Unknotting numbers for handlebody-knots and Alexander quandle colorings

MASAHIDE IWAKIRI
(Saga University)

2014.12.24
at Tokyo Woman’s Christian University
§1. Handlebody-knots

**handlebody-knot** $\iff$ an embedded handlebody in $\mathbb{R}^3$

$H \cong H' \iff \exists$ ori. pres. homeo. $h$ of $\mathbb{R}^3$ with $h(H) = H'$. 

\[
\begin{center}
\includegraphics[width=0.5\textwidth]{handlebody-knot-diagram.png}
\end{center}
\]
handlebody-knot $H$ \overset{\text{spine}}{\overset{\text{regular nbd.}}{\leftrightarrow}} \text{tri. spatial graph } G

genus one handlebody-knot $\iff$ classical knot

diagram of $H$ $\overset{\text{def.}}{\iff}$ diagram of $G$

trivial handlebody-knot
R-moves for handlebody-knots

Theorem 1 (Ishii). $H_i$: a handlebody-knot represented by a diagram $D_i$ for $i = 1, 2$. Then $H_1 \cong H_2 \iff D_1, D_2$ are related by a finite number of R-moves.
§2. Crossing changes

A crossing change of a handlebody-knot $H$ is that of a spatial trivalent graph representing $H$. 
Proposition 2. Any handlebody-knot can be deformed into trivial one by crossing changes.

Proof. Any handlebody-knot is represented by a spatial embedding $f$ of $T_n$.

$T_n$ is a ”trivializable graph”. Thus $f(T_n)$ can be trivial one by crossing changes.
$u(H)$ is the min. $\#$ of crossing changes needed to obtain a trivial handlebody-knot from $H$. $u(H)$ is called the unknotting number of $H$.

$u(4_1)=1$
§3. A quandle

A quandle is a non-empty set $X$ with $*: X \times X \to X$ satisfying the following axioms.

- $\forall x \in X$, $x * x = x$.
- $\forall x \in X$, $S_x : X \to X$ defined by $S_x(y) = y * x$ is a bijection.
- $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

**Example 3.** (1) $(R_n, *)$ is a dihedral quandle when $R_n = \mathbb{Z}_n$ with $x * y = 2y - x$. (2) $(M, *)$ is an Alexander quandle when $M$ is a $\Lambda$-module with $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. $R_n \cong \Lambda/(n, t + 1)$. 
**quandle of type** $m$

A quandle $X$ is of **type** $m$ if

$$m = \min\{k > 0 \mid S_x^k = \text{id} \text{ for } \forall x \in X\}.$$ 

$$x \star^l y := S_y^l(x).$$
§4. Colorings ([Ishii], [Ishii-I], [Ishii-I-Jang-Oshiro])

\(D\) : diagram with normal orientation to each edge.

\(\mathcal{A}(D) := \{\text{over-arcs of } D\}\). \(A\) : abelian group.

A map \(\varphi : \mathcal{A}(D) \to A\) is an \(A\)-flow of \(D\) if

\[
\varepsilon_1 k + \varepsilon_2 l + \varepsilon_3 j = 0
\]

\[
\varepsilon_i = \begin{cases} 
1 & \text{anticlockwise} \\
-1 & \text{clockwise}
\end{cases}
\]

Flow\((D; A) := \{A\text{-flows of } D\}\).

\((D, \varphi)\): a handlebody-knot diagram with \(\varphi \in \text{Flow}(D; A)\).
$X$: A finite quandle of type $m$.

For $(D, \varphi)$ with $\varphi \in \text{Flow}(D; \mathbb{Z}_m)$, a map $C : A(D) \rightarrow X$ is an $X$-coloring of $D$ if

$$
\begin{array}{cc}
x & \xrightarrow{k} & x^*y \\
y & \rightarrow & y
\end{array}
$$

$$
\begin{array}{c}
x \\
\downarrow \\
x
\end{array} \quad \begin{array}{c}
x \\
\downarrow \\
x
\end{array}
$$

$\text{Col}_X(D, \varphi) := \{X\text{-colorings of } D\}$

**Proposition 4.**

$D \xrightarrow{R\text{-moves with flow}} D' \quad \Rightarrow \quad \text{Col}_X(D, \varphi) \leftrightarrow 1:1 \quad \text{Col}_X(D', \varphi')$. 

10
Example of an $X$-coloring where $X = \Lambda/(3, t + 1)(\cong R_3)$:

\[ \begin{array}{c}
1 \\
1 \\
2 \\
1 \\
0 \\
0 \\
\end{array} \]

$\text{Col}_X(H)$ is the multiset $\{\#\text{Col}_X(D, \varphi) \mid \varphi \in \text{Flow}(L; \mathbb{Z}_m)\}$. $\text{Col}_X(H)$ is an invariant of handlebody-knots.
Set $\Lambda_p := \mathbb{Z}[t, t^{-1}]/(p)$ where $p$ is an odd prime integer and $\mathbb{F}_q = \Lambda_p/(h(t))$ where $h(t)$ is an irreducible polynomial in $\Lambda_p$ with $h(t) \neq t, t-1$. Then $\mathbb{F}_q$ is a finite field of order $q = p^n$ where $n$ is the span of $f$.

For the Alexander quandle $\mathbb{F}_q$, $\text{Col}_{\mathbb{F}_q}(D, \varphi)$ forms a linear space over $\mathbb{F}_q$ whose generators are in $A(D)$ and relators are given for each crossings and vertices.
\[ X = \Lambda/(3, t + 1)(\cong R_3). \]

\[
\text{Col}_{R_3}(D_{4_1}, \varphi) = \langle a, b, c, d, e, f, g \mid a = b = c, e * b = d, a = b, e = f = g, c * g = d, f = g \rangle
\]

\[
= \langle a, e \mid e * a = a * e \rangle
\]

\[= \langle a, e \rangle. \]
\S 5. Main Theorem

**Proposition 5** (Clark-Elhamdadi-Saito-Yeatman). Let $D$ be a diagram of a knot $K$. Then $\dim \mathrm{Col}_{F_q}(D) - 1 \leq m(K)$ where $m(K)$ is the Nakanishi index of $K$.

**Corollary 6.** Let $D$ be a diagram of a knot $K$. Then $\dim \mathrm{Col}_{F_q}(D) - 1 \leq u(K)$.

**Theorem 7** (Main theorem). Let $H$ be a handlebody-knot. Let $m = 2$ or $3$ and $D$ be a diagram of $H$ with $\varphi \in \mathrm{Flow}(D; \mathbb{Z}_m)$. If $F_q$ is of type $m$, then $\dim \mathrm{Col}_{F_q}(D, \varphi) - 1 \leq u(H)$.
Example 8. $u(A_n) = u(B_n) = u(C_n) = n$. 

§6. Examples
$X = R_3 = \Lambda_3/(t + 1)$ is a quandle of type 2

$\dim \text{Col}_{R_3}(D_{A_n}, \varphi) = n + 1$. Hence $n \leq u(A_n)$. 
Thus, $u(A_n) \leq n$. Hence $u(A_n) = n$. 
\[ \dim \text{Col}_{R_3}(D_{B_n}, \varphi) = n + 1. \] Hence \( n \leq u(B_n). \)
Thus, \( u(B_n) \leq n \). Hence \( u(B_n) = n \).
\[ \dim \text{Col}_{R_3}(D_{C_n}, \varphi) = n + 1. \text{ Hence } n \leq u(C_n). \]
Thus, $u(C_n) \leq n$. Hence $u(C_n) = n$. 
Irreducibility

A 2-sphere $S$ in $\mathbb{R}^3$ is an $n$-decomposing sphere for a handlebody-knot $H$ if

1. $S \cap H$ consists of $n$ essential disks in $H$, and
2. $S \cap E(H)$ is incompressible and not $\partial$-parallel in $E(H)$.

$H$ is reducible if $\exists$ 1-decomposing sphere for $H$.

$H$ is irreducible if $H$ is not reducible.

(a) $A_n$ is reducible.
(b) $B_n$ is irreducible by using results in [Ishii-Kishimoto-Ozawa].
§7. Proof of Theorem 7

**Theorem 7** Let $H$ be a handlebody-knot. Let $m = 2$ or $3$ and $D$ be a diagram of $H$ with $\varphi \in \text{Flow}(D; \mathbb{Z}_m)$. If $F_q$ is of type $m$, then $\dim \text{Col}_{F_q}(D, \varphi) - 1 \leq u(H)$. 
Col_{F_q}(D, \varphi) = \langle A(D) \mid R(D) \rangle_{F_q} \text{ where } \\
R(D) = \left\{ \begin{array}{l}
z = x \cdot^l y \text{ (at crossing)}, \\
x = y = z \text{ (at vertex)} \end{array} \right\} \times \begin{array}{c}
x \\
y \\
z \\
y_1 \\
\end{array}
\\n\textbf{Lemma 9.} SA(D) := \{\text{semi-arcs of } D\}, \\
SR(D) = \left\{ \begin{array}{l}
z = x \cdot^l y, y = w \text{ (at crossing)}, \\
x = y = z \text{ (at vertex)} \end{array} \right\}. \times \begin{array}{c}
x \\
w \\
z \\
y \\
\end{array}
\\Then \text{ Col}_{F_q}(D, \varphi) = \langle SA(D) \mid SR(D) \rangle_{F_q}.
$SA(D) = \{a, b, \ldots, k\}$.

$SR(D) = \begin{cases} 
a = b = c, e = f = g, a = i, e = h, b = i, 
h \ast i = d, d = k, f = j, g = j, c \ast g = k 
\end{cases}$. 
**Proposition 10.** Let $c$ be a crossing such that the two flows $k, l$ are $kl = 0$, $k = l$ or $k + l = 0$. Let $D'$ be a diag. obtained from $D$ by one c. c. at $c$. Then the difference of dim. of $\text{Col}_{\mathbb{F}_q}(D, \varphi)$ and $\text{Col}_{\mathbb{F}_q}(D', \varphi')$ is $0, 1$.

Type 2 $\iff \varphi \in \text{Flow}(D; \mathbb{Z}_2) \iff kl = 0$ or $k = l = 1$.

Type 3 $\iff \varphi \in \text{Flow}(D; \mathbb{Z}_3) \iff kl = 0$, $k = l = 1, 2$ or $k + l = 0$. 
crossing with flows $k, l$ ($k + l = 0$)

\[
\begin{align*}
x \star_l y &= z \\
y &= w \\
z &= x \star_l y = t^l x + (1 - t^l) y = t^l x + y - t^l w.
\end{align*}
\]

\[
\bar{\mathcal{SR}}(D) := \mathcal{SR}(D) \cup \{z - y = t^l(x - w)\} \setminus \{x \star_l y = z\}.
\]

\[
\text{Col}_{\mathbb{F}_q}(D, \varphi) \cong \langle \mathcal{SA}(D) \mid \bar{\mathcal{SR}}(D) \rangle_{\mathbb{F}_q}.
\]
crossing with flows $k, l$ ($k + l = 0$)

\[ x^l y = z \quad \iff \quad x = z \]

\[ y = w \quad \iff \quad y ^ k x = w \]

\[ w = y ^ k x = t^k y + (1 - t^k)x = t^k y + x - t^k z. \]

\[ \widetilde{SR}(D') := SR(D') \cup \{x - w = t^k(z - y)\}\{y ^ k x = w\}. \]

\[ \text{Col}_{\mathbb{F}_q}(D', \phi) \cong \langle SA(D') \mid \widetilde{SR}(D') \rangle_{\mathbb{F}_q}. \]
crossing with flows $k, l$ ($k + l = 0$)

\[ z - y = t^l(x - w) \iff t^k(z - y) = t^{k+l}(x - w) = x - w. \]

\[ \tilde{\mathcal{SR}}(D) = \tilde{\mathcal{SR}}(D') \cup \{y = w\} \setminus \{x = z\}. \]

Thus, \( \dim \text{Col}_{\mathbb{F}_q}(D, \varphi) = \dim \text{Col}_{\mathbb{F}_q}(D', \varphi) + r \) \( (r = -1, 0, 1) \). \( \square \)
Remark
(1) An ordinary classical knot coloring is a genus one handlebody-knot coloring with all flows 1. Thus, our proof gives Corollary 6.

(2) Our proof also implies that \( \frac{1}{2}(\dim \text{Col}_{F_q}(D, \varphi) - 1) \leq u(H) \) without condition for type. (We don’t have good example.)