \in \mathcal{P}(\bar{b}^\beta, \bar{d}^\delta)\), and \(\bar{c}''' \in \mathcal{P}(\bar{d}^\delta, \bar{a}^\alpha)\) such that each pair represent adjacent vertices in \(\mathcal{A}(N)\).
Outline of proof of Main theorem

\(\alpha, \beta, \delta\): one of the endpoints of \(\bar{a}, \bar{b}, \bar{d}\).

**Lemma 4.6**

There exist \(\bar{c'} \in \mathcal{P}(\bar{a}^\alpha, \bar{b}^\beta)\), \(\bar{c''} \in \mathcal{P}(\bar{b}^\beta, \bar{d}^\delta)\), and \(\bar{c''' \in \mathcal{P}(\bar{d}^\delta, \bar{a}^\alpha)}\) such that each pair represent adjacent vertices in \(A(N)\).

By Claim 4.5 and Lemma 4.6, vertex \(\bar{c'}\) of \(AC(N)\) is at distance \(\leq 9\) from all sides of \(T\), in particular from a vertex of \(G = ab\), which is a curve.
Outline of proof of Main theorem
Recall the retraction \( r : \mathcal{AC}(N) \to \mathcal{C}(N) \).

Since \( r \) is 2-Lipschitz,

\( r(\bar{c}') \) becomes 17-center of the triangle \( T \) in \( \mathcal{C}(N) \).
Thank you for your attention!
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