Stable maps of 3-manifolds into the plane and two-bridge links

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Stable map

Definition

M, N : smooth manifolds. $f, g : M \to N$: smooth maps.

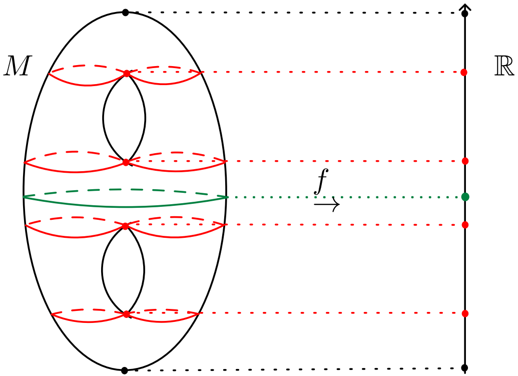
$f \sim g$: right-left equivalent $\overset{\text{def}}{\iff}$ ${}^{\exists}\Phi : M \to M$ and $\phi : N \to N$: diffeo. s.t. $g = \phi \circ f \circ \Phi^{-1}$;

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \Phi\big\downarrow & \circlearrowleft & \big\downarrow\phi \\ M & \xrightarrow[g]{} & N \end{array}$$

f is called a stable map $\overset{\text{def}}{\iff}$ ${}^{\exists}U_f$: a neighborhood of f in $C^\infty(M, \mathbb{R}^2)$ s.t. ${}^{\forall}g \in U_f, f \sim g$.

Example

A Morse function $f : M \to \mathbb{R}$ is a stable map.



M : a closed orientable smooth 3-manifold. As a generalization of a Morse function we consider stable maps $M \to \mathbb{R}^2$.

Throughout the following, M denotes a closed orientable smooth 3-manifold.

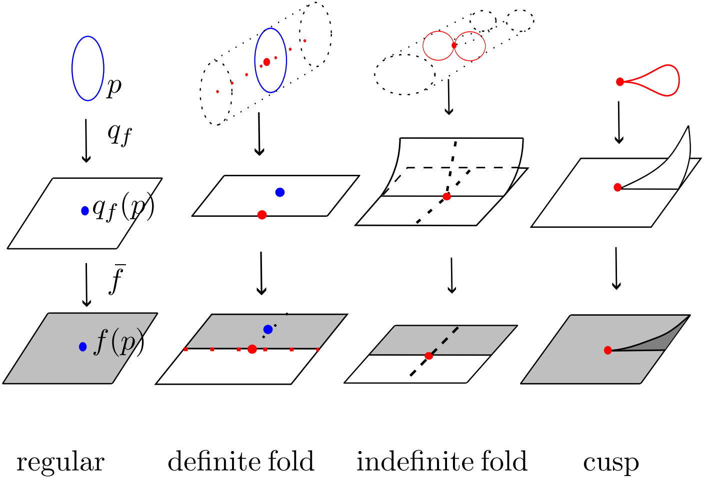
Fact [c.f. Levine]

A stable map $f : M \to \mathbb{R}^2$ is locally given in one of the following:

1. (i) $(u, x, y) \mapsto (u, x) \cdots$ regular point $\in M$
2. (ii) $(u, x, y) \mapsto (u, x^2 + y^2) \cdots$ definite fold point
3. (iii) $(u, x, y) \mapsto (u, x^2 - y^2) \cdots$ indefinite fold point
4. (iv) $(u, x, y) \mapsto (u, y^2 + ux - x^3) \cdots$ cusp point

ii, iii and iv are singular points.
and f globaly satisfies

1. (I) $f^{-1} \circ f(p) \cap S(f) = p$ for a cusp point p
2. (II) the restriction of f to $S(f)-$ $\{$cusp points$\}$ is an immersion with only normal crossings.


p
q_f
$q_f(p)$
$\bar{f}$
$f(p)$
regular
definite fold
indefinite fold
cusp

Definition

$f : M \to \mathbb{R}^2$: a smooth map.

$S(f) := \{p \in M \mid \mathrm{rank}(df_p) < 2\}$: the set of singular points of f.

the sets of definite fold, indefinite fold and cusp points are denoted by $S_0(f)$, $S_1(f)$ and $C(f)$, respectively.

Fact [Levine,1965]

The cusp points of each stable map $M \to \mathbb{R}^2$ can be eliminated by homotopy.

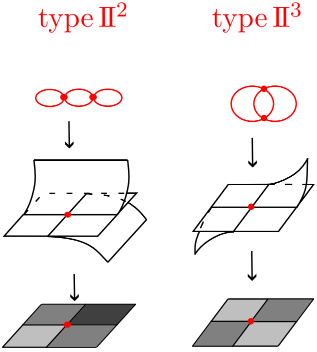
Theorem [Saeki,1996]

${}^{\exists}f : M \to \mathbb{R}^2$ s.t. $C(f) = \emptyset$ and $f(S_1(f))$ is simple closed curves $\overset{\text{iff}}{\Longleftrightarrow}$ M : a graph manifold.

Definition

$f : M \to \mathbb{R}^2$: a stable map
the singular fibers over non-simple crossings are of two types, type II^2 and type II^3.

$\mathrm{II}^2(f)$: the set of singular fibers of type II^2 of f
$\mathrm{II}^3(f)$: the set of singular fibers of type II^3 of f.

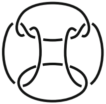


Theorem [cf. Ishikawa-Koda]

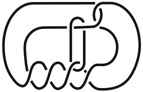
${}^\forall L$: a link in M. Then ${}^\exists f : M \to \mathbb{R}^2$ s.t. $C(f) = \emptyset$, $L \subset S_0(f)$ and $\mathrm{II}^3(f) = \emptyset$.

Theorem [Ishikawa-Koda,2017]

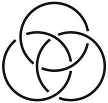
L : a link in S^3, $^\exists f : S^3 \to \mathbb{R}^2$: a stable map s.t. $C(f) = \emptyset$, $L \subset S_0(f)$ and $|\mathrm{II}^2(f)| = 1$ and $|\mathrm{II}^3(f)| = 0$ $\stackrel{\text{iff}}{\Longleftrightarrow}$ the exterior of L $\stackrel{\text{diffeo}}{\approx}$ a 3-manifold obtained by Dehn filling the exterior of one of the six links $L_1, L_2, \cdots, L_6$ in S^3 along some of (possibly none of) boundary tori, where $L_1, L_2, \cdots, L_6$ are illustrated in the following.



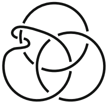
L_1



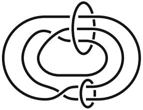
L_2



L_3



L_4

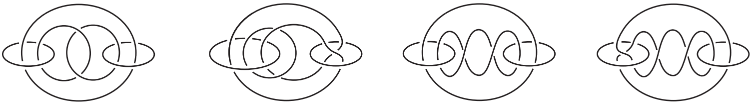


L_5

L_6

Theorem [Furutani-Koda,2021]

L : a link in S^3, ${}^{\exists}f : S^3 \to \mathbb{R}^2$: a stable map s.t. $C(f) = \emptyset$, $L \subset S_0(f)$ and $|\mathrm{II}^2(f)| = 0$ and $|\mathrm{II}^3(f)| = 1 \overset{\text{iff}}{\Longleftrightarrow}$ the exterior of $L \overset{\text{diffeo}}{\approx}$ a 3-manifold obtained by Dehn filling the exterior of one of the four links L'_1, L'_2, L'_3 and L'_4 in S^3 along some of (possibly none of) boundary tori.



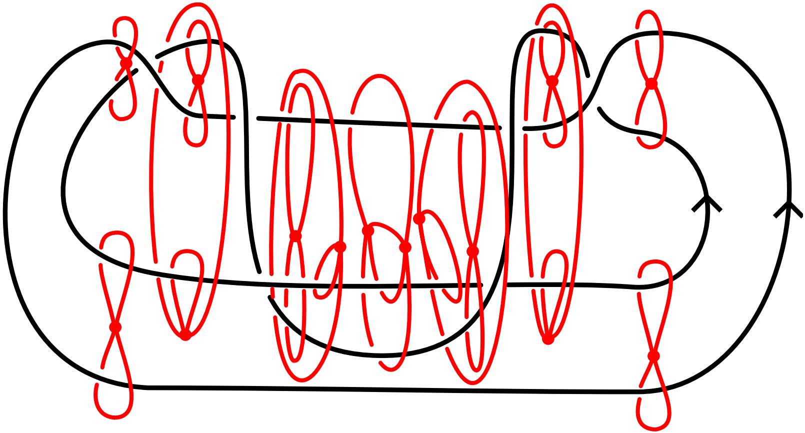
L'_1 L'_2 L'_3 L'_4

Theorem [Ichihara-K.]

$^\forall L$: a two-bridge 2-componets link in S^3.
Then $^\exists f : S^3 \to \mathbb{R}^2$: a stable map s.t. $S_0(f) = L$ and $\mathrm{II}^2(f) = \emptyset$.

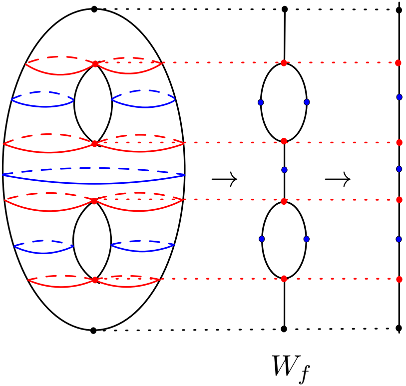
	Saeki	I-K	F-K	Ichihara-K
Link	Graph link	Six links	Four links	Two bridge link
	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Downarrow$
type II^2	0	1	0	0
type II^3	0	0	1	$*$

Example (Whitehead link)



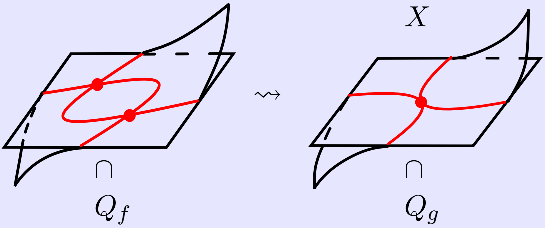
Definition

$f : M \to N$: a smooth map. $^\forall p_1, p_2 \in M$, $p_1 \sim p_2 \overset{\text{def}}{\Longleftrightarrow}$ they are contained in the same componet of the fibers of f.
$W_f := M/_\sim$: the Stein factorization of f.



Proposition [Furutani-Koda,2017]

$f : S^3 \to \mathbb{R}^2$: a stable map. Then ${}^{\exists}g : S^3 \to \mathbb{R}^2$ s.t. the Stein factorization Q_g is obtained from Q_f by replacing each part homeomorphic to X with the model in the figure.



Thank you
for your attention.