向き付け不可能曲面のファイン曲線グラフの自己同型群

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Plan

§ 1. Background and main result

§ 2. Outline of proof

§ 3. Issue for nonorientable surfaces

§ 1. Background and main result

$S = S_{g,p}^{b}$: a conn. ori. surf. of genus g with b bdry comps. and p punctures.

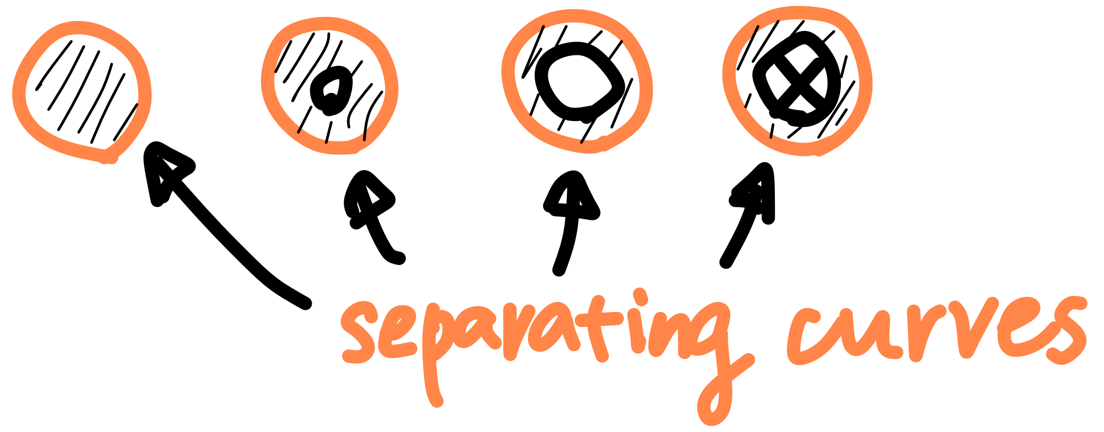
$N = N_{g,p}^{b}$: 〃 nonori. surf 〃.

$F = S$ or N.

Def (fine curve graph by Bowden – Hensel – Webb, 2022)

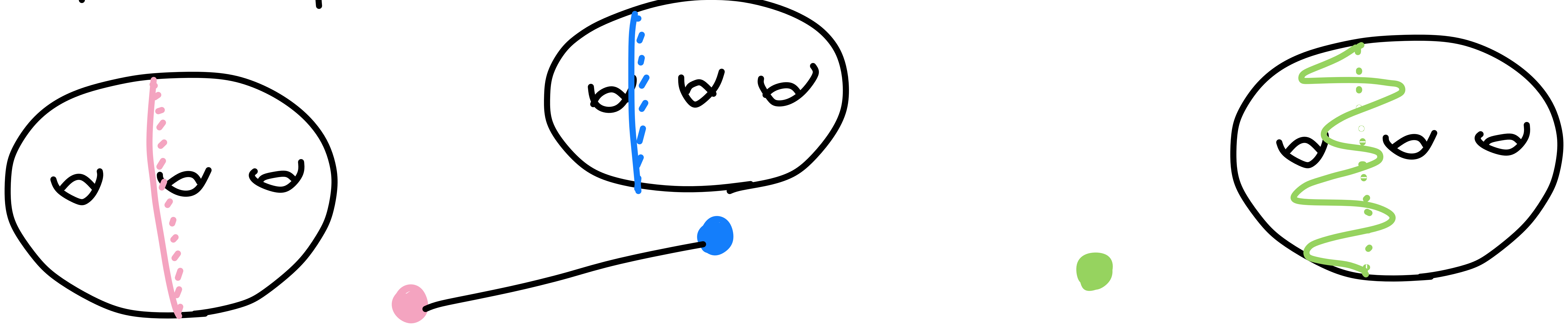
The fine curve graph $\mathcal{C}^{\dagger}(F)$ of F :

NOT:



- a vertex : an ess. s.c.c. on F.
- an edge $\{\alpha, \beta\}$: a pair of vertices that are disjoint on F.

e.g.



<u>cf.</u>

(Classical) curve graph $C(F)$ of F

$\underset{\text{def}}{\iff} \begin{cases} \cdot \text{ a vertex : a homotopy class of an ess. s.c.c. on } F. \\ \cdot \text{ an edge } \{a, b\} \text{: a pair of vertices that can be realized disjointly on } F. \end{cases}$

- $\mathrm{Mod}(F) \curvearrowright C(F)$
- $\mathrm{Diff}(F) \curvearrowright C^{\dagger}(F)$, $\mathrm{Homeo}(F) \curvearrowright C^{\dagger}(F)$

<u>Thm.</u> (Long – Margalit – Pham – Verberne – Yao, 2021 arXiv, to appear Trans. AMS)

S_g : a closed orientable surface of $g \geq 2$.

The natural map η : $\mathrm{Homeo}(S_g) \to \mathrm{Aut}\, C^{\dagger}(S_g)$ is an isomorphism.

<u>Main Thm.</u> (Kimura – K., 2023, arXiv)

N_g : a closed nonorientable surface of $g \geq 4$.

The natural map η : $\mathrm{Homeo}(N_g) \to \mathrm{Aut}\, C^{\dagger}(N_g)$ is an isomorphism.

§ 2. Outline of proof

$\mathcal{E}C^{\dagger}(F)$: an extended fine curve graph of F

$\overset{\text{def}}{\Longleftrightarrow}$ vertices are ess. s.c.c.s and iness. s.c.c.s bounding a disk on F.

Lem.

For any closed surf. F, the natural map ν : $\mathrm{Homeo}(F) \longrightarrow \mathrm{Aut}\,\mathcal{E}C^{\dagger}(F)$ is an isomorphism.

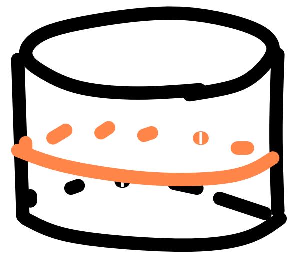
Goal : To prove the composition "extention hom"

$$\mathrm{Homeo}(F) \xrightarrow{\eta} \mathrm{Aut}\, C^{\dagger}(F) \underset{\cong}{\xrightarrow{\mathcal{E}}} \mathrm{Aut}\,\mathcal{E}C^{\dagger}(F) \xrightarrow{\nu^{-1}} \mathrm{Homeo}(F)$$

is identity.

Main : construct !

§ 3. Issue for nonori. surfaces



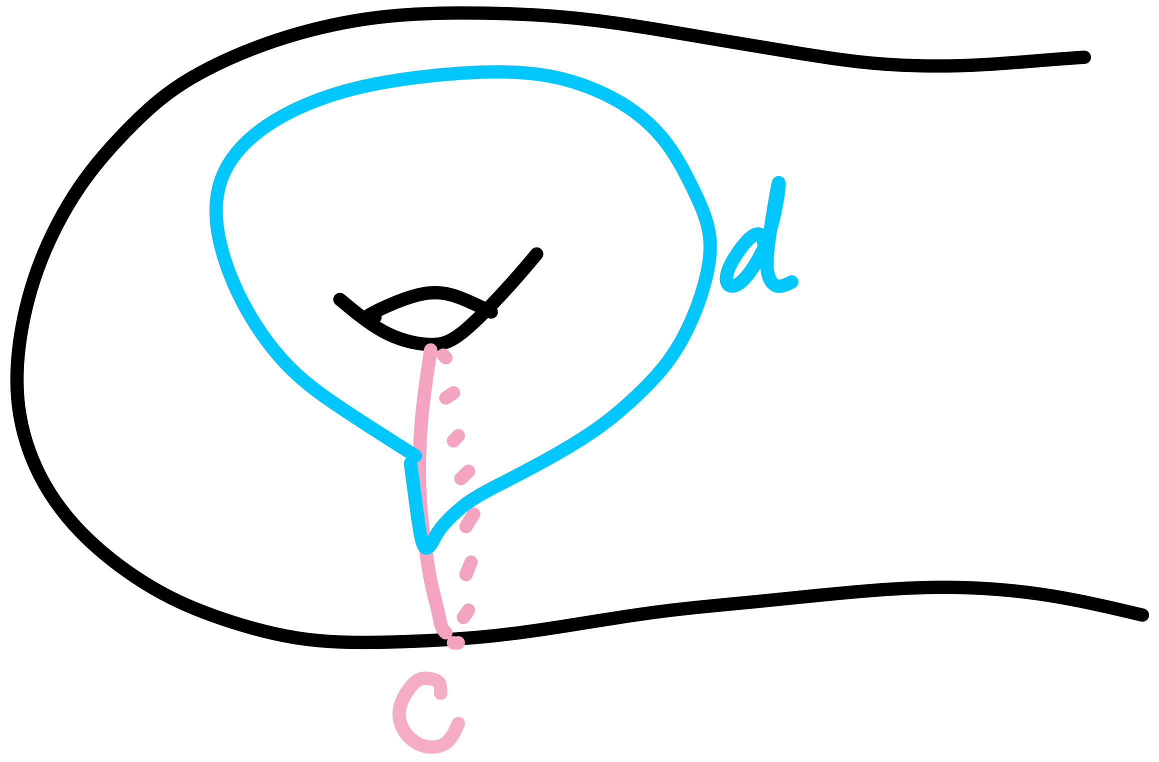
two-sided s.c.c.

one-sided s.c.c.

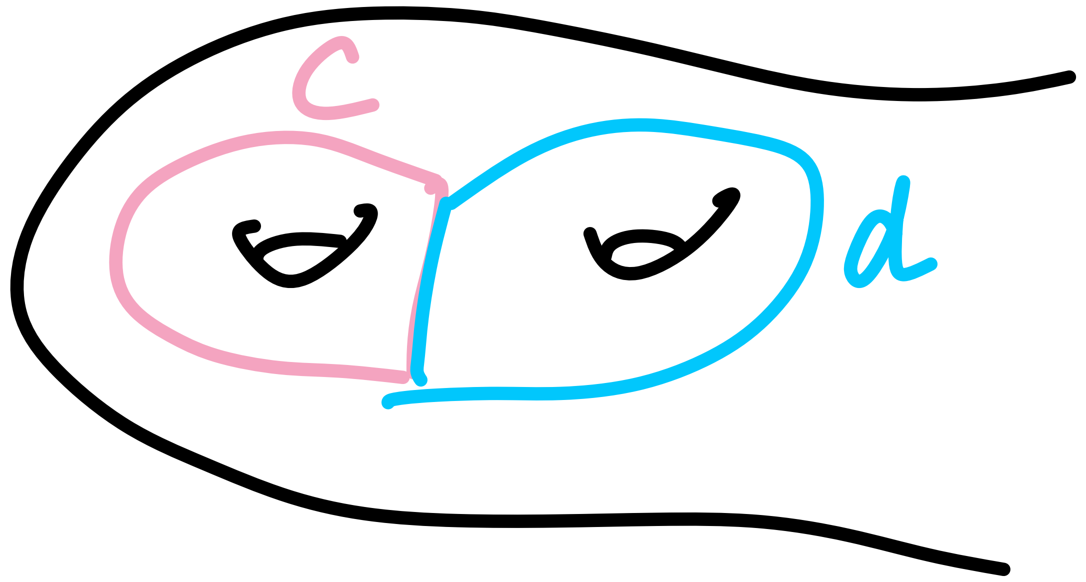
Def.

$c, d \in \mathcal{C}^{t}(N)$, c, d : two-sided.

- $\{c, d\}$ is a <u>torus pair</u> if
  - $c \cap d$ is a single interval,
  - c and d are crossing.

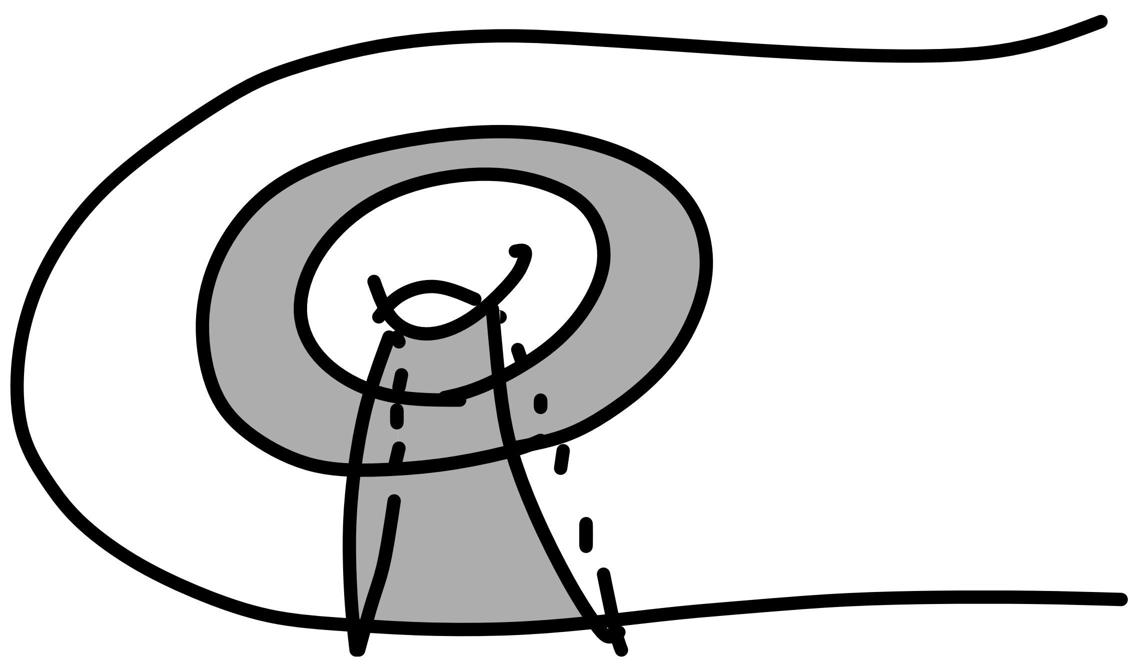


- $\{c, d\}$ is a <u>pants pair</u> if
  - $c \cap d$ is a single interval,
  - c and d are noncrossing.



- The hull of a collection of curves in a surface is the union of the curves along with disks bounded by the curves.

e.g.



<u>Issue</u> We need the following lemma.

<u>Lem</u>
$N = N_g$ $(g \geq 4)$. $\alpha \in \mathrm{Aut}\, C^{+}(N)$ preserves the set of torus pairs.

◎ The case of orientable surfaces

They prove Lem in two steps.

Step 1 : α preserves the union of the torus pairs and the pants pairs.

Step 2 : α preserves the set of the torus pairs.

In Step 1, they prove the following.

$\{c, d\}$: a pair of intersecting vertices of $C^{\dagger}(S)$.

Then, the following three statements are equivalent.

(1) The pair $\{c, d\}$ is a torus pair or a pants pair.

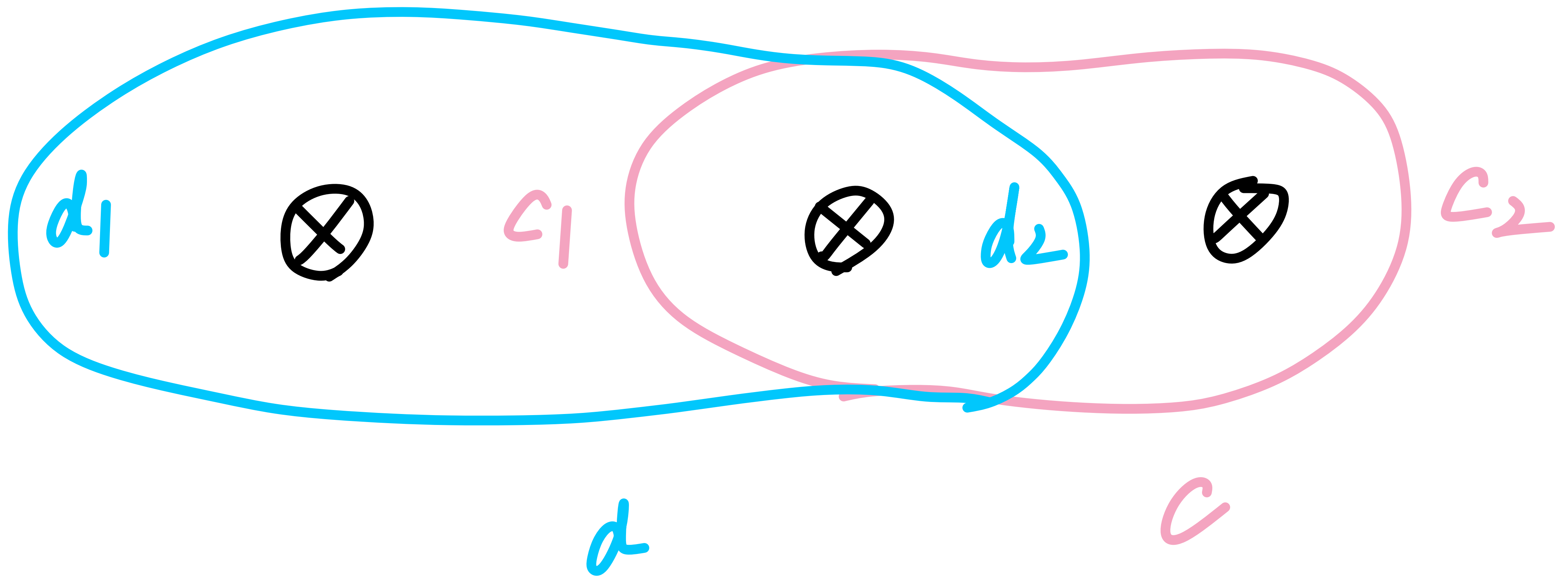
(2) There is at most one other vertex of $C^{\dagger}(S)$ that lies in the hull of $\{c, d\}$.

(3) There is at most one other vertex of $C^{\dagger}(S)$ whose link contains the link of $\{c, d\}$.

@ For nonorientable surfaces

$(1) \cancel{\Leftrightarrow} (2)$!

e.g.



$e_1 = c_1 \cup d_1$: iness.

$e_2 = c_1 \cup d_2$: iness.

$e_3 = c_2 \cup d_1$: ess.

$e_4 = c_2 \cup d_2$: iness.

c, d : ess.

$\{c, d\}$ is not a torus pair nor a pants pair.

However, there is just one other ess. s.c.c. in the hull of $\{c, d\}$.

Our solution

We add a condition to $c, d \in C^{\dagger}(N)$.

We can prove the following.

$\{c, d\}$: a pair of intersecting curves which are ess. and nonseparating.

Then, the following three statements are equivalent.

(1) The pair $\{c, d\}$ is a torus pair or a pants pair.

(2) There is at most one other vertex of $C^{\dagger}(S)$ that lies in the hull of $\{c, d\}$.

(3) There is at most one other vertex of $C^{\dagger}(S)$ whose link contains the link of $\{c, d\}$.

REASON

- why the added condition "nonseparating" does not affect the proof :

The set of torus pairs consisting of nonsep. s.c.c.s.

= The set of torus pairs.