Twisted Torus Knots and Essential Surfaces

by

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Mathematics of Knots IV

December 24-27, 2011, Tokyo Woman’s Christian University

1. Twisted torus knots $T(p, q; r, s)$

Let $p, q$ be coprime integers with $p > q > 1$, $r, s$ integers with $p > r > 1$ and $s \neq 0$. Consider the torus knot $T(p, q)$, take $r$-strands in the parallel $p$-strings of $T(p, q)$, and perform $s$-times full twists on the $r$-strands.

Then we get the knot illustrated in the next figure, denoted by $T(p, q; r, s)$. It is called the twisted torus knot of type $(p, q; r, s)$ by J.C. Dean ([C-D-R]).

![Twisted Torus Knot Diagram](image)

2. Essential surfaces

Let $K$ be a knot in the 3-sphere $S^3$, $N(K)$ the regular neighborhood of $K$ in $S^3$, and $E(K) = cl(S^3 - N(K))$ be the exterior. Let $F$ be a surface (i.e. a connected 2-manifold) properly embedded in $E(K)$. Then we say that $F$ is essential in $E(K)$ if $F$ is closed, incompressible and not parallel to the torus $\partial E(K)$ or if $F$ is bounded, incompressible and not parallel to an annulus $\subset E(K)$. Then we ask the following :

**Problem 1** Which twisted torus knots have closed essential surfaces?

In particular,
Problem 2 Which twisted torus knots have essential tori?

Concerning these problems, the first result is:

Theorem 1 ([M1]) If \( r = 2 \), then \( T(p, q; r, s) \) has no closed essential surfaces.

In addition, on essential tori in the exteriors, we have:

Theorem 2 ([M-Y]) If \( r = qm \) and \( p = qa + 1 \) with \( n \geq m > 1 \), then \( T(p, q; r, s) \) is a \( q \)-cable knot along \( T(m, ms + 1) \). Hence \( T(qa + 1, q; qm, s) \) has an essential torus if \( (m, s) \neq (2, -1) \).

Theorem 3 ([SL1]) Suppose \( r = qm \) with \( m > 1 \), then \( T(p, q; r, s) \) is a \( q \)-cable knot along \( T(m, ms + 1) \). Hence \( T(p, q; qm, s) \) has an essential torus if \( (m, s) \neq (2, -1) \).

Example Put \( n = m = 2 \) and \( q = 2 \). Then \( p = 5 \), \( r = 4 \) and \( T(5, 2; 4, s) \) has an essential torus \((s \neq 0, -1)\).

Hence we can ask:

Problem 3 Are there twisted torus knots with \( r = 3 \) which have closed essential surfaces?

By Theorem 3, it is understandable to conjecture that if \( T(p, q; r, s) \) has an essential torus then \( r = qm \) for some \( m \). But the situation is not so simple as we see in the next section.

3. Composite twisted torus knots

On the primeness of twisted torus knots, we have:

Theorem 4 ([M2]) Let \( e > 0, k_1 > 1, k_2 > 1 \) be integers, and put \( p = (e + 1)(k_1 + k_2) + 1 \), \( q = e(k_1 + k_2) + 1 \), \( r = p - k_1 \) and \( s = -1 \). Then \( T(p, q; r, s) \) is the connected sum of \( T(k_1, ek_1 + 1) \) and \( T(k_2, -(e + 1)k_2 - 1) \).

Example Put \( e = 1, k_1 = 3, k_2 = 2 \), then \( p = 11, q = 6, r = 8 \) and we have

\[
T(11, 6; 8, -1) = T(3, 4) \# T(2, -5)
\]

Theorem 4 shows that there are twisted torus knots with essential tori even if \( r \neq qm \) because composite knots have essential tori.
\( T(11, 6; 8, -1) - (1) \)

\( T(11, 6; 8, -1) - (2) \)

\( T(11, 6; 8, -1) - (3) \)

\( T(11, 6; 8, -1) = T(3, 4) \# T(2, -5) \)

\( T(3, 4) \)

\( T(2, -5) \)
4. Tangle decompositions of twisted torus knots

As a generalization of Theorem 4 we get the following:

**Theorem 5 ([M3])**  Let $e > 0$, $k_1 > 1$, $k_2 > 1$, $x_1 > 0$, $x_2 > 0$ be integers with $\gcd(x_1, x_2) = 1$. Put

$$p = ((e + 1)(k_1 + k_2 - 1) + 1)x_1 + (e + 1)x_2,$$
$$q = (e(k_1 + k_2 - 1) + 1)x_1 + ex_2,$$
$$r = ((e + 1)(k_1 + k_2 - 1) - k_1 + 2)x_1 + ex_2$$

and $s = -1$.

Then we have:

(1) $T(p, q; r, s)$ has an $x_1$-string essential tangle decomposition.

(2) The decomposition is obtained by the $x_1$-string fusion of the torus knot $T((k_1 - 1)x_1 + x_2, e((k_1 - 1)x_1 + x_2) + x_1)$ and the torus link $T(k_2x_1, -(e + 1)k_2 + 1)x_1$.

(3) $T(p, q; r, s)$ has an essential torus in the exterior whose companion is the torus knot $T(k_2, -(e + 1)k_2 - 1)$.

By Theorem 5, for any integer $n > 0$, by putting $x_1 = n$ we get infinitely many twisted torus knots with $n$-string essential tangle decompositions.

**Example**  Put $e = 1$, $k_1 = 2$, $k_2 = 2$, $x_1 = 2$, $x_2 = 3$, then $p = 20$, $q = 11$, $r = 15$ and we see that $T(20, 11; 15, -1)$ is the 2-string fusion of $T(5, 7)$ and $T(4, -10)$ as illustrated in the next figure. By tubing the decomposing 2-sphere along the torus link $T(4, -10)$, we have an essential torus whose companion is the torus knot $T(2, -5)$.

![Diagram of torus knots and links](image-url)
5. T-links and Lorenz links

Let \(1 < r_1 < r_2 < \cdots < r_{k-1} < r_k\) be integers, and \(s_i > 0\) \((i = 1, 2, \ldots, k)\) integers. Then by combining torus knots \(T(r_1, s_1), T(r_2, s_2), \ldots, T(r_k, s_k)\), we get the link illustrated in the next figure. This link is called a T-link, and denoted by \(T((r_1, s_1), (r_2, s_2), \ldots, (r_k, s_k))\). Then Birman-Kofman showed in [B-K] that “every T-link is a Lorenz link”.

**Example**

\(T((3, 2), (5, 3), (7, 5))\)

By the definition of T-links, we see that \(T(p; q; r, s)\) is the T-link \(T((r, rs), (p, q))\) for \(s > 0\). By combining the above result [B-K] and the result of R. Williams, we have:

**Theorem 6 ([B-K], [W])**  *Lorenz links are prime, and hence T-links are prime.*

This theorem says that if \(T(p, q; r, s)\) is composite then \(s < 0\). Moreover, Sy.Lee has shown the following:

**Proposition 7 ([SL2])**  If \(T(p, q; r, s)\) with \(r \neq qm\) has an essential torus, then \(|s| \leq 2\).

This implies that if \(T(p, q; r, s)\) is composite then \(s = -1\) or \(-2\).

6. Conjecture and more problems

By Theorem 3 and Theorem 5, we conjecture the following:

**Conjecture**  *If a twisted torus knot \(T(p, q; r, s)\) has an essential torus, then it is a knot in Theorem 3 or a knot in Theorem 5.*

\(i.e.,\)  

1. \(r = qm\) \((m > 1)\) or
2. \(p = ((e + 1)(k_1 + k_2 - 1) + 1)x_1 + (e + 1)x_2,\)
   \(q = (e(k_1 + k_2 - 1) + 1)x_1 + ex_2,\)
   \(r = ((e + 1)(k_1 + k_2 - 1) - k_1 + 2)x_1 + ex_2\) and \(s = -1\).

On the knot types of twisted torus knots, we can ask:
Problem 4  Characterize the knot types of twisted torus knots which are trivial knots ?

Problem 5  Characterize the knot types of twisted torus knots which are torus knots ?

8. Tunnel numbers of twisted torus knots

On the tunnel numbers of twisted torus knots we have :

Fact  If \( r = 2 \), then \( T(p, q; r, s) \) has tunnel number one.

Theorem 8 ([JL])  If \( r = 3 \), then \( T(p, q; r, s) \) has tunnel number one.

Theorem 9 ([SL1], [M4])  Suppose \( r = qm \) with \( m > 1 \), then \( T(p, q; r, s) \) has tunnel number one if and only if \( p - qm = 1 \).

Hence we ask :

Problem 7  Characterize the knot types of tunnel number one twisted torus knots.

References


[SL1] S. Y. Lee, Twisted torus knots \( T(p, q; kq, s) \) are cable knots, to appear in J. Knot Theory Ramifications.


[M4] K. Morimoto, On tunnel numbers of twisted torus knots, Konan University
