On a generalization of the Fox formula for twisted Alexander invariants

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Contents

$K \subset S^3$: knot,
M_n: n-fold cyclic cover branched over K,
$\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$: Alexander polynomial of K.

Theorem (Fox formula)

If $\Delta_K(t)$ and $t^n - 1$ have no common roots in $\mathbb{C}$, then

$$\#H_1(M_n;\mathbb{Z}) = \left| \prod_{i=1}^{n} \Delta_K(\zeta_n^i) \right|,$$

where $\#G$ denotes the order of a group G and ζ_n is a primitive n-th root of unity.

Theorem (asymptotic growth formula)

If $\Delta_K(t)$ and $t^n - 1$ have no common roots in $\mathbb{C}$ for all $n \in \mathbb{Z}_{>0}$, then

$$\lim_{n\to\infty} \frac{1}{n} \log(\# H_1(M_n, \mathbb{Z})) = \log \mathbb{M}(\Delta_K(t)),$$

where $\mathbb{M}(\Delta_K(t))$ is the Mahler measure of $\Delta_K(t)$.

For $f(t) \in \mathbb{Z}[t^{\pm 1}]$, we define the *Mahler measure* $\mathbb{M}(f(t))$ of $f(t)$ by

$$\mathbb{M}(f(t)) := \exp\left(\int_0^1 \log|f(e^{2\pi\sqrt{-1}x})|dx\right).$$

§0 Introduction

Our aim is to generalize Fox formulas and asymptotic growth formulas for

twisted Alexander invariants

associated to representations of knot groups over rings of S-integers

$$\mathcal{O}_{F,S} := \{a \in F \mid v_{\mathfrak{p}}(a) \geq 0 \text{ for all } \mathfrak{p} \in S_F \setminus S\}.$$

(F: number field, S: set of finite primes of F)

($v_{\mathfrak{p}}$: additive valuation of F at $\mathfrak{p}$, S_F: set of all finite primes of F)

Note that holonomy representations are representations of hyperbolic knot groups over rings of S-integers.

Our motivation is coming from Arithmetic Topology.

Number theory	**Knot theory**
Iwasawa asymptotic formula for p-ideal class groups	asymptotic growth formula for knot modules
asymptotic formula for Tate–Shafarevich/Selmer groups	asymptotic growth formula for twisted knot modules

§1 Twisted Alexander invariants

$K \subset S^3$: knot,
$X_K := S^3 \setminus K$,
$G_K := \pi_1(X_K)$.

R: commutative Noetherian UFD,
$\rho : G_K \to \mathrm{GL}_m(R)$: representation.

$\alpha : G_K \to G_K^{\mathrm{ab}} \simeq \langle t \rangle$: abelianization homomorphism.

$G_K = \langle g_1, \ldots, g_q \mid r_1 = \cdots = r_{q-1} = 1 \rangle$,
F_q: free group on $g_1, \ldots, g_q$,
$\pi : R[F_q] \to R[G_K]$: natural homomorphism of group R-algebras.

$$\Phi := (\rho \otimes \alpha) \circ \pi : R[F_q] \longrightarrow \mathrm{M}_m(R[t^{\pm 1}]).$$

§1 Twisted Alexander invariants

P:= (big) $q \times (q-1)$ matrix, whose (i,j) component is

$$\Phi\left(\frac{\partial r_j}{\partial g_i}\right) \in \mathrm{M}_2(R[t^{\pm 1}]).$$

P_h: square matrix obtained by deleting the h-th row from P.

The *twisted Alexander invariant* of K associated to ρ:

$$\Delta_K(\rho; t) := \frac{\det(P_h)}{\det\big(\Phi(g_h - 1)\big)} \ \big(\in Q(R)(t)\big).$$

Example (K: trefoil)

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$,

$$\rho : G_K \to \mathrm{SL}_2(\mathbb{Z});\ \rho(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \rho(g_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$r_1 = g_1 g_2 g_1 (g_2 g_1 g_2)^{-1}$,
$\frac{\partial r_1}{\partial g_1} = 1 + g_1 g_2 - g_2$,
$P_1 = I_2 + t^2 \cdot \rho(g_1)\rho(g_2) - t \cdot \rho(g_2)$,
$\Phi(g_1 - 1) = t \cdot \rho(g_1) - I_2$.

$\det(P_1) = (t^2 + 1)(t - 1)^2$,
$\det\big(\Phi(g_1 - 1)\big) = (t - 1)^2$
$\rightsquigarrow \Delta_K(\rho; t) = t^2 + 1 \in \mathbb{Z}[t^{\pm 1}]$.

X_∞: infinite cyclic cover of X_K,
V_ρ: representation space of ρ.

Proposition (Kirk–Livingston)

For any representation $\rho : G_K \to \mathrm{GL}_m(R)$, *we have*

$$\Delta_{K,\rho}(t) = \frac{\Delta_0(H_1(X_\infty; V_\rho))}{\Delta_0(H_0(X_\infty; V_\rho))}.$$

Corollary

A: PID.
For any irreducible representation $\rho : G_K \to \mathrm{GL}_m(A)$, *we have*

$$\Delta_{K,\rho}(t) \doteq \Delta_0(H_1(X_\infty; V_\rho)).$$

In particular, $\Delta_{K,\rho}(t)$ *is a Laurent polynomial over* A.

$\rho : G_K \to \mathrm{GL}_m(A)$ is an *irreducible* representation
$\overset{\text{def.}}{\Leftrightarrow}$ $G_K \overset{\rho}{\to} \mathrm{GL}_m(A) \to \mathrm{GL}_2(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$: irreducible ($\forall \mathfrak{p} \in \mathrm{Spec}(A)$).

Theorem (T.)

F: number field,
S: finite set of finite primes of F such that $\mathcal{O}_{F,S}$ is a PID,
$\rho : G_K \to \mathrm{GL}_m(\mathcal{O}_{F,S})$: irreducible representation,
$\Delta_{K,\rho}(t) \in \mathcal{O}_{F,S}[t^{\pm 1}]$: twisted Alexander invariant of K asso. to ρ.
If $\Delta_{K,\rho}(t) \neq 0$, and $\Delta_{K,\rho}(t)$ and $t^n - 1$ have no common roots in $\overline{F}$, then we have

$$\# H_1(X_n; V_\rho) =_S \left| \mathrm{N}_{F/\mathbb{Q}} \left(\prod_{i=1}^{n} \Delta_{K,\rho}(\zeta_n^i) \right) \right|,$$

where X_n is the n-fold cyclic cover of X_K,
$\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$ is a norm map, and
ζ_n is a primitive n-th root of unity.

$$m =_S n \overset{\text{def}}{\Leftrightarrow} m = np_1^{e_1} \cdots p_r^{e_r} \ (e_1, \ldots, e_r \in \mathbb{Z}, (p_i) = \mathfrak{p}_i \cap \mathbb{Z}, S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\})$$

(Ideas of proof)

Wang sequences

Lemma

For any representation $\rho : G_K \to \mathrm{GL}_m(R)$*, we have an exact sequence*

$$0 \to C_*(X_\infty; V_\rho) \overset{t_\#^n - 1}{\to} C_*(X_\infty; V_\rho) \overset{p_{n\#}}{\to} C_*(X_n; V_\rho) \to 0.$$

$$\cdots \to H_1(X_\infty; V_\rho) \overset{t_*^n - 1}{\to} H_1(X_\infty; V_\rho) \overset{p_{n*}}{\to} H_1(X_n; V_\rho) \overset{\partial_{1,*}}{\to} H_0(X_\infty; V_\rho) \to \cdots$$

Corollary

A: PID.

For any irreducible representation $\rho : G_K \to \mathrm{GL}_m(A)$*, we have*

$$H_1(X_n; V_\rho) \simeq H_1(X_\infty; V_\rho)/(t^n - 1)H_1(X_\infty; V_\rho).$$

Resultants

$$f = f(t) = a\prod_{i=1}^{m}(t-\xi_i),\ \ g = g(t) = b\prod_{j=1}^{n}(t-\zeta_j) \in \mathcal{O}_{F,S}[t].$$

We define the *resultant* $\mathrm{Res}(f,g)$ for f, $g \in \mathcal{O}_{F,S}[t]$ by

$$\mathrm{Res}(f,g) := a^m b^n \prod_{i,j}(\xi_i - \zeta_j) = a^m \prod_i g(\xi_i).$$

The resultant can be generalized for Laurent polynomials.

$$h := c_{-k}t^{-k} + \cdots + c_n t^n \in \mathcal{O}_{F,S}[t^{\pm 1}],$$

where c_{-k} and c_n are non-zero, denote $\widetilde{h} := t^k h \in \mathcal{O}_{F,S}[t]$.
Then we define the *resultant* $\mathrm{Res}(f,g)$ for f, $g \in \mathcal{O}_{F,S}[t^{\pm 1}]$ by

$$\mathrm{Res}(f,g) := \mathrm{Res}(\widetilde{f},\widetilde{g}).$$

§2 Fox formulas for twisted Alexander invariants

$\mathrm{Res}(f,g) \neq 0 \Leftrightarrow f$ and g have no common roots in $\overline{F}$.

Lemma

F: number field,
S: finite set of finite primes of F so that $\mathcal{O}_{F,S}$ is a PID,
N: finitely generated $\mathcal{O}_{F,S}[t^{\pm 1}]$-module
having no submodule of finite length with $\mathrm{rank}_F(N \otimes_{\mathcal{O}_{F,S}} F) < \infty$.
Then
$N/(t^n - 1)N$ is a torsion $\mathcal{O}_{F,S}$-module
$\Leftrightarrow \Delta_0(N)$ and $t^n - 1$ have no common roots in $\overline{F}$.
When $\Delta_0(N)$ and $t^n - 1$ have no common roots in $\overline{F}$, we have

$$\# N/(t^n - 1)N =_S |\mathrm{N}_{F/\mathbb{Q}}(\mathrm{Res}(t^n - 1,\ \Delta_0(N)))|.$$

§3 Asymptotic growth formulas

Assume $\Delta_{K,\rho}(t)$ and $t^n - 1$ have no common roots in $\overline{F}$ ($^\forall n \in \mathbb{Z}_{>0}$).
Set $\overline{\Delta_{K,\rho}}(t) := \mathrm{N}_{F/\mathbb{Q}}(\Delta_{K,\rho}(t)) \in \mathbb{Z}_{S_0}[t^{\pm 1}]$, where
$S_0 = \{\mathfrak{p}_1 \cap \mathbb{Z}, \ldots, \mathfrak{p}_r \cap \mathbb{Z}\}$, and $\mathbb{Z}_{S_0}$ is the ring of S_0-integers of $\mathbb{Q}$.
Then we have

$$\#H_1(X_n, V_\rho) =_S \prod_{i=1}^{n} \left|\overline{\Delta_{K,\rho}}(\zeta_n^i)\right|.$$

$$|\#H_1(X_n, V_\rho)|_p = \prod_{i=1}^{n} \left|\overline{\Delta_{K,\rho}}(\zeta_n^i)\right|_p.$$

($|\cdot|_p$: p-adic absolute value on $\overline{\mathbb{Q}_p}$ normalized by $|p|_p = p^{-1}$)

For $f(t) \in \mathbb{Z}_{S_0}[t^{\pm 1}]$, we define
the *Mahler measure* $\mathbb{M}(f(t))$ of $f(t)$ by

$$\mathbb{M}(f(t)) := \exp\left(\int_0^1 \log|f(e^{2\pi\sqrt{-1}x})|dx\right).$$

For $f(t) \in \overline{\mathbb{Q}_p}[t^{\pm 1}] \setminus \{0\}$ with no root on roots of unity, we define
the *Ueki p-adic Mahler measure* $\mathbb{M}_p(f(t))$ of $f(t)$ by

$$\mathbb{M}_p(f(t)) := \exp\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log|f(e^{\frac{2\pi\sqrt{-1}}{n}i})|_p\right).$$

§3 Asymptotic growth formulas

Theorem (T.)

Assume $\Delta_{K,\rho}(t)$ *and* $t^n - 1$ *have no common roots in* $\overline{F}$ *for all* $n \in \mathbb{Z}_{>0}$. *When* $(p) \notin S_0$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log |\# H_1(X_n, V_\rho)|_p = \log \mathbb{M}_p(\overline{\Delta_{K,\rho}}(t)).$$

When $S = \emptyset$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log(\# H_1(X_n, V_\rho)) = \log \mathbb{M}(\overline{\Delta_{K,\rho}}(t)).$$

Let K be the figure-eight knot, whose knot group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2^{-1} g_1^{-1} g_2 g_1 = g_2 g_1 g_2^{-1} g_1^{-1} g_2 \rangle.$$

$$\rho : G_K \to \mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right);\ \rho(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \rho(g_2) = \begin{pmatrix} 1 & 0 \\ \frac{1+\sqrt{-3}}{2} & 1 \end{pmatrix},$$

where $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is the ring of integers of $\mathbb{Q}(\sqrt{-3})$. Then we have $\Delta_{K,\rho}(t) = \frac{1}{t^2}(t^2 - 4t + 1) \doteq t^2 - 4t + 1$ and hence we have

$$\begin{aligned} \#H_1(X_K; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-2) = 4, \\ \#H_1(X_2; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-12) = 144, \\ \#H_1(X_3; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-50) = 2500, \\ \#H_1(X_4; V_\rho) &= \mathrm{N}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(-192) = 36864. \end{aligned}$$

Since $\overline{\Delta_{K,\rho}}(t) = (t^2 - 4t + 1)^2$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\#H_1(X_n, V_\rho)) = 2\log(2 + \sqrt{3}).$$

§5 Future work

For the case of former Example, we have

$$\Delta_{K,\mathrm{Ad}(\rho)}(t) = \frac{1}{t^3}(t-1)(t^2-5t+1).$$

<u>dual Selmer module</u>

$\mathrm{Sel}^*(\rho_p^*) := \mathrm{Ker}\big(H^1(G_p, \rho_p^*) \to H^1(I_p, \rho_p^*)\big)^*.$

<u>twisted knot module</u> = homological Selmer module

$\mathrm{Coker}\big(H_1(\langle\mu\rangle, \rho_K) \to H_1(G_K, \rho_K)\big) =: \mathrm{Sel}(\rho_K).$

For the case of former Example, we have

$$\mathrm{Sel}(\mathrm{Ad}(\rho)) \simeq \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right] / \sqrt{-3}\,\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right].$$

On the other hand, we have

$$\mathbb{T}_{X_K,\mu}(\rho) = \pm\frac{\sqrt{-3}}{2} \quad (\mathbb{T}_{X_K,\mu} : \text{Porti torsion at meridian } \mu).$$

$$\#\mathrm{Sel}(\mathrm{Ad}(\rho)) \overset{?}{\sim} \mathbb{T}_{X_K,\mu}(\rho)$$