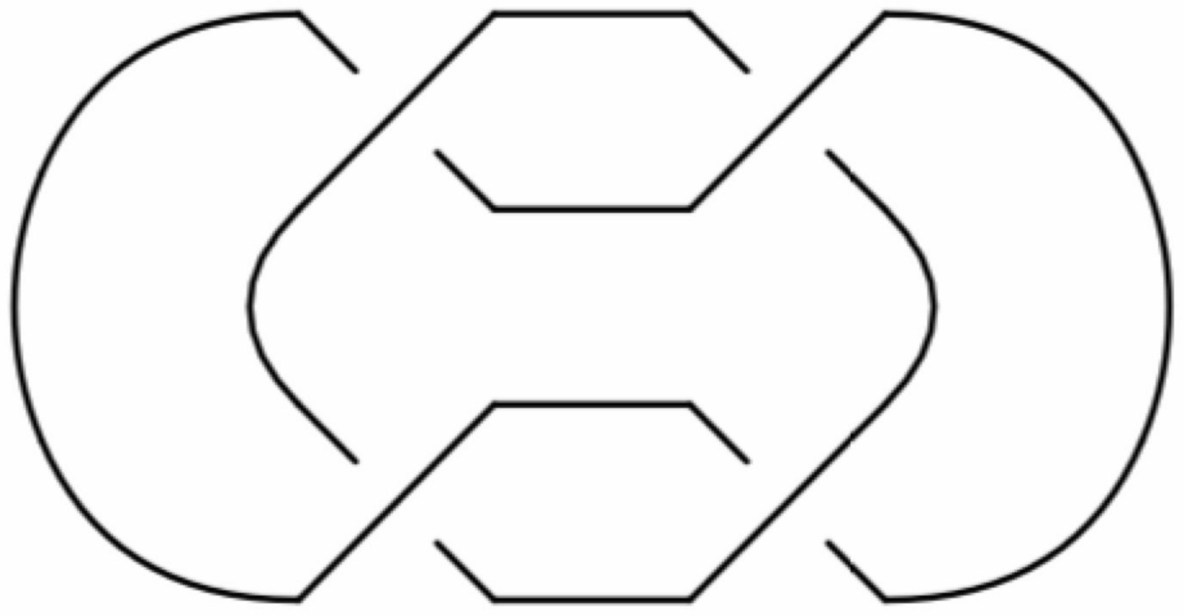
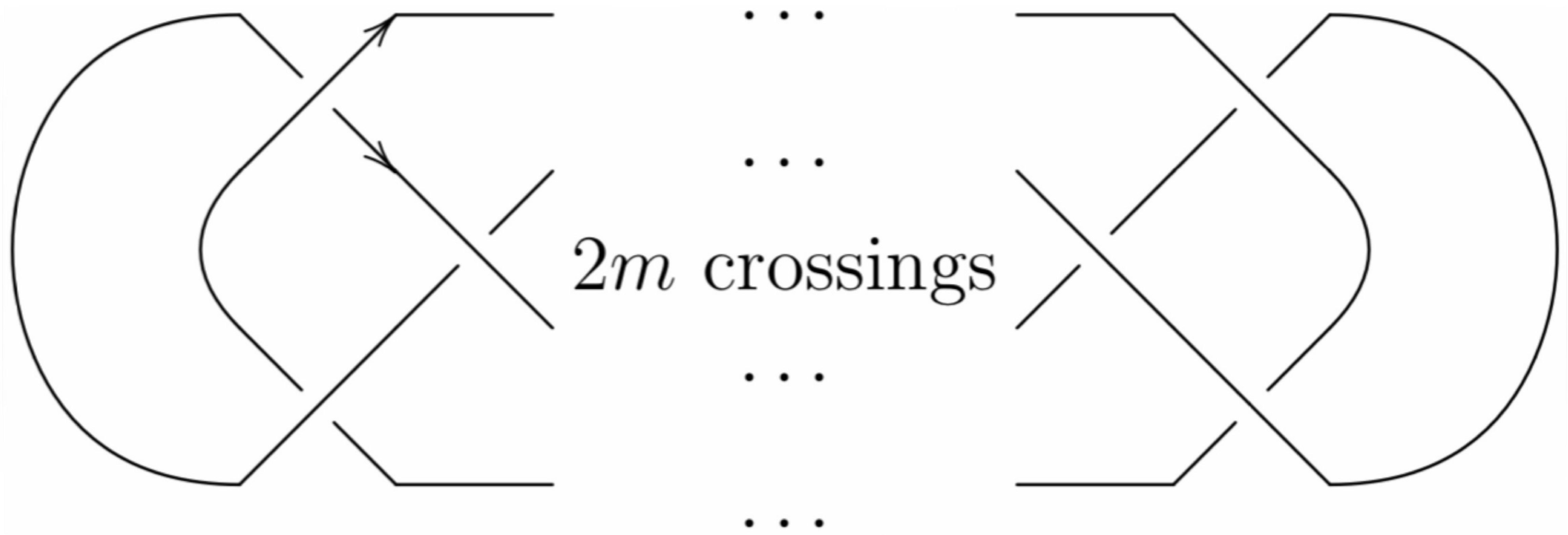
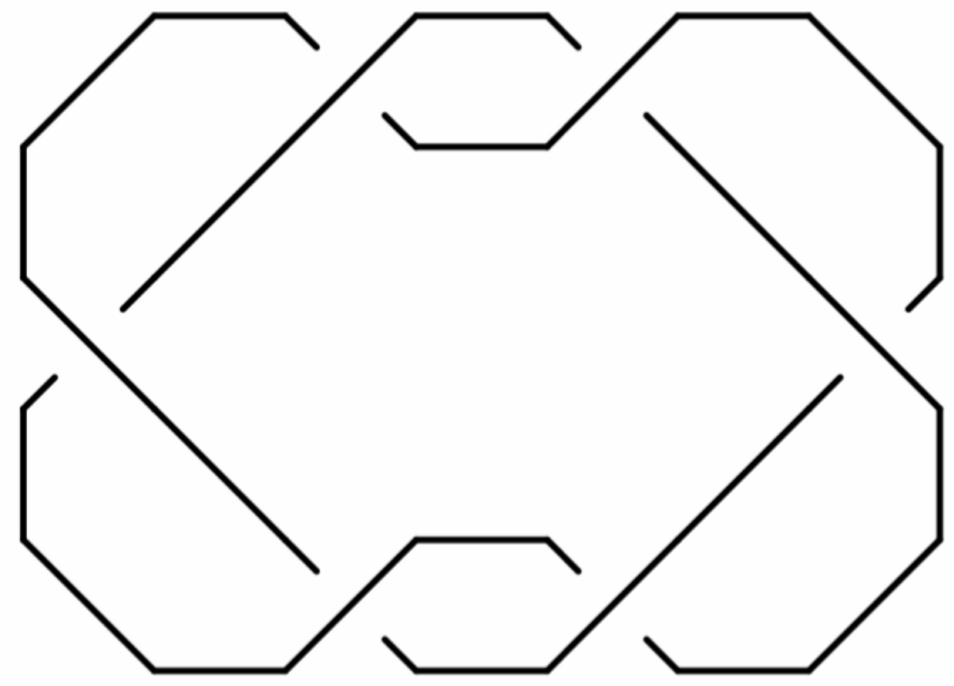
# The Iwasawa-type formula of $\mathbb{Z}_p^d$-covers of links in homology 3-spheres



Sohei Tateno joint work with Jun Ueki 2023/12/25

# Contents

Fix a prime number $p$.

Notation

For a finite group $G$, let
$e(G)$ = ( how many times $p$ divides the order of $G$ ).

Example

- $e(\mathbb{Z}/p^2\mathbb{Z}) = 2$
- $e(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}) = 4$

## Historical backgrounds

For each number field $k$, an abelian group called the ideal class group $Cl(k)$ is defined.

The finite number $h(k) := \#Cl(k)$ is an important algebraic invariant called the class number of $k$.

**Theorem [Kummer, 1847]**

Let $\zeta_p$ denote a $p$-th root of unity. If $p \neq 2$ and $p \nmid h(\mathbb{Q}(\zeta_p))$, then The Fermat Last Conjecture holds for $n = p$.

Let $\mathbb{Z}_p$ denote the group $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$.

Theorem(Iwasawa's class number formula) [Iwasawa, 1959]

Let $k_\infty/k$ be a $\mathbb{Z}_p$-extension and let $k_{p^n}/k$ be the $\mathbb{Z}/p^n\mathbb{Z}$-subextensions.

Then $^{\exists!}\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$, depending only on $k_\infty/k$, such that, for every $n \gg 0$,

$$e(Cl(k_{p^n})) = \mu p^n + \lambda n + \nu.$$

Example

$$k := \mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\zeta_{p^2}) \subset \mathbb{Q}(\zeta_{p^3}) \subset \ldots \subset \bigcup_{n\geq 1} \mathbb{Q}(\zeta_{p^n}) =: k_\infty.$$

A closed connected orientable 3-manifold $M$ is called a rational homology 3-sphere($\mathbb{Q}HS^3$) if $H_i(M,\mathbb{Q}) \simeq H_i(S^3,\mathbb{Q})$ for all $i \geq 0$.

Theorem [Hillman-Matei-Morishita, 2006], [Kadokami-Mizusawa, 2008], [Ueki, 2017]

Let $L$ be a link in a $\mathbb{Q}HS^3$ $M$. Let $(M_{p^n} \to M)_n$ be a compatible system of $\mathbb{Z}/p^n\mathbb{Z}$-covers branched along $L$. Suppose every $M_{p^n}$ is a $\mathbb{Q}HS^3$.

Then $^{\exists!}\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$, depending only on $(M_{p^n} \to M)_n$ and $p$, such that, for every $n \gg 0$,

$$e(H_1(M_{p^n})) = \mu p^n + \lambda n + \nu.$$

## class field theory

$k$ : number field

$l$ : maximal unramified Galois extension of $k$

We have $\mathrm{Gal}(l/k)^{\mathrm{ab}} \cong Cl(k)$.

## Hurewicz theorem

$X$ : path-connected space

We have $\pi_1(X)^{\mathrm{ab}} \cong H_1(X)$.

## Remark

We have

$M$ is $\mathbb{Q}HS^3 \Longleftrightarrow H_1(M)$ is finite

$M$ is $\mathbb{Z}HS^3 \Longleftrightarrow H_1(M) = 0$

Hence

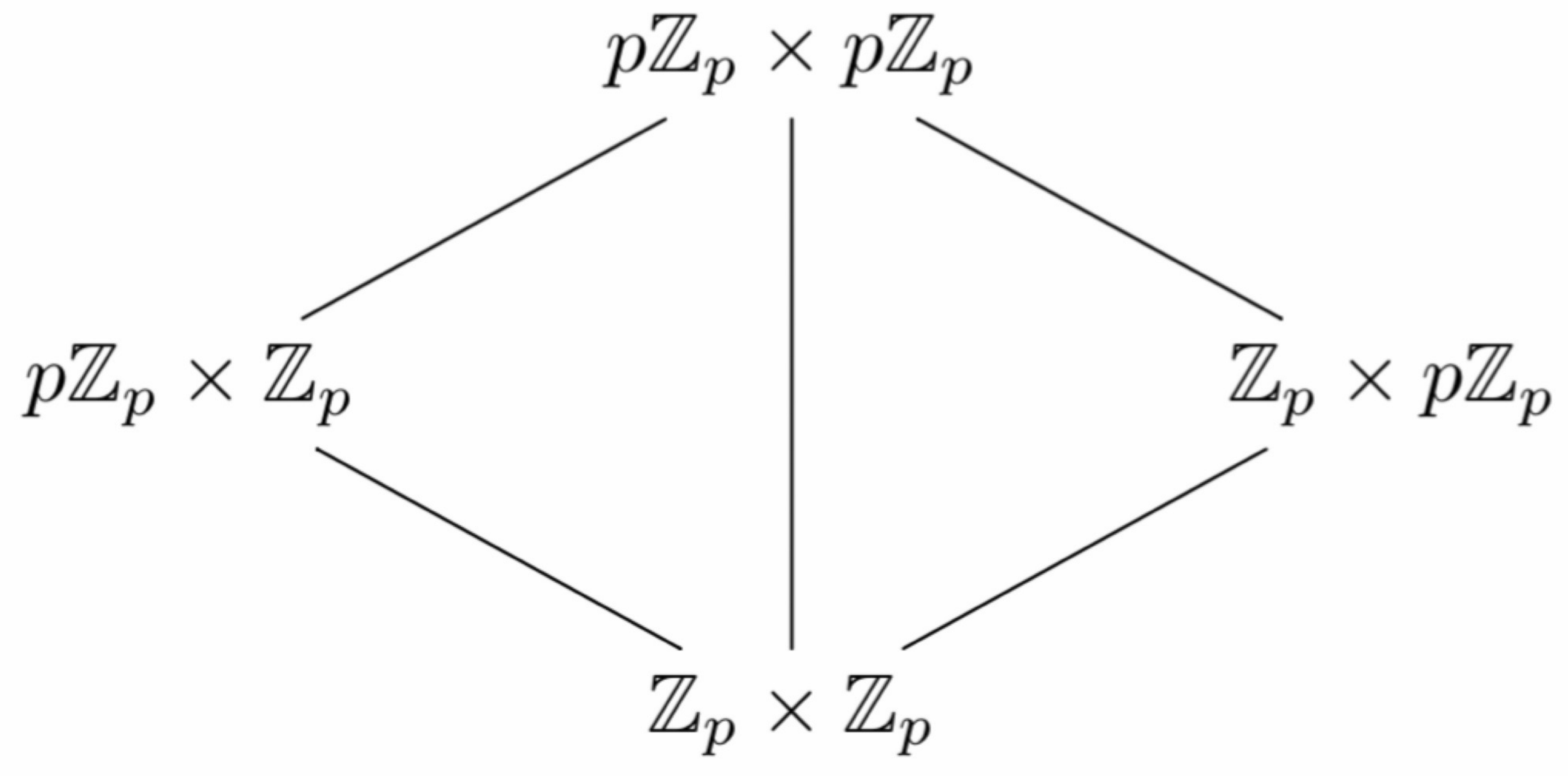
$S^3 \in \{\mathbb{Z}HS^3\} \subset \{\mathbb{Q}HS^3\}$

$\mathbb{Q} \in$ {number fields with $Cl(k) = 0$} $\subset$ {number fields}

## Theorem [Cuoco-Monsky, 1981]

Let $k_\infty/k$ be a $\mathbb{Z}_p^d$-extension and let $k_{p^n}/k$ be the $(\mathbb{Z}/p^n\mathbb{Z})^d$-subextensions.

Then $^{\exists!}\mu, \lambda \in \mathbb{Z}_{\geq 0}$, depending only on $k_\infty/k$, s.t.




$e(Cl(k_{p^n})) = (\mu p^n + \lambda n + O(1))p^{(d-1)n}$,

where $O$ is the Bachmann-Landau notation.

## Our main result

Let $(M, L)$ be a pair of a $\mathbb{Q}HS^3$ and a link. Put $X := M \backslash L$.

Let $(X_{p^n} \to X)_n$ be the compatible system of $(\mathbb{Z}/p^n\mathbb{Z})^d$-covers of $X$.

Let $M_{p^n}$ be the Fox completions of $X_{p^n}$.

Let $W := \{\xi \in \mathbb{C} \mid \xi^{p^n} = 1 \text{ for some } n \geq 0\}$.

## Main result

Suppose that $M$ is a $\mathbb{Z}HS^3$ and the Alexander polynomial does not vanish on $(W \backslash \{1\})^d$. Then $^{\exists !}\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\mu_{d-1}, \ldots, \mu_1, \lambda_{d-1}, \ldots, \lambda_1, \nu \in \mathbb{Q}$, depending only on $L$ and $p$, such that, for $^{\forall} n \gg 0$,

$$e(H_1(M_{p^n})) = \mu p^{dn} + \lambda n p^{(d-1)n} + \mu_{d-1} p^{(d-1)n} + \lambda_{d-1} n p^{(d-2)n} + \ldots + \mu_1 p^n + \lambda_1 n + \nu \,.$$

## $p$-adic numbers

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow{\varphi_2} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{\varphi_3} \mathbb{Z}/p^3\mathbb{Z} \longleftarrow \cdots$$

$\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \{\{a_n\}_n \in \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z} \mid \varphi_n(a_n) = a_{n-1}\}$ is the ring of $p$-adic integers.

### Example

$p := 5$

$a := (2,\ 7,\ 57,\ldots) \in \mathbb{Z}_5$

$2 \in \mathbb{Z}/5\mathbb{Z}$

$7 = 2 + 1 \cdot 5 \in \mathbb{Z}/5^2\mathbb{Z}$

$57 = 2 + 1 \cdot 5 + 2 \cdot 5^2 \in \mathbb{Z}/5^3\mathbb{Z}$

$1 = (1,\ 1,\ldots)$ is the identity of $\mathbb{Z}_5$.

$-1 = (-1, -1,\ldots) = (4,\ 24,\ 124,\ldots) = (2^2,\ 7^2,\ 57^2,\ \ldots) = (2,\ 7,\ 57,\ldots)^2 = a^2$

Therefore, $a = \sqrt{-1} \in \mathbb{Z}_5$.

$\mathbb{Z}_5 := \varprojlim_n \mathbb{Z}/5^n\mathbb{Z} = \{\{a_n\}_n \in \prod_{n\geq 1} \mathbb{Z}/5^n\mathbb{Z} \mid \varphi_n(a_n) = a_{n-1}\}$

$\sqrt{-1} = (2,\ 7,\ 57,\ldots) \in \mathbb{Z}_5$

$L := K_1 \cup K_2$

$X := S^3 \backslash L$

$\alpha_1, \alpha_2$ : meridians of $K_1, K_2$

Define $\tau : \pi_1(X) \to \mathbb{Z}_5$ by $\alpha_1 \mapsto 1,\ \alpha_2 \mapsto \sqrt{-1}$.

$\tau_n : \pi_1(X) \xrightarrow{\tau} \mathbb{Z}_5 \twoheadrightarrow \mathbb{Z}/5^n\mathbb{Z}$

$X_{5^n}$ : spaces corresponding to $\ker \tau_n$

Then $(X_{5^n} \to X)_n$ form a compatible system of $\mathbb{Z}/5^n\mathbb{Z}$-covers

This is not derived from any $\mathbb{Z}$-cover. Indeed, if so, then

$\pi_1(X) \to \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ induces $\pi_1(X) \to \mathbb{Z} \to \mathbb{Z}_5$.

This contradicts $\mathrm{im}\tau = \mathbb{Z}^2$.

## Example

$L = K_1 \cup K_2 \cup K_3$

$X := S^3 \backslash L$

$\alpha_1, \alpha_2, \alpha_3$ : meridians of $K_1, K_2, K_3$

Define $\tau : \pi_1(X) \to \mathbb{Z}_5^2$ by $\alpha_1 \mapsto (1,0)$, $\alpha_2 \mapsto (\sqrt{-1},0)$, $\alpha_3 \mapsto (0,1)$.

$\tau_n : \pi_1(X) \xrightarrow{\tau} \mathbb{Z}_5^2 \twoheadrightarrow (\mathbb{Z}/5^n\mathbb{Z})^2$

$X_{5^n}$ : spaces corresponding to $\ker \tau_n$

Then $(X_{5^n} \to X)_n$ form a compatible system of $(\mathbb{Z}/5^n\mathbb{Z})^2$-covers.

## Main result

$e(H_1(M_{p^n})) = \mu p^{dn} + \lambda n p^{(d-1)n} + \mu_{d-1} p^{(d-1)n} + \lambda_{d-1} n p^{(d-2)n} + \ldots + \mu_1 p^n + \lambda_1 n + \nu$ .

# Sketch of the proof of our main result

$\tau : \pi_1(X) \to \mathbb{Z}_p^d$ a homom. corresp. to the $\mathbb{Z}_p^d$-cover.

$\alpha_1, \ldots, \alpha_c$ : meridians of the components $K_1, \ldots, K_c$ of $L$

$\mathsf{v}_i := \tau(\alpha_i) = (v_{i1}, \ldots, v_{id})$. Put $W(n) := \{\xi \in \mathbb{C} \mid \xi^{p^n} = 1\}$.

For $\zeta = (\zeta_1, \ldots, \zeta_d) \in W(n)^d$, put $\zeta^{\mathsf{v}_i} := \zeta_1^{v_{i1}} \cdots \zeta_d^{v_{id}}$.

By works of Mayberry-Murasugi and Porti, we can show

$$e(H_1(M_{p^n})) = \mathrm{ord}_p \left( N_{p^n} \prod_{L' \subset L} \prod_{\substack{\zeta \in W(n)^d \\ \zeta^{\mathsf{v}_j} \neq 1\,(j \in \{i_1, \ldots i_{c(L')}\}) \\ \zeta^{\mathsf{v}_j} = 1\,(j \notin \{i_1, \ldots i_{c(L')}\})}} \Delta_{L'}(\zeta^{\mathsf{v}_{i_1}}, \ldots, \zeta^{\mathsf{v}_{i_{c(L')}}}) \right) = \mathrm{ord}_p(N_{p^n}) + \sum_{L' \subset L} \sum_{\substack{\zeta \in W(n)^d \\ \zeta^{\mathsf{v}_j} \neq 1\,(j \in \{i_1, \ldots i_{c(L')}\}) \\ \zeta^{\mathsf{v}_j} = 1\,(j \notin \{i_1, \ldots i_{c(L')}\})}} \mathrm{ord}_p(\Delta_{L'}(\zeta^{\mathsf{v}_{i_1}}, \ldots, \zeta^{\mathsf{v}_{i_{c(L')}}})),$$

where $N_{p^n}$ are positive integers that divide $p^{dn}$. By a work of Monsky for estimates, we complete the proof.

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