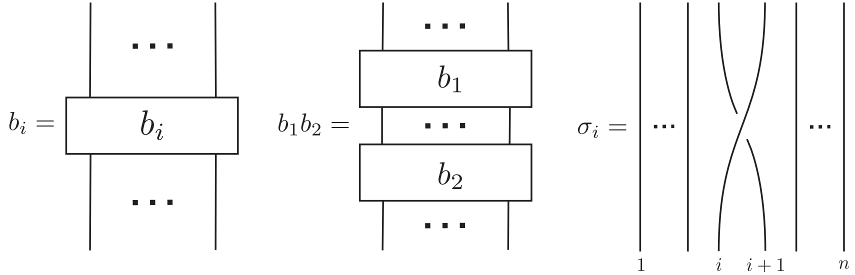
Quasitoric 組み紐群の最小生成系について

大森　源城

芝浦工業大学

2023 年 12 月 26 日
結び目の数理 VI @東京女子大学

B_n: the classical braid group of $n \geq 1$ strands.



Fact

- $B_1 = 1$, $B_2 = \langle \sigma_1 \rangle \cong \mathbb{Z}$.
- $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \rightleftarrows \sigma_j \ (|j-i| \geq 2),\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$.
- *For $n \geq 3$, B_n is generated by two element.* (← $B_n = \langle \sigma_1,$ [illegible] $\rangle$)
- $H_1(B_n; \mathbb{Z}) \cong \mathbb{Z}$ $(n \geq 2)$.

Theorem (Alexender (1923))

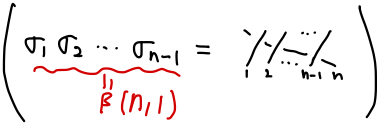
Every link is represented by a closure of a braid.

Definition

- A *torus link* (resp. a *torus knot*) is a link (resp. a knot) which is included in a standardly embedded torus in S^3 or $\mathbb{R}^3$.
- A *toric braid* is a braid whose closure is a torus link.

For $n \geq 2$ and $m \geq 1$,

$$\beta(n,m) := (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^m \in B_n.$$



Remark

A closure of $\beta(n,m)$ is a torus link.

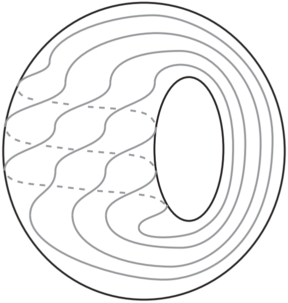
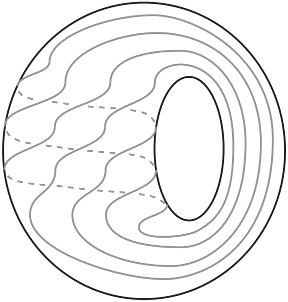


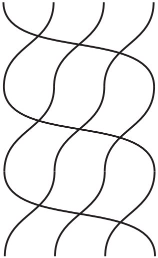
Figure: The closure of $\beta(4,3)$.

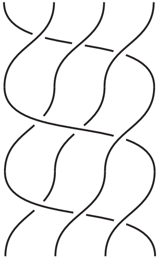
Manturov ('02) introduced a <u>n-*quasi toric braid*</u> in B_n that is a generalization of $\beta(n,m)$ and has a form

$$(\sigma_1^{\varepsilon_1^1}\sigma_2^{\varepsilon_2^1}\cdots\sigma_{n-1}^{\varepsilon_{n-1}^1})(\sigma_1^{\varepsilon_1^2}\sigma_2^{\varepsilon_2^2}\cdots\sigma_{n-1}^{\varepsilon_{n-1}^2})\cdots(\sigma_1^{\varepsilon_1^m}\sigma_2^{\varepsilon_2^m}\cdots\sigma_{n-1}^{\varepsilon_{n-1}^m})\in B_n,$$

where $\varepsilon_i^j \in \{\pm 1\}$.







$(\sigma_1\sigma_2\sigma_3)(\sigma_1^{-1}\sigma_2^{-1}\sigma_3)(\sigma_1^{-1}\sigma_2\sigma_3)$

Theorem (Manturov ('02), Lamm ('99))

A link represented by a closure of a n-braid is also represented by a closure of a n-quasitoric braid.

$QB_n := \{n\text{-quasi toric braids}\} \subset B_n$.

Theorem (Manturov ('02))

QB_n is a subgroup of B_n.

※ $QB_1 = B_1 = 1$, $QB_2 = B_2 = \langle \sigma_1 \rangle \cong \mathbb{Z}$.

Main theorems:

Theorem (O. (arXiv:2301.07917))

- *For $n \geq 3$ is odd, QB_n is generated by $\frac{n+1}{2}$ elements.*
- *For $n \geq 4$ is even, QB_n is generated by $\frac{n+2}{2}$ elements.*

Theorem (O. (arXiv:2301.07917))

$$H_1(QB_n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\frac{n-1}{2}} \oplus \mathbb{Z}_n & \text{if } n \geq 3 \text{ is odd,} \\ \mathbb{Z}^{\frac{n}{2}} \oplus \mathbb{Z}_{\frac{n}{2}} & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

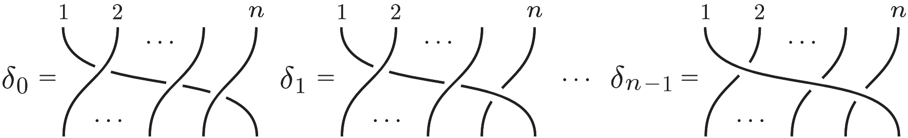
$\rightsquigarrow$ The generating set for QB_n above is minimal!!

These results above for $n = 3$ are obtained from results of Shigeta ('23).

An explicit minimal generating set

$$\delta_0 := \beta(n,1) = \sigma_1\sigma_2\cdots\sigma_{n-1} \in QB_n,$$
$$\delta_i := \sigma_1\cdots\sigma_{n-i-1}\sigma_{n-i}^{-1}\cdots\sigma_{n-1}^{-1} \in QB_n \quad (1 \le i \le n-1),$$

i.e. $\delta_0 = \sigma_1\cdots\sigma_{n-1}$, $\delta_1 = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^{-1}$, $\delta_2 = \sigma_1\cdots\sigma_{n-3}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}$, $\ldots$, $\delta_{n-2} = \sigma_1\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}$, $\delta_{n-1} = \sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-1}^{-1}$.



Theorem (O. (arXiv:2301.07917))

- *For* $n \ge 3$ *is odd,* QB_n *is generated by* δ_i $(0 \le i \le \frac{n-1}{2})$.
- *For* $n \ge 4$ *is even,* QB_n *is generated by* δ_i $(0 \le i \le \frac{n}{2})$.

S_n: the symmetric group of degree n.

The action $B_n \curvearrowright \{1, 2, \ldots, n\}$ induces the surjective homomorphism

$$\Psi\colon B_n \to S_n.$$

$\sigma_i \mapsto (i\ i{+}1)$

$PB_n := \ker \Psi$: the *pure braid group*.

Then, we have the following exact sequence:

$$1 \longrightarrow PB_n \longrightarrow B_n \xrightarrow{\Psi} S_n \longrightarrow 1.$$

$\rho := (1\ 2\ \cdots\ n) \in S_n$.
Since $\Psi(\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\cdots\sigma_{n-1}^{\varepsilon_{n-1}}) = \rho \in S_n$ for any $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1} \in \{\pm 1\}$,
$\Psi(QB_n) = \langle \rho \rangle \cong \mathbb{Z}_n$.

Proposition (Manturov ('02))

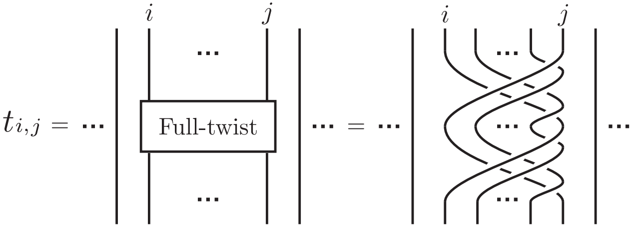
$PB_n \subset QB_n$.

Thus, we have the exact sequence

$$1 \longrightarrow PB_n \longrightarrow QB_n \xrightarrow{\Psi} \mathbb{Z}_n[\rho] \longrightarrow 1.$$

A finte presentation for PB_n

For $1 \leq i < j \leq n$,



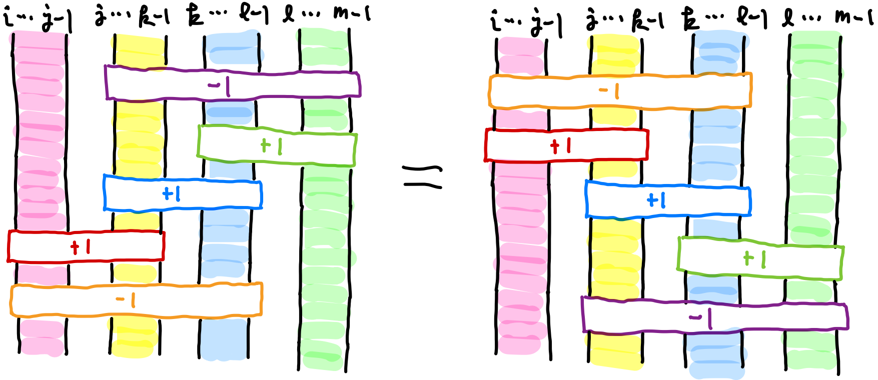
Proposition (Namanya ('23), O.)

For $n \geq 1$, PB_n admits the presentation with generators $t_{i,j}$ for $1 \leq i < j \leq n$ and the following defining relations:

1. $t_{i,j}t_{k,l} = t_{k,l}t_{i,j}$ *for* $j < k$, $k \leq i < j \leq l$, *or* $l < i$,
2. $t_{j,m-1}^{-1}t_{k,m-1}t_{j,l-1}t_{i,k-1}t_{i,l-1}^{-1} = t_{i,l-1}^{-1}t_{i,k-1}t_{j,l-1}t_{k,m-1}t_{j,m-1}^{-1}$ *for* $1 \leq i < j < k < l < m \leq n$.

(左辺) = $t_{j,m-1}^{-1}\, t_{k,m-1}\, t_{j,l-1}\, t_{i,k-1}\, t_{i,l-1}^{-1}$

(右辺) = $t_{i,l-1}^{-1}\, t_{i,k-1}\, t_{j,l-1}\, t_{k,m-1}\, t_{j,m-1}^{-1}$



A finite presentation for QB_n

Proposition

For $n \geq 1$, QB_n admits the presentation with generators δ_0 and $t_{i,j}$ for $1 \leq i < j \leq n$, and the following defining relations:

$\delta_0 = \sigma_1\sigma_2\cdots\sigma_{n-1}$

1. $t_{i,j}t_{k,l} = t_{k,l}t_{i,j}$ for $j < k$, $k \leq i < j \leq l$, or $l < i$,
2. $t_{j,m-1}^{-1}t_{k,m-1}t_{j,l-1}t_{i,k-1}t_{i,l-1}^{-1} = t_{i,l-1}^{-1}t_{i,k-1}t_{j,l-1}t_{k,m-1}t_{j,m-1}^{-1}$ for $1 \leq i < j < k < l < m \leq n$,
3. $\delta_0^n = t_{1,n}$,
4. $$\delta_0 t_{i,j}\delta_0^{-1} = \begin{cases} t_{i+1,j+1} & \text{for } j < n, \\ t_{1,n} & \text{for } (i,j) = (1,n), \\ t_{2,n}^{-1}t_{1,i}^{-1}t_{2,i}t_{i+1,n}t_{1,n} & \text{for } j = n \text{ and } i > 1. \end{cases}$$

Theorem (O. (again))

Put $X = t_{1,n-1}\delta_0^{-n+2}$.

$$H_1(QB_n;\mathbb{Z}) \cong \begin{cases} \left\langle t_{1,j}\ (2 \leq j \leq \frac{n-1}{2}),\ \delta_0,\ X \,\middle|\, X^n = 1 \right\rangle & \text{if } n \geq 3 \text{ is odd}, \\ \left\langle t_{1,j}\ (2 \leq j \leq \frac{n}{2}),\ \delta_0,\ X \,\middle|\, X^{\frac{n}{2}} = 1 \right\rangle & \text{if } n \geq 4 \text{ is even}. \end{cases}$$

Future works

Recall: $B_n = \Psi^{-1}(S_n)$, $PB_n = \Psi^{-1}(\{1\})$, and $QB_n = \Psi^{-1}(\langle \rho \rangle)$.

- For $n \geq 3$, B_n is generated by two elements.
- For $n \geq 3$, PB_n is generated by $\binom{n}{2}$ elements.
- For $n \geq 3$ is odd, QB_n is generated by $\frac{n+1}{2}$ elements.
- For $n \geq 4$ is even, QB_n is generated by $\frac{n+2}{2}$ elements.

$\mathrm{Mod}_{0,n}$: the mapping class group of 2-sphere with n marked points.
$\bar{\Psi}\colon \mathrm{Mod}_{0,n} \to S_n$: the natural sujective homomorphism.

- For $n \geq 3$, $\mathrm{Mod}_{0,n} = \bar{\Psi}^{-1}(S_n)$ is generated by two elements.
- For $n \geq 3$, $\mathrm{PMod}_{0,n} = \bar{\Psi}^{-1}(\{1\})$ is generated by $\binom{n-1}{2} - 1$ elements. ← the pure MCG
- For $n \geq 1$, $\mathrm{LMod}_{2n+2} = \bar{\Psi}^{-1}(W_{2n+2})$ is generated by three elements.

↑ the liftable MCG $\cong (S_n \times S_n) \rtimes \mathbb{Z}_2$

Problem

- For given subgroup H of S_n, determine the minimum number of generators for $\Psi^{-1}(H)$ or $\bar{\Psi}^{-1}(H)$.
- When does the number depend on n?

Thank you for your attention!!