4 次元多様体のエキゾチック対に対するトライセクション種数

高橋 夏野
大阪大学 情報科学研究科

結び目の数理 VI
東京女子大学
2023 年 12 月 24 日

Introduction

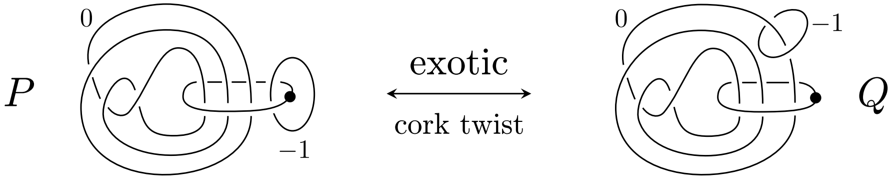
- Trisection : A decomposition of a 4-manifolds into three 1-handlebodies.
- Trisection genus : A 4-dim analogue of Heegaard genus for 3-manifolds.

Conjecture (Lambert-Cole–Meier 2020)

Any exotic pair of 4-manifolds has the same trisection genus.

Theorem (T. 2023)

$\exists(P, Q)$: exotic pair of 4-manifolds with trisection genus 4.



§1. Trisection

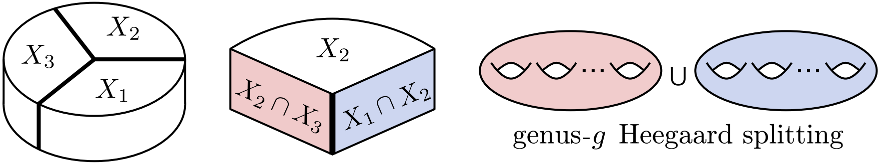
X : closed connected oriented smooth 4-manifold.

Definition (Gay–Kirby 2016)

$X = X_1 \cup X_2 \cup X_3$: (g, k)-trisection

$\overset{\text{def}}{\Leftrightarrow}$

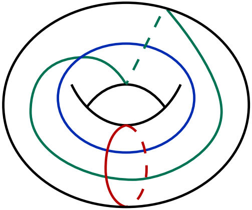
- $X_i \cong \natural^k (S^1 \times D^3)$: genus-k $4D$ 1-handlebody.
- $\partial X_i = (X_i \cap X_{i+1}) \cup (X_i \cap X_{i-1})$: genus-g Heegaard splitting.



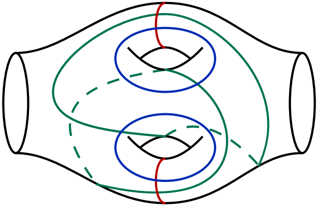
Remark

A relative trisection is a trisection for a 4-manifolds “with boundary”.

A trisection diagram $(\Sigma; \alpha, \beta, \gamma)$ is a description of a trisected 4-manifold.



A genus-1 trisection diagram of the complex projective plane $\mathbb{C}P^2$.



A genus-2 relative trisection diagram of $S^2 \times D^2$.

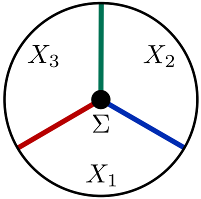
Theorem

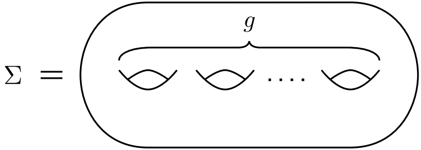
$$\frac{\{(\text{relative) trisections}\}}{\text{diffeo}} \overset{1:1}{\longleftrightarrow} \frac{\{(\text{relative) trisection diagrams}\}}{\text{diffeo of surfaces \& slides of curves}}$$

Trisection genus

Definition

The trisection genus of a 4-manifold X is defined as
$\mathrm{tg}\,(X) := \min\{g \in \mathbb{Z} \mid X$ admits a genus-g (relative) trisection$\}$.





Theorem (Haken 1968)

$\forall M, N$: 3-manifolds, $\mathrm{Hg}\,(M\#N) = \mathrm{Hg}\,(M) + \mathrm{Hg}\,(N)$.

Conjecture (Lambert-Cole–Meier 2020)

$\forall X, Y$: 4-manifolds, $\mathrm{tg}\,(X\#Y) = \mathrm{tg}\,(X) + \mathrm{tg}\,(Y)$.

Conjectures about trisection genus

If the trisection genus is additive, then the following conjecture holds:

Conjecture (Lambert-Cole–Meier 2020)

(X, Y) : exotic pair of 4-manifolds $\Rightarrow \mathrm{tg}\,(X) = \mathrm{tg}\,(Y)$.

If this is true, there are no exotic structures on certain small 4-manifolds.
(e.g. $S^4, \mathbb{C}P^2, S^1 \times S^3, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2 \times S^2, \dots$)

Theorem (Meier–Zupan 2018 and Lambert-Cole–Meier 2020)

$\exists$ infinitely many exotic pairs of closed 4-manifolds with the same tg.

Example (Spreer–Tillmann 2018)

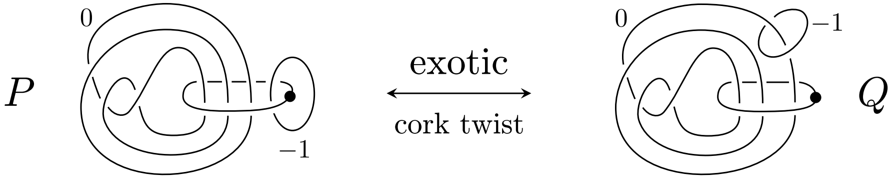
$\mathrm{tg}\,(K3 \# \overline{\mathbb{C}P^2}) = \mathrm{tg}\,(20\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}) = 23.$

<u>Relative case</u> : Previously, it had not been found exotic 4-manifolds with boundary($\neq S^3$) that have the same trisection genus.

§2. Main result

Theorem (T. 2023)

$\exists(P, Q)$: exotic pair of 4-manifolds with ∂ s.t. $\mathrm{tg}\,(P) = \mathrm{tg}\,(Q) = 4$.



Overview of proof

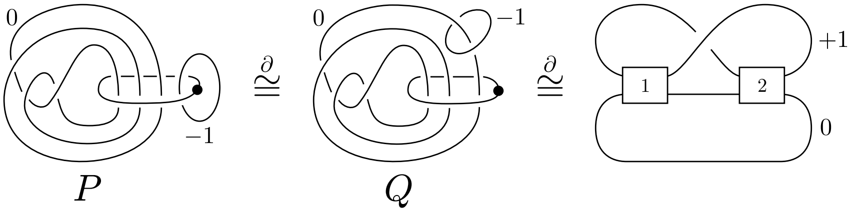
- Prove that $\mathrm{tg}\,(P)$ and $\mathrm{tg}\,(Q)$ are greater than or equal to 4.
- Give explicit genus-4 relative trisection diagrams of P and Q.

Theorem (T. 2022)

X : 4-manifold with boundary.
∂X is a hyperbolic 3-manifold $\Rightarrow \mathrm{tg}\,(X) \geq \chi(X) + 2$.

By the following conditions, we have $\mathrm{tg}\,(P) = \mathrm{tg}\,(Q) \geq 4$.

- $\chi(P) = \chi(Q) = 2$.
- The boundaries of P and Q are hyperbolic.

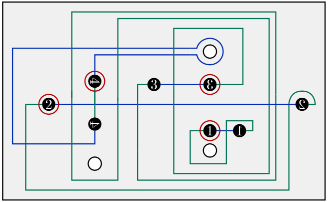


A construction of genus-4 relative trisections

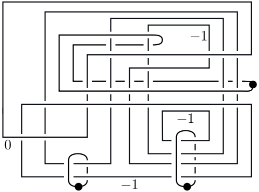
(1) Give genus-4 relative trisection diagrams.

(2) Draw the corresponding handlebody diagrams.

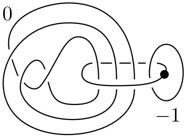
(3) Prove that each of the 4-manifolds coincides with P and Q.



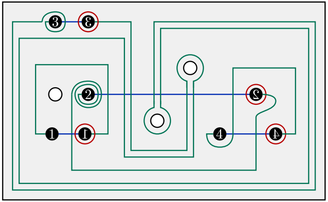
→



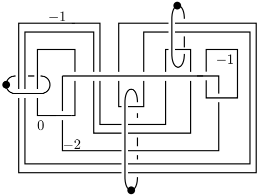
→



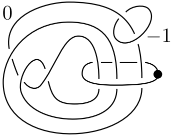
P



→

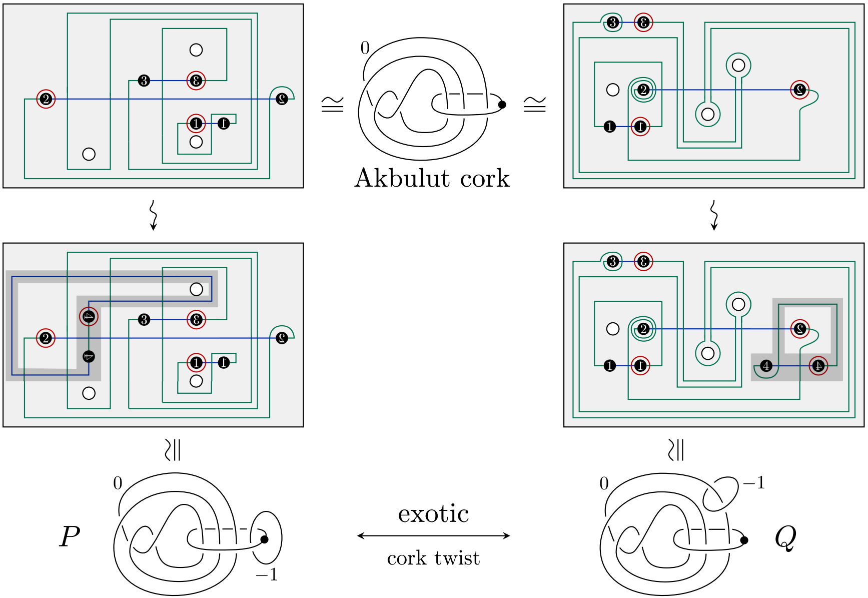


→



Q

A construction of genus-4 relative trisections



Trisections for two families of contractible 4-manifolds

Theorem (T. 2023)

- The cork M_n admits a genus-3 trisection, and $\mathrm{tg}\,(M_n) = 3$.
- The contractible 4-manifold $W^-(l,k)$ admits a genus-3 trisection, and $\mathrm{tg}\,(W^-(l,k)) = 3$ if $l + k \notin \{2,3,4,5\}$.

※ $W^-(0,0)$ and M_1 are diffeomorphic to the Akbulut cork.