Constructing pseudo Goeritz matrix from Dehn coloring of virtual knots

Unhou Kin

December 25, 2023

Nagoya City Univercity D1

Table of contents

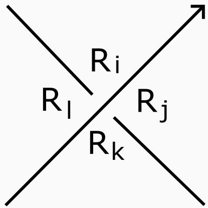
- Dehn coloring of classical knots
- The pseudo Goeritz matrices of virtual knots
- Main result

- **A Dehn coloring matrix of classical knots(1/3)**

D : A classical knot diagram with m complementary regions

$R_1, \cdots, R_m$: Complementary regions of D

$Z_1, \cdots, Z_m$: Variables corresponding to the region $R_1, \cdots, R_m$ of D



$$\boxed{Z_i + Z_j - Z_k - Z_l = 0} \quad \cdots ☆$$

where the variables Z_i, Z_j, Z_k, Z_l correspond to the regions R_i, R_j, R_k, R_l in figure

- **A Dehn coloring matrix of classical knots(2/3)**

Dehn coloring equations of D :

The system of ☆ equations

Dehn coloring matrix of D :

The coefficient matrix of the Dehn coloring equations

Theorem [M. Horiuchi, K. Ichihara, E. Matsudo, S. Yoshida]

A Goeritz matrix can constract from a Dehn coloring matrix of classical knots.

• A Dehn coloring matrix of classical knots(3/3)

A **Dehn n-coloring** of D :
a map $C : \{R_1, \cdots, R_m\} \to \mathbb{Z}/n\mathbb{Z}$

Theorem [Alexander Madaus, Maisie Newman, Heather M. Russell]

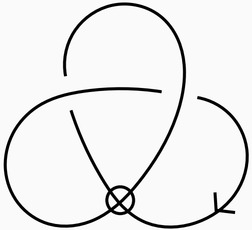
The set of Dehn n-colorings is an invariant of classical knots.

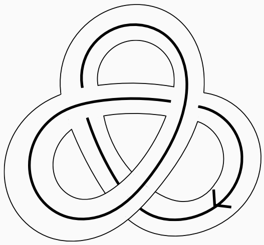
- **An abstract knot diagram**

D : a virtual knot diagram

An abstract knot diagram of D : a pair $(\sum, D_{\sum})$

where the $\sum$ is a compact, orientable or non-orientabie surface and $D_{\sum}$ is a knot diagram in $\sum$ like in figure





- **Two pseudo Goeritz matrices of virtual knots(1/2)**

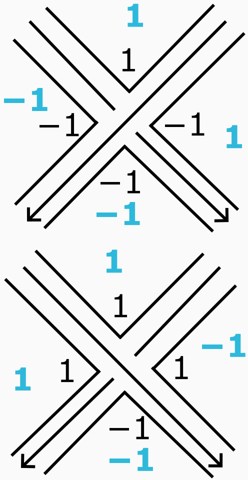
D : a virtual knot diagram

A **semi-arc** of D

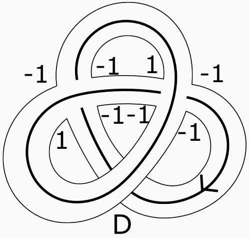
:= an arc of D between two classical crossing or loop without classical crossing of D

The first (or second) local region index of D

:= a numbering of local regions of $\mathbb{R} \setminus D$ around a classical crossing depicted as in figure.

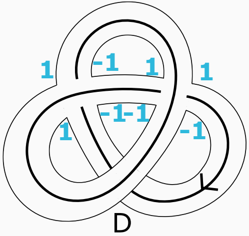


- **Example of the local region indices**



D : a virtual knot diagram as in figure

The labeling of region indices of D are given as in figure.



- **The pseudo Goeritz matrices of virtual knots(2/2)**

$a_1 \cdots a_r$: semi-arcs of D

The **first (or second) pseudo Goeritz matrix** :

$$G_1(D) \text{ (or } G_2(D)) = \begin{bmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & & \vdots \\ p_{r1} & \cdots & p_{rr} \end{bmatrix}$$

$$p_{ij} = \begin{cases} \text{the sum of indices of local regions between} \\ \text{semi-arcs } a_i \text{ and } a_j & (i \neq j) \\ -\sum_{k, k \neq i} p_{ik} & (i = j) \end{cases}$$

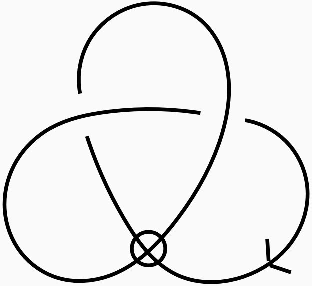
- **Theorem and Example**

Theorem [N. Kamada]

The torsion invariant of $G_1(D)$ (or $G_2(D)$) is an invariant of virtual knots.

$$G_1(D) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G_2(D) = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



D

• A Dehn coloring matrix of virtual knots(1/3)

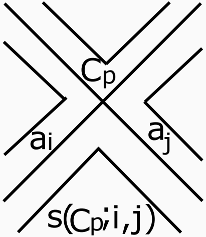
D : A virtual knot diagram

$a_1 \cdots a_r$: Semi-arcs of D

$C_1, \cdots, C_n$: Real crossings of D

$(\sum, D_{\sum})$: An abstract knot diagram of D

$R_1, \cdots, R_m$: Complementary regions of $(\sum, D_{\sum})$

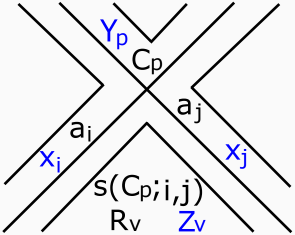


$s(C_p; i, j)$: A local region index of the region between a_i and a_j of D

• A Dehn coloring matrix of virtual knots(2/3)

$$x_j - x_i + s(C_p; i, j)(Y_p - Z_v) = 0$$

where the variables x_i, x_j, Y_p, Z_v correspond to the a_i, a_j, C_p, R_v in figure



Dehn coloring equations of D :

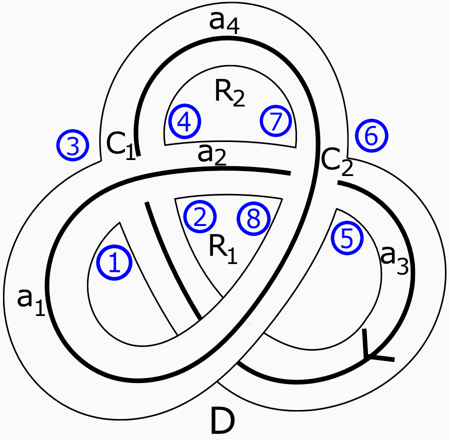
The system of the equations for real crossings and four local regions around them

Dehn coloring matrix of D :

The coefficient matrix of the Dehn coloring equations

- **Example**

$$
\begin{cases}
x_3 - x_1 + (Y_1 - Z_1) = 0 & \cdots (1) \\
x_2 - x_3 - (Y_1 - Z_1) = 0 & \cdots (2) \\
x_1 - x_4 - (Y_1 - Z_1) = 0 & \cdots (3) \\
x_4 - x_2 - (Y_1 - Z_2) = 0 & \cdots (4) \\
x_3 - x_1 - (Y_2 - Z_1) = 0 & \cdots (5) \\
x_4 - x_3 - (Y_2 - Z_1) = 0 & \cdots (6) \\
x_2 - x_4 + (Y_2 - Z_2) = 0 & \cdots (7) \\
x_1 - x_2 - (Y_2 - Z_1) = 0 & \cdots (8)
\end{cases}
$$



• A Dehn coloring matrix of virtual knots(3/3)

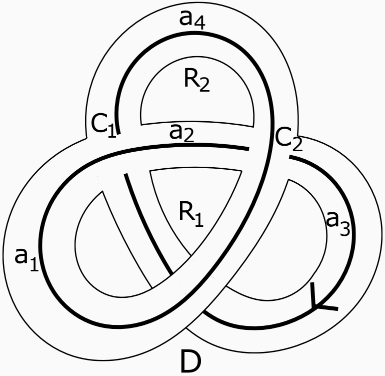
A **Dehn n-coloring** of D :
a map C : $\{R_1, \cdots, R_m, C_1, \cdots, C_n, a_1 \cdots a_r\} \to \mathbb{Z}/n\mathbb{Z}$

Theorem

The set of Dehn n-colorings is an invariant of virtual knots.

• Example and main result

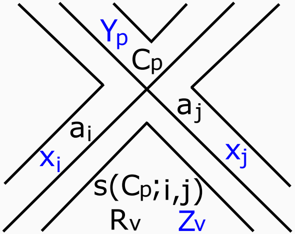
$$\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$



Main result

A pseudo Goeritz matrix can constract from a Dehn coloring matrix of virtual knots.

- **Constructing pseudo Goeritz matrix from Dehn coloring matrix(1/2)**



D : A virtual knot diagram

$a_1 \cdots a_r$: Semi-arcs of D

$C_1, \cdots, C_n$: Real crossings of D

$R_1, \cdots, R_m$: Complementary regions of $(\sum, D_\sum)$

$s(C_p; i, j)$: A local region index of the region between a_i and a_j of D

$M_D(= p_{ij})$: Dehn coloring matrix of D where the first to the r-th columns correspond to the $a_1 \cdots a_r$

• Constructing pseudo Goeritz matrix from Dehn coloring matrix(2/2)

$\mathbb{P}_i$: A row vector of M_D where $p_{ij} \neq 0$

$$\mathbb{G}_j = \sum_k -\frac{s(C_p; k, j)}{p_{kj}} \mathbb{P}_k$$

$$M_G = \begin{bmatrix} \mathbb{G}_1 \\ \vdots \\ \mathbb{G}_r \end{bmatrix} = \begin{bmatrix} \mathbb{G}^1 & \cdots & \mathbb{G}^r & \mathbb{G}^{r+1} & \cdots & \mathbb{G}^{r+n+m} \end{bmatrix}$$

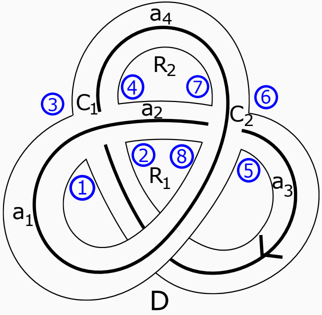
$\begin{bmatrix} \mathbb{G}^1 & \cdots & \mathbb{G}^r \end{bmatrix}$ is equal to the pseudo Goeritz matrix of D

$\begin{bmatrix} \mathbb{G}^{r+1} & \cdots & \mathbb{G}^{r+n+m} \end{bmatrix} = \begin{bmatrix} \mathbb{O} & \cdots & \mathbb{O} \end{bmatrix}$

• Example(1/3)

M_D is given by the followings.

$$
M_D = \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & c_1 & c_2 & R_1 & R_2 \end{array} \\ \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \\ \textcircled{5} \\ \textcircled{6} \\ \textcircled{7} \\ \textcircled{8} \end{array}
$$



- **Example(2/3)**

From $\mathbb{G}_1$ to $\mathbb{G}_4$ are given by the follow

$$\begin{aligned}\mathbb{G}_1 = &\begin{bmatrix}-1 & 0 & 1 & 0 & 1 & 0 & -1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0 & -1 & -1 & 0 & 1 & 0\end{bmatrix} - \\ &\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & -1 & 1 & 0\end{bmatrix} + \begin{bmatrix}1 & -1 & 0 & 0 & 0 & -1 & 1 & 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbb{G}_2 = &\begin{bmatrix}0 & 1 & -1 & 0 & -1 & 0 & 1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1 & 0 & 1 & -1 & 0 & 0 & 1\end{bmatrix} - \\ &\begin{bmatrix}0 & 1 & 0 & -1 & 0 & 1 & 0 & -1\end{bmatrix} - \begin{bmatrix}1 & -1 & 0 & 0 & 0 & -1 & 1 & 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbb{G}_3 = &-\begin{bmatrix}-1 & 0 & 1 & 0 & 1 & 0 & -1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1 & -1 & 0 & -1 & 0 & 1 & 0\end{bmatrix} \\ &+\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & -1 & 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0 & -1 & 1 & 0 & -1 & 1 & 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbb{G}_4 = &-\begin{bmatrix}1 & 0 & 0 & -1 & -1 & 0 & 1 & 0\end{bmatrix} + \begin{bmatrix}0 & -1 & 0 & 1 & -1 & 0 & 0 & 1\end{bmatrix} \\ &+\begin{bmatrix}0 & 0 & -1 & 1 & 0 & -1 & 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1 & 0 & -1 & 0 & 1 & 0 & -1\end{bmatrix}\end{aligned}$$

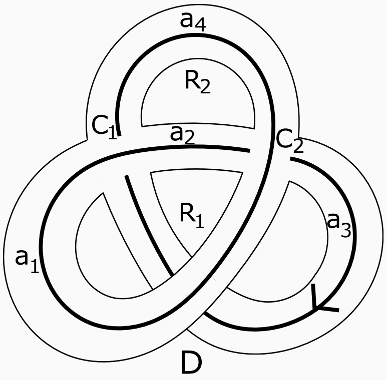
- **Example(3/3)**

M_G is given by the followings.

$$M_G = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$G_1(D)$ is given by the followings.

$$G_1(D) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$



Thank you for your attention !