SHORT-SS4: $H^3$ and $H^4$ regularities of the Poisson equation on polygonal domains

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Abstract. This paper presents two equalities of $H^3$ and $H^4$ semi-norms for the solutions of the Poisson equation in a two-dimensional polygonal domain. These equalities enable us to obtain higher order constructive a priori error estimates for finite element approximation of the Poisson equation with validated computing.

Keywords: Poisson equation, a priori estimates

1 Introduction

Consider the Poisson equation

$$\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases} \tag{1a-1b}$$

with a multiply-connected polygonal domain $\Omega \subset \mathbb{R}^2$. The regularities of solutions of the equation (1a)-(1b) depend on the shape of $\Omega$ and $f$. For example, when $\Omega$ is convex and $f \in L^2(\Omega)$, it is well-known (e.g. Grisvard (1985)) that there exists a unique solution $u \in H^1_0(\Omega) \cap H^2(\Omega)$ of (1a)-(1b).

Recently, Hell, Ostermann and Sandbichler (2014, Lemma 2.4), and Hell and Ostermann (2014, Proposition 3) showed the following results.

**Lemma 1.** Let $\Omega = (0,1)^2$. Then all solutions to (1a)-(1b) lie in $H^3(\Omega)$ for $f \in H^1_0(\Omega)$. Moreover, for $f \in H^1_0(\Omega) \cap H^2(\Omega)$ the solution of (1a)-(1b) lies in $H^4(\Omega)$.

**Remark 1.** The assumption $f \in H^1_0(\Omega)$ is essential at Lemma 1. For example, Hell and Ostermann (2014) pointed out that, in the case of $f = 1$, the solution is not in $H^3(\Omega)$ even though $f \in C^\infty(\Omega)$. 
2 A priori error estimations

Higher regularities of the solutions for the Poisson equation such as Lemma 1 will lead us to higher order error estimations for finite element approximate solutions of (1a)-(1b). For example, a result by Nakao, Yamamoto and Kimura (1998) strongly suggests that when \( f \in H^1_0(\Omega) \) and a solution \( u \) of (1a)-(1b) lies in \( H^3(\Omega) \), for \( P2 \) (or \( Q2 \)) finite element approximation \( u_h \) of \( u \), there exists numerically determined \( C_2 > 0 \) satisfying

\[
\| u - u_h \|_{H^1_0(\Omega)} \leq C_2 h^2 |u|_{H^3(\Omega)}. \tag{2}
\]

Here, \( h \) shows the mesh size, \( \| u \|_{H^1_0(\Omega)} \) and \( |u|_{H^3(\Omega)} \) are \( H^1_0 \) norm and \( H^3 \) semi-norm of \( u \) defined by

\[
\| u \|_{H^1_0(\Omega)} := |u|_{H^1(\Omega)} = \| \nabla u \|_{L^2(\Omega)} = \sqrt{\| u_{x_1} \|_{L^2(\Omega)}^2 + \| u_{x_2} \|_{L^2(\Omega)}^2},
\]

\[
|u|_{H^3(\Omega)} := \sqrt{\| u_{x_1 x_1 x_1} \|_{L^2(\Omega)}^2 + 3 \| u_{x_1 x_2 x_2} \|_{L^2(\Omega)}^2 + \| u_{x_2 x_2 x_2} \|_{L^2(\Omega)}^2},
\]

respectively. Moreover, if \( u \) has sufficient regularities and \( u_h \) is a \( P3 \) (or \( Q3 \)) finite element approximation, there also exists \( C_3 > 0 \) such that

\[
\| u - u_h \|_{H^1_0(\Omega)} \leq C_3 h^3 |u|_{H^4(\Omega)} \tag{3},
\]

where \( |u|_{H^4(\Omega)} \) is \( H^4 \) semi-norm of \( u \) defined by

\[
|u|_{H^4(\Omega)} := \left( \| u_{x_1 x_1 x_1 x_1} \|_{L^2(\Omega)}^2 + 4 \| u_{x_1 x_1 x_1 x_2} \|_{L^2(\Omega)}^2 \right. \\
\left. + 6 \| u_{x_1 x_1 x_2 x_2} \|_{L^2(\Omega)}^2 + 4 \| u_{x_1 x_2 x_2 x_2} \|_{L^2(\Omega)}^2 \right)^{1/2}.
\]

3 Main theorem

We present a priori estimates replaced by \( f \) in the right-hand side of (2) and (3) instead of \( H^3 \) and \( H^4 \) semi-norms of \( u \), respectively.

Let \( D^1(-\Delta) \) and \( D^2(-\Delta) \subset H^1_0(\Omega) \) be the Banach spaces defined by

\[
D^1(-\Delta) := \{ u \in H^1_0(\Omega) : -\Delta u \in H^1_0(\Omega) \},
\]

\[
D^2(-\Delta) := \{ u \in H^1_0(\Omega) : -\Delta u \in H^1_0(\Omega) \cap H^2(\Omega) \},
\]

respectively. Note that \( D^n(-\Delta) \ (n \in \{1, 2\}) \) is the set of solutions of the Poisson equation (1a)-(1b).

**Theorem 1.** It is true that

\[
|u|_{H^1(\Omega)} = \| \nabla (\Delta u) \|_{L^2(\Omega)^2}, \quad \forall u \in D^1(-\Delta) \cap H^2(\Omega). \tag{4}
\]
Remark 2. Using (4) and (2) we obtain an a priori error estimate with $O(h^2)$:
\[
\| u - u_h \|_{H_0^1(\Omega)} \leq C_2 h^2 \| f \|_{H_0^1(\Omega)}.
\]

Theorem 2. It is true that
\[
|u|_{H^4(\Omega)} = \| \Delta^2 u \|_{L^2(\Omega)}, \quad \forall u \in D^2(-\Delta) \cap H^4(\Omega).
\]

Remark 3. Using (5) and (3) we obtain an a priori error estimate with $O(h^3)$:
\[
\| u - u_h \|_{H_0^1(\Omega)} \leq C_3 h^3 \| \Delta f \|_{L^2(\Omega)}.
\]

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References