結び目 n-カンドルの2次カンドルホモロジー群

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結び目の数理 VI
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Summary

- K: an oriented knot in $S^3 \rightsquigarrow Q(K)$: the **knot quandle** of K
  The knot quandle is a complete invariant for oriented knots up to orientation.
- Eisermann established the covering theory of quandles, and computed the second quandle homology group $H_2^Q(Q(K))$.
- The **knot n-quandle** $Q_n(K)$ is a quotient of $Q(K)$ ($n \in \mathbb{Z}_{>1}$).
- Knot n-quandles are more treactable than knot quandles.

Main result

We determine the second quandle homology group $H_2^Q(Q_n(K))$.

Theorem

$$H_2^Q(Q_2(K)) \cong \begin{cases} 0 & (K \risingdotseq 0_1), \quad \mathbb{Z}/2\mathbb{Z} \quad (K \risingdotseq M(1/2, */3, */3)), \\ \mathbb{Z}/4\mathbb{Z} & (K \risingdotseq M(1/2, */3, */5)), \quad \mathbb{Z} \quad (K : \text{otherwise}). \end{cases}$$

($K \risingdotseq K' \Leftrightarrow K$ and K' are equivalent up to 2-bridge knot summands.)

Theorem

$$H_2^Q(Q_3(K)) \cong \begin{cases} 0 & (K = 0_1), \quad \mathbb{Z}/2\mathbb{Z} \quad (K = 3_1), \\ \mathbb{Z}/6\mathbb{Z} & (K = 5_1), \quad \mathbb{Z} \quad (K : \text{otherwise}). \end{cases}$$

Theorem

$$H_2^Q(Q_4(K)) \cong \begin{cases} 0 & (K = 0_1), \quad \mathbb{Z}/4\mathbb{Z} \quad (K = 3_1), \\ \mathbb{Z} & (K : \text{otherwise}). \end{cases}$$

Theorem

$$H_2^Q(Q_5(K)) \cong \begin{cases} 0 & (K = 0_1), \qquad \mathbb{Z}/10\mathbb{Z} \quad (K = 3_1), \\ \mathbb{Z} & (K : \text{otherwise}). \end{cases}$$

Theorem

$$\forall n > 5, H_2^Q(Q_n(K)) \cong \begin{cases} 0 & (K = 0_1), \\ \mathbb{Z} & (K : \text{otherwise}). \end{cases}$$

Corollary

(1) $H_2^Q(Q_n(K)) \cong 0 \Leftrightarrow K = 0_1 \ (n \in \mathbb{Z}_{>2})$.
(2) $H_2^Q(Q_n(K)) \cong H_2^Q(Q_n(3_1)) \Leftrightarrow K = 3_1 \ (n = 3, 4, 5)$.
(3) $H_2^Q(Q_3(K)) \cong H_2^Q(Q_3(5_1)) \Leftrightarrow K = 5_1$.

Quandle

Definition [Joyce '82, Matveev '82]

X: a non-empty set, $* : X^2 \to X$: a binary operation
$X = (X, *)$: a **quandle**
$\Leftrightarrow$ • $\forall x \in X, x * x = x$. • $\forall y \in X, S_y : X \to X; x \mapsto x * y$: a bijection.
• $\forall x, y, z \in X, (x * y) * z = (x * z) * (y * z)$.

Example K: an ori. knot in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, $E(K) = S^3 \setminus \mathrm{int} N(K)$.
$Q(K) := \{\alpha : I \to E(K) \mid \alpha(0) \in \partial E(K), \alpha(1) = \infty\}/\text{homotopy}$
$\alpha * \beta := \alpha \cdot \beta^{-1} \cdot (\text{a meridian loop at } \beta(0) \text{ in the } +\text{-direction}) \cdot \beta$.

Knot n-quandle

K, K': ori. 1-knots.

Fact
- $Q(K') \cong Q(K) \Leftrightarrow K' \sim K$ or $-K$! [Joyce '82, Matveev '82].
- $|Q(K)| < \infty \Leftrightarrow K = 0_1$. ($|Q(0_1)| = 1$.)

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Definition

$n \in \mathbb{Z}_{\geq 2}$, $Q_n(K) := Q(K)/x \sim S_y^n(x)$ $(S_y(x) = x * y)$.
$Q_n(K) := (Q_n(K), *)$: the **knot n-quandle** of K.

Fact
- $Q_2(4_1) \cong Q_2(5_1)$.
- $|Q_2(3_1)| = 3$, $|Q_3(3_1)| = 4$.
- $|Q_n(K)| = 1 \Leftrightarrow K = 0_1$ [Winker '84].
- $\forall X$: a finite quandle, $\exists n \in \mathbb{Z}_{\geq 2}$ s.t. $\mathrm{Hom}(Q(K), X) \overset{1:1}{\leftrightarrow} \mathrm{Hom}(Q_n(K), X)$.

Covering, extension and universal covering

$X, \tilde{X}$: connected quandles, Λ: a group.

- $p : \tilde{X} \twoheadrightarrow X$: a **covering** $\Leftrightarrow p(\tilde{y}) = p(\tilde{z})$ implies that $\forall \tilde{x} \in \tilde{X}, \tilde{x} * \tilde{y} = \tilde{x} * \tilde{z}$.
- $\tilde{X}$: an **extension** of X by a group Λ ($\Lambda \curvearrowright \tilde{X}$)
  $\Leftrightarrow \exists p : \tilde{X} \twoheadrightarrow X$ s.t.
  $$\begin{cases} \forall \lambda \in \Lambda, \forall \tilde{x}, \tilde{y} \in \tilde{X}, (\lambda \cdot \tilde{x}) * \tilde{y} = \lambda \cdot (\tilde{x} * \tilde{y}) \text{ and } \tilde{x} * (\lambda \cdot \tilde{y}) = \tilde{x} * \tilde{y}. \\ \forall x \in X, \Lambda \curvearrowright p^{-1}(x)\text{: free and transitive.} \end{cases}$$
- $p : \tilde{X} \twoheadrightarrow X$: a **universal covering**
  $\Leftrightarrow \forall \bar{p} : \bar{X} \twoheadrightarrow X$: a covering, $\exists \phi : \tilde{X} \to \bar{X}$: a quandle hom. s.t. $p = \bar{p} \circ \phi$.

Note $p : \tilde{X} \twoheadrightarrow X$: a quandle homomorphism.
p: a universal covering $\Rightarrow \tilde{X}$: an extension of X (by ${}^{\exists}\Lambda$) $\Rightarrow p$: a covering

- If $\tilde{X}$: an extension of X by a group Λ, "$\tilde{X} \cong X \times \Lambda$".
- If $p : \tilde{X} \twoheadrightarrow X$: a universal covering, $H_2^Q(X) \cong \Lambda_{\mathrm{ab}}$.
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Theorem [Eisermann '03]

K: an ori. knot, $\hat{K}$: the long knot obtained from K.
If K is nontrivial, then the following hold:
(1) $Q(\hat{K})$: an extension of $Q(K)$ by $\mathbb{Z}(= \langle l_K \rangle < \pi_1(\mathbb{R}^3 \backslash K))$.
(2) $\exists p : Q(\hat{K}) \twoheadrightarrow Q(K)$: a universal covering.

Corollary [Eisermann '03]

$H_2^Q(Q(K)) = 0 \Leftrightarrow K = 0_1$.

Goal To show the knot n-quandle version of the Eisermann's results.

K: an ori. knot, $n \in \mathbb{Z}_{\geq 2}$.
$\tau^n K$: the n-**twist spun** K (=the 2-knot obtained from K by n-twist spinning).
M_K^n: the n-fold branched covering space of S^3 branched along K.

Theorem

If K is nontrivial and $n \geq 2$, then the following hold:
(1) $Q(\tau^n K)$: an extension of $Q_n(K)$ by $\langle l_K \rangle$ $(< \pi_1(M_K^n))$,
where $l_K \in \pi_1(M_K^n) \cong \mathrm{Ker}(\pi_1(E(K)) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}; m_K \mapsto 1)/\langle\langle m_K^n \rangle\rangle$.
(2) $\exists p : Q(\tau^n K) \twoheadrightarrow Q_n(K)$: a universal covering.

Corollary

$H_2^Q(Q_n(K)) \cong \langle l_K \rangle < \pi_1(M_K^n)$.

To compute $H_2^Q(Q_n(K))$, it is sufficient to determine the order of l_K.
$\widetilde{K}$: the branching set of $M_K^n \Rightarrow l_K = [\widetilde{K}] \in \pi_1(M_K^n)$.

<u>K: prime</u> ($\Leftrightarrow M_K^n$: irreducible)

(i) $|\pi_1(M_K^n)| = \infty \Rightarrow$ the universal covering space of M_K^n is $\mathbb{R}^3$.
$p : \mathbb{R}^3 \to M_K^n$: the universal covering.
If l_K is trivial, each connected component of $p^{-1}(\widetilde{K})$ is S^1.
$\uparrow$ This contradicts to the Smith theory.

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(ii) $|\pi_1(M_K^n)| < \infty \Rightarrow$ the universal covering space of M_K^n is S^3.
(ii)-(a) $\underline{n = 3, 4, 5}$
Using results of [Inoue '23] and [Crans et. al. '19],
we can compute the order of $l_K \in \pi_1(M_K^n)$.

(ii)-(b) $\underline{n = 2}$
$p : S^3 \to M_K^2$: the universal covering, $L := p^{-1}(\widetilde{K})$: an ori. link in S^3.
$\Rightarrow$ $\text{(order of } l_K) = |\pi_1(M_K^2)|/|\{\text{components of } L\}|$.
In [Sakuma '90], the link L has been studied.

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$\underline{K\text{: composite}}$

$$\begin{aligned} K = K_1 \sharp K_2 \ &\Rightarrow (M_K^n, \widetilde{K}) \cong (M_{K_1}^n, \widetilde{K_1}) \sharp (M_{K_2}^n, \widetilde{K_2}) \\ &\Rightarrow l_K = l_{K_1} \cdot l_{K_2} \in \pi_1(M_{K_1}^n) * \pi_1(M_{K_2}^n) \cong \pi_1(M_K^n). \end{aligned}$$

Hence, if l_{K_1} and l_{K_2} are nontrivial, l_K is not a torsion element.
Remark K: prime.
$l_K \in \pi_1(M_K^n)$: trivial $\Leftrightarrow$ K: 2-bridge knot and $n = 2$.

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Thank you for your attention.