Shadow-complexity & trisection genus

Hironobu Naoe (Chuo Univ.)
joint work w/ Masaki Ogawa (Tohoku Univ. MathCCS)

結び目の数理 VI @東京女子大学

December 24, 2023

Table of contents

1 Trisections

2 Shadows

3 Results

$*$ Suppose any manifold is compact, connected, oriented and smooth.

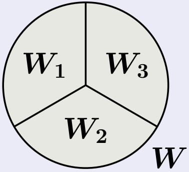
Trisections (1/2)

W : a closed 4-manifold.

Definition

A *trisection* $\mathcal{T}$ of W is a decomposition $W = W_1 \cup W_2 \cup W_3$ s.t. for some $g, k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$,

- $W_i \cong \natural k_i (S^1 \times B^3)$,
- $W_i \cap W_j \cong \natural g (S^1 \times B^2)$, and
- $W_1 \cap W_2 \cap W_3 \cong \Sigma_g$.



$g(\mathcal{T}) := g$ is called the *genus* of $\mathcal{T}$.

Theorem (Gay and Kirby, '16)

Any closed 4-manifold admits a trisection.

Trisections (2/2)

Definition

The *trisection genus* of W is

$$g(W) := \min\{\, g(\mathcal{T}) \mid \mathcal{T} \text{ is a trisection of } W\}.$$

Note $g : \{\text{closed 4-mfd.}\} \to \mathbb{Z}_{\geq 0}$ is an invariant.

Classification

- $g(W) = 0 \iff W \cong S^4$.
- $g(W) = 1 \iff W \cong \pm\mathbb{C}P^2$ or $S^1 \times S^3$.
- [Meier and Zupan, '17]
  $g(W) = 2 \iff W \cong \pm 2\mathbb{C}P^2,\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2},\ 2(S^1 \times S^3),$
  $\pm\mathbb{C}P^2 \# (S^1 \times S^3)$ or $S^2 \times S^2$.
- (Conjecture by Meier) An irreducible 4-mfd. w/ $g = 3$ is either $\mathcal{S}_p$ or $\mathcal{S}'_p$ for some $p \in \mathbb{Z}$, where $\mathcal{S}_p$ and $\mathcal{S}'_p$ are Pao's manifolds.

Table of contents

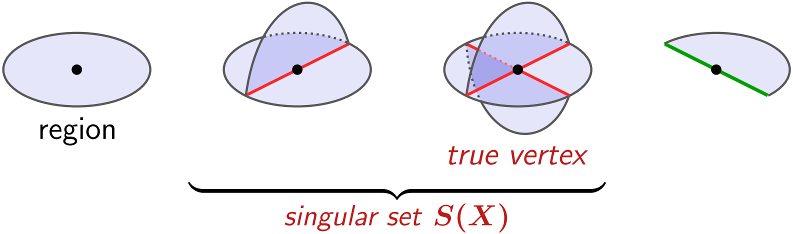
1 Trisections

2 Shadows

3 Results

Shadows (1/2)

Local models of a *simple polyhedron* X :



Definition

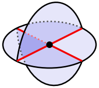
A *shadow* of a closed 4-manifold W is a simple polyhedron $X \subset W$ s.t.

- $^{\forall}p \in X$, $\mathrm{Nbd}(p; X) \subset {}^{\exists}B^3 \subset W$ (local-flatness), and
- $W \setminus N(X) \cong \natural k(S^1 \times B^3)$.

Remark

- A shadow is a 2-skeleton of a 4-manifold.
- Any closed 4-manifold admits a shadow [Turaev, '94].

Shadows (2/2)

$c(X) :=$ # of true vertices  of X.

The *shadow-complexity* and the *special shadow-complexity* of W are

$$\mathrm{sc}(W) := \min\{\, c(X) \mid X \text{ is a shadow of } W \,\}$$

$$\mathrm{sc}^{\mathrm{sp}}(W) := \min\left\{ c(X) \;\middle|\; \begin{array}{l} X \text{ is a shadow of } W \text{ s.t.} \\ \text{all regions are open } 2\text{-disks} \end{array} \right\}.$$

Note $\mathrm{sc},\ \mathrm{sc}^{\mathrm{sp}} : \{\text{closed } 4\text{-mfd.}\} \to \mathbb{Z}_{\geq 0}$ are invariants.

- [Costantino, '06]
  $\mathrm{sc}^{\mathrm{sp}}(W) = 0$
  $\iff \mathrm{sc}^{\mathrm{sp}}(W) \leq 1$
  $\iff W \cong S^4,\ \pm\mathbb{C}P^2,\ \pm 2\mathbb{C}P^2,\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ or $S^2 \times S^2$.
- [Martelli, '11] $\mathrm{sc}(W) = 0$.
- [Koda, Martelli and N., '22] $\mathrm{sc}^*(W) \leq 1$.

Table of contents

1 Trisections

2 Shadows

3 Results

Results (1/6)

X : simple polyhedron

Fix $r \in \mathbb{R}_{\geq 0}$ (*weight*). Define $c_r(X)$ as follows:

if X is not a closed surface, set

$$c_r(X) := c(X) + \sum_{R\,:\,\text{region}} r(1 - \chi(R)),$$

and set $c_r(S^2) := 0$.

Remark X is a closed surface except for S^2
$\Longrightarrow$ X can not be a shadow of any closed 4-manifold.

Definition

The *r-weighted shadow-complexity* of W is

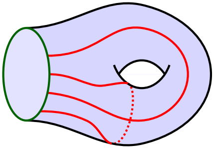
$$\mathrm{sc}_r(W) := \min\{c_r(X) \mid X \text{ is a shadow of } W\}$$

Note $\mathrm{sc}_r : \{\text{closed 4-mfd.}\} \to \{a + rb \mid a, b \in \mathbb{Z}_{\geq 0}\}$ is an invariant.

Results (2/6)

The value $1 - \chi(R)$ in the definition

$$c_r(X) = c(X) + \sum_{R\,:\,\text{region}} r(1 - \chi(R))$$



means the number of arcs cutting R open into a disk.

The collection Γ of such arcs is called a *cut system* for X.

Note $S(X) \cup \Gamma$ is a 1-skeleton of X and also of W.

Proposition 1 (Ogawa-N.)

For any $r, r' \in \mathbb{R}_{\geq 0}$ *with* $r \leq r'$,

$$\mathrm{sc}(W) \leq \mathrm{sc}_r(W) \leq \mathrm{sc}_{r'}(W) \leq \mathrm{sc}^{\mathrm{sp}}(W).$$

Moreover, $\mathrm{sc}(W) = \mathrm{sc}_0(W)$, *and* $\mathrm{sc}_r(W) = \mathrm{sc}^{\mathrm{sp}}(W)$ *if* $r \geq 2$.

Results (3/6)

Proposition 2 (Ogawa-N.)

For any $r \in \mathbb{R}_{>0}$, sc_r is finite-to-one.
(i.e. for any $a \in \mathbb{R}$, $\#\{W \mid \mathrm{sc}_r(W) = a\} < \infty$.)

- sc is NOT finite-to-one (e.g. $\mathrm{sc}(k\mathbb{C}\mathrm{P}^2) = 0$ for any $k \in \mathbb{Z}$).
- $\mathrm{sc}^{\mathrm{sp}}$ is finite-to-one [Costantino, '06] (c.f. [Martelli, '05]).

Theorem (Martelli, '05)

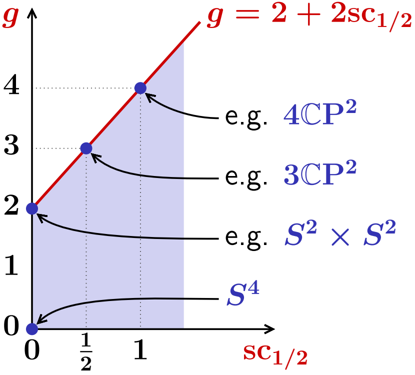
For a link $L \subset \partial(k(S^1 \times B^3))$, there exist at most finitely many closed 4-manifolds (up to diffeo.) obtained from $k(S^1 \times B^3)$ by ataching 2-handles along L and some 3- and 4-handles.

Theorem 3 (Ogawa-N.)

For any closed 4*-manifold* W *and any* $r \geq 1/2$,

$$g(W) \leq 2 + 2\mathrm{sc}_r(W).$$

Remark When $r = 1/2$, the inequality is the best possible:



Results (5/6)

Theorem 3 (Ogawa-N.)

For any closed 4-manifold W *and any* $r \geq 1/2$,

$$g(W) \leq 2 + 2\mathrm{sc}_r(W).$$

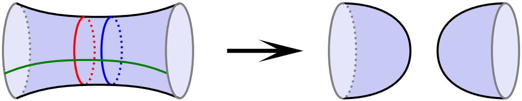
Sketch of proof

X and Γ define a trisection $\mathcal{T}$: $\begin{cases} W_1 := N(S(X) \cup \Gamma), \\ W_2 := N(\text{regions} \cup \tau), \\ W_3 := W \setminus \mathrm{Int}(W_1 \cup W_2), \end{cases}$

where $\tau \subset \partial W_1$ is an unknotting tunnel of $\partial(\text{regions})$.

$\rightsquigarrow g(\mathcal{T}) = 1 + \#\tau$

$\qquad = 1 + (2\mathrm{rank}\pi_1(S(X)) + \#\Gamma) = 3 + 2c_{1/2}(X)$

$\rightsquigarrow$ destabilization 

□

Results (6/6)

We focus on the case $r = 1/2$ due to $g(W) \leq 2 + 2\mathrm{sc}_{1/2}(W)$.

Recall $\mathrm{sc}_{1/2}(W) \in \frac{1}{2}\mathbb{Z}_{\geq 0} = \{0,\ \frac{1}{2},\ 1,\ \frac{3}{2}, \ldots\}$.

Theorem 4 (Ogawa-N.)

$\mathrm{sc}_{1/2}(W) = 0$

$\iff W \cong S^4,\ \pm\mathbb{C}P^2,\ \pm 2\mathbb{C}P^2,\ \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ *or* $S^2 \times S^2$.

($\iff \mathrm{sc}^{\mathrm{sp}}(W) = 0$ [Costaintino, '06])

Theorem 5 (Ogawa-N.)

$\mathrm{sc}_{1/2}(W) = 1/2$

$\iff W \cong \pm 3\mathbb{C}P^2,\ 2\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\ \mathbb{C}P^2\#2\overline{\mathbb{C}P^2},$

$S^1 \times S^3,\ \pm\mathbb{C}P^2\#(S^1 \times S^3),\ \mathcal{S}_2,\ \mathcal{S}_2'$ *or* $\mathcal{S}_3$.

Summary

Inequality between $g(W)$ and $\mathrm{sc}_r(W)$

For any closed 4-manifold W and any $r \geq 1/2$,

$$g(W) \leq 2 + 2\mathrm{sc}_r(W).$$

trisection genus ↗

↖ r-weighted shadow-complexity

Classification of 4-maifolds w/ $\mathrm{sc}_{1/2} \leq 1/2$

- $\mathrm{sc}_{1/2}(W) = 0$

  $\Longleftrightarrow W \cong S^4,\ \pm\mathbb{C}P^2,\ \pm 2\mathbb{C}P^2,\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ or $S^2 \times S^2$.

- $\mathrm{sc}_{1/2}(W) = 1/2$

  $\Longleftrightarrow W \cong \pm 3\mathbb{C}P^2,\ 2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2},\ \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2},$
  $S^1 \times S^3,\ \pm\mathbb{C}P^2 \# (S^1 \times S^3),\ \mathcal{S}_2,\ \mathcal{S}_2'$ or $\mathcal{S}_3$.

Thank you very much!