Finite type invariants and n-similarity of virtual knots via forbidden moves

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Contents

Main Results

Result 1 ([Ito–S. , 2017])

For a given K, for any natural number n and ℓ, there exists K_n^ℓ such that $K \# K_n^\ell$ is GPV_n- similar to K.

Result 2

For a given K, for any natural number n and ℓ, there exists K_n^ℓ such that $K \# K_n^\ell$ is F_n- similar to K.

Result 3

$$\{v \mid v = v_i^{\mathrm{GPV}} (i \leq 2n+1)\} \subset \{v \mid v = v_i^F (i \leq n)\}$$

Result 4

Non-trival F_n invariants of every order exist.
Further, $\{F_i (1 \leq i \leq n+1)\}$ is strictly stronger than $\{F_i (1 \leq i \leq n)\}$.

Virtual knots and local moves

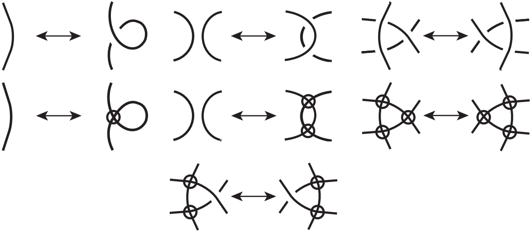
D : a virtual knot diagram

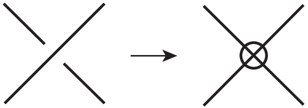
$\overset{\text{def}}{\Leftrightarrow}$ D : a knot diagram with  and 

real   virtual

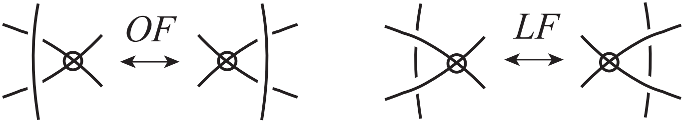
K : a virtual knot

$\overset{\text{def}}{\Leftrightarrow}$ K : an eq. class of virtual knot diagrams under GR-moves





Virtualization



Forbidden moves

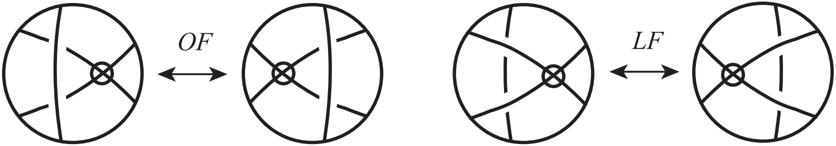
Fact 1

Moves of virtualization are unknotting operations for virtual knots ([Goussarov, Polyak and Viro, ’00]).

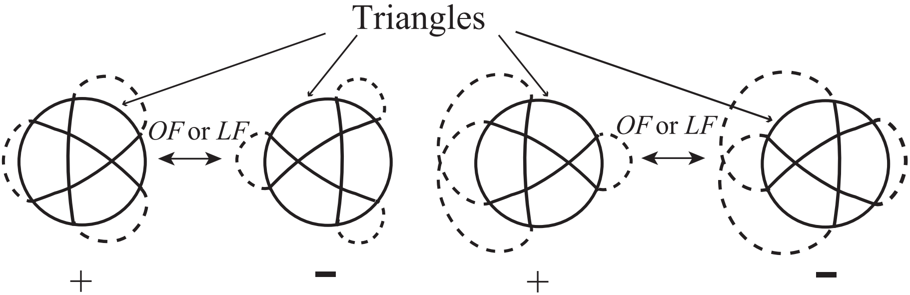
Fact 2

Forbidden moves are unknotting operations for virtual knots ([Kanenobu, ’01], [Nelson, ’01]).

Notation



Triangles

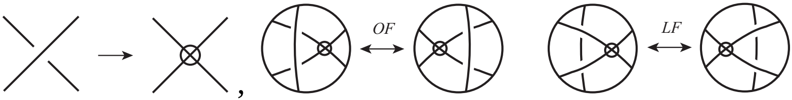


Definition 3 (GPV$_n$-invariant and F_n-invariant)

$\mathcal{VK}$: the set of all virtual knots
G : an abelian group
$v : \mathcal{VK} \to G$ is a finite-type inv. by virtualization, called GPV$_n$ (Forbidden moves, called F_n-inv.) of order $\leq n$ if for any virtual knot diag. D and $n+1$ real crossings (disjoint triangles) $d_1, d_2, \ldots, d_{n+1}$,

$$\sum_{\delta} (-1)^{|\delta|} v(D_\delta) = 0,$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_{n+1})$ runs over $(n+1)$-tuples of 0 or 1, $|\delta|$ = #1's in δ, and D_δ obtained from D by applying a virtualization (a forbidden move) to d_i with $\delta_i = 1$.



Definition 4 (n-trivial, n-similar by virtualization and forbidden moves, i.e, Virtual ver. for [Ohyama, '90, Taniyama, '92])

K, K' : virtual knots

D : a diag. of K

K is GPV_n-similar / GPV_n-trivial (F_n-similar / F_n-trivial) to K' by A_i $(1 \leq i \leq n)$

$\overset{\text{def.}}{\Leftrightarrow}$ $^{\exists} A_1, A_2, \ldots, A_n$: non-empty sets of real crossings (triangles) of D s. t.

1. $A_i \cap A_j = \emptyset$ $(i \neq j, {}^{\forall} i, j)$,
2. a diagram of K'/trivial knot is obtained by applying virtualizations (forbidden moves) to crossings (triangles) in any non-empty subfamily of $\{A_i \mid 1 \leq i \leq n\}$.

GPV$_n$-trivial knots and F_n-trivial virtual knots

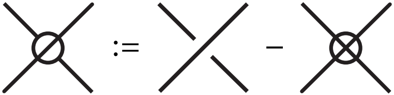
Theorem 5 ([Ito–S. , 2017])

For any natural number n and ℓ, there exists K_n^ℓ such that K_n^ℓ is a GPV$_n$*-trivial virtual knot.*

Theorem 6 ([Ito–S. , 2017])

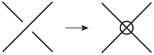
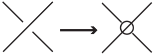
For any natural number n, there exists K_n such that K_n is a F_n-trivial virtual knot.

Relation 1



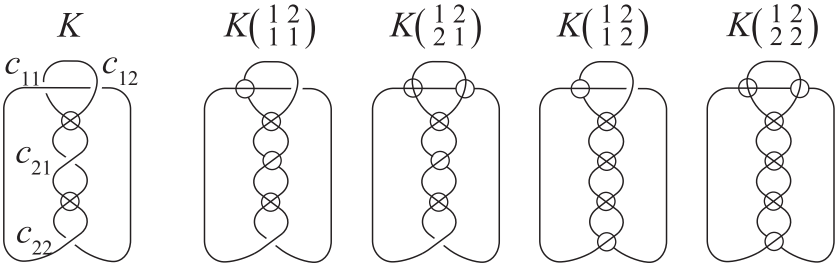
$A_i := \{c_{i1}, c_{i2}, \ldots, c_{i\alpha(i)}\}$.

$K\begin{pmatrix} 1 & 2 & \ldots & k \\ i_1 & i_2 & \ldots & i_k \end{pmatrix}$ is a diagram with semi-virtual crossings obtained from replacing

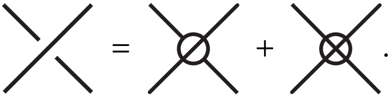
 at $\begin{cases} c_{11}, \ldots, c_{1i_1-1} \\ c_{21}, \ldots, c_{2i_2-1} \\ \vdots \\ c_{k1}, \ldots, c_{ki_k-1} \end{cases}$ and  at $\begin{cases} c_{1i_1} \\ c_{2i_2} \\ \vdots \\ c_{ki_k} \end{cases}$

(cf. [Y. Ohyama, ’90]).

Example 7



Examples of $K\begin{pmatrix}1 & 2\\ i_1 & i_2\end{pmatrix}$.

Note that 

Lemma 8

If K is GPV$_n$*-similar to K',*

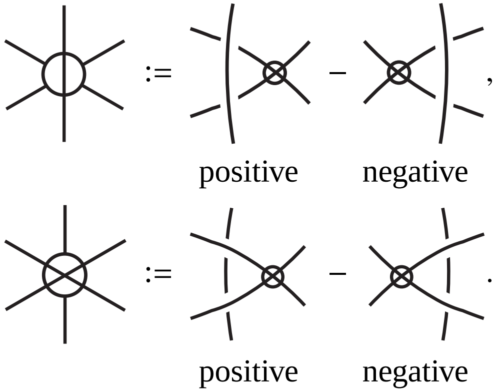
$$v_m^{\mathrm{GPV}}(K) = v_m^{\mathrm{GPV}}(K') + \sum_{1\le i_j\le \alpha(j), 1\le j\le n} v_m^{\mathrm{GPV}}\left(K\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}\right).$$

Corollary 9

If K is GPV$_n$*-similar to K', then*

$$v_m^{\mathrm{GPV}}(K) = v_m^{\mathrm{GPV}}(K') \qquad (m < n).$$

Relation 2



Lemma 10

If K is F_n-similar to K',

$$v_m^F(K) = v_m^F(K') + \sum_{\substack{1\le i_j\le\alpha(j)\\ 1\le j\le n}} \varepsilon_{1i_1}\varepsilon_{2i_2}\cdots\varepsilon_{ni_n} v_m^F\left(K\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}\right).$$

Corollary 11

If K is F_n-similar to K', then

$$v_m^F(K) = v_m^F(K') \qquad (m < n).$$

Main Results

Result 1 ([Ito–S. , 2017])

For a given K, for any natural number n and ℓ, there exists K_n^ℓ such that $K\#K_n^\ell$ is GPV_n- similar to K.

Result 2

For a given K, for any natural number n and ℓ, there exists K_n^ℓ such that $K\#K_n^\ell$ is F_n- similar to K.

Result 3

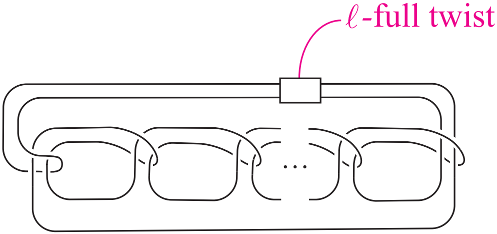
$\{v \mid v = v_i^{\mathrm{GPV}} (i \leq 2n+1)\} \subset \{v \mid v = v_i^F (i \leq n)\}$

Result 4

Non-trival F_n invariants of every order exist.
Further, $\{F_i (1 \leq i \leq n+1)\}$ is strictly stronger than $\{F_i (1 \leq i \leq n)\}$.

GPV$_n$-similarity and GPV$_n$-invariants

K_n^ℓ :



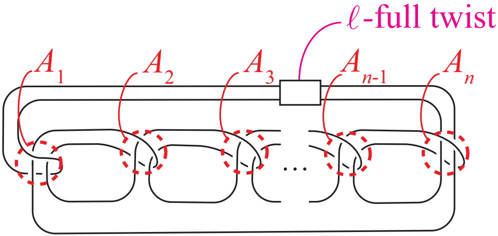
[Kanenobu, '86, Math. Ann.],
[Ohyama-Ogushi, '90 , Tokyo J. Math.]

Corollary 12

Let m, n be a given pair of positive integers satisfying $m \leq n - 1$ and fixed. For any virtual knot K, any positive integer ℓ, there exist infinitely many classical knots K_n^ℓ such that $v_m^{\mathrm{GPV}}(K \# K_n^\ell) = v_m^{\mathrm{GPV}}(K)$.

GPV$_n$-similarity and GPV$_n$-invariants

K_n^ℓ :



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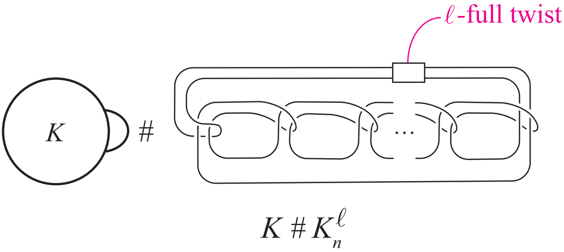
GPV$_n$-similarity and GPV$_n$-invariants



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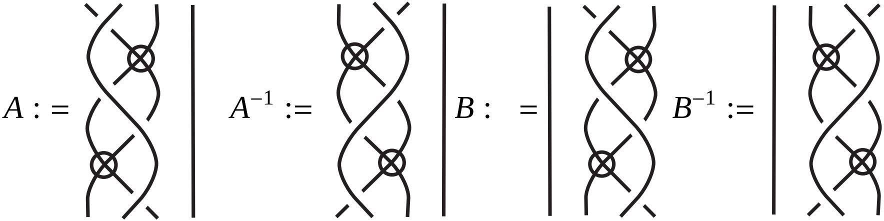
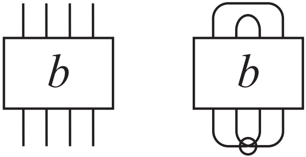
F_n-similarity and F_n-invariants

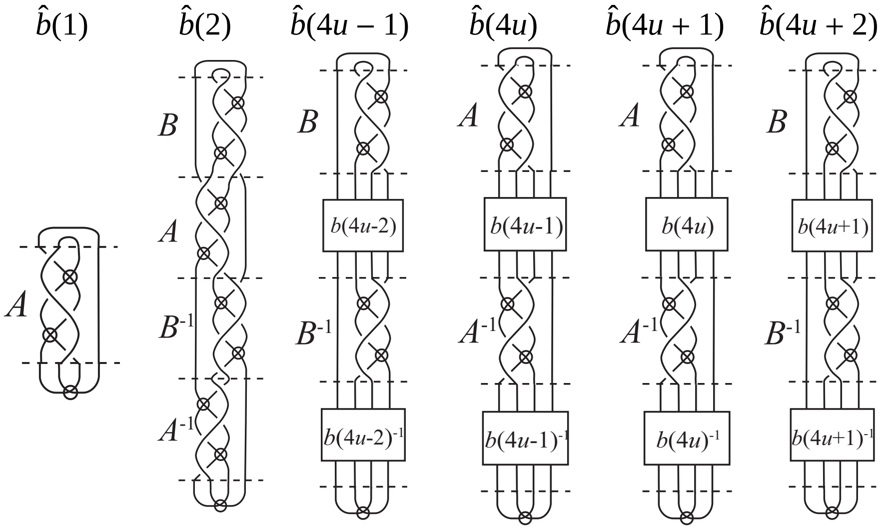


(cf. [S. Kamada, ’00])

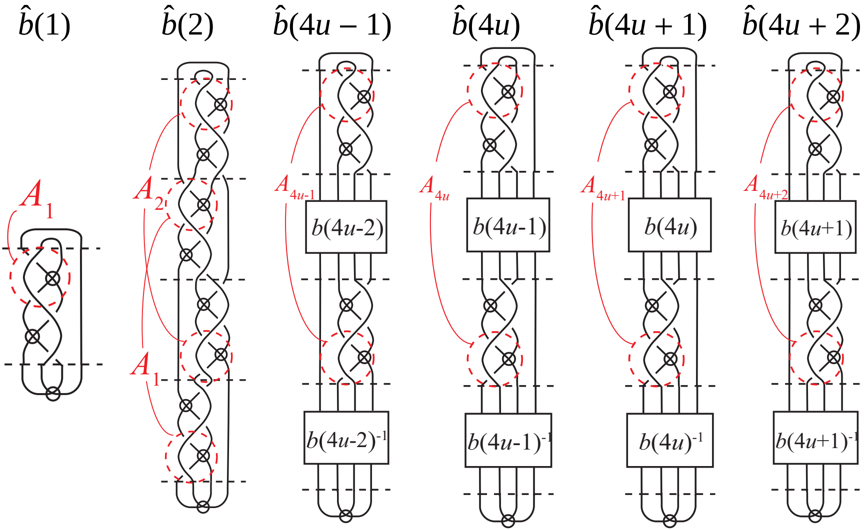


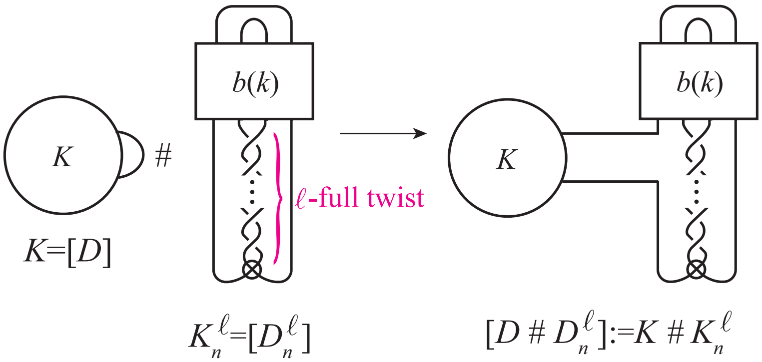
A braid b and its closure $\hat{b}$

$$b(1) = A,\ b(2) = [B, b(1)],$$
$$b(4u-1) = [B, b(4u-2)],\ b(4u) = [A, b(4u-1)],$$
$$b(4u+1) = [A, b(4u)],\ b(4u+2) = [B, b(4u+1)],\ (u \geq 1).$$



$b(1) = A$, $b(2) = [B, b(1)]$,
$b(4u-1) = [B, b(4u-2)]$, $b(4u) = [A, b(4u-1)]$,
$b(4u+1) = [A, b(4u)]$, $b(4u+2) = [B, b(4u+1)]$, $(u \geq 1)$.



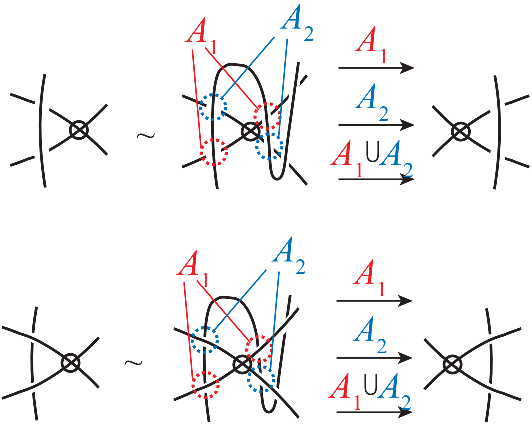


Corollary 13

Let m, n be a given pair of positive integers satisfying $m \leq n-1$ and fixed. For any virtual knot K, there exists ℓ_0, and for any positive integer ℓ ($\geq \ell_0$), there exist infinitely many virtual knots K_n^ℓ such that $v_m^F(K\#K_n^\ell) = v_m^F(K)$.

High degree of GPV$_n$-invariant and F_n-invariant

Lemma 14



By Lemma 14,
K is F_n-similar to $K' \Rightarrow K$ is GPV_{2n}-similar to K'.

Proposition 15

$\{v \mid v = v_i^{\mathrm{GPV}} (i \leq 2n+1)\} \subset \{v \mid v = v_i^F (i \leq n)\}$

Let $\mathcal{P}$: Polayak algebra, i.e., GPV_n-inv. $\in \mathcal{P}_n^*$.

Theorem 16 ([Goussarov, Polyak, Viro, '00])

Let D be any diagram of a virtual knot K. The formula $K \mapsto I(D) \in \mathcal{P}$ defines a complete invariant of virtual knots.

By using Theorem 16, we have Result .

Result 3

All F_n invariants define a complete invariant of virtual knots.

Theorem 17 ([Chmutov, Khoury, Rossi, '09])

For $n \geqq 1$, the coefficient c_{2n} of z^{2n} in the Conway polynomial of a knot K with the Gauss diagram G is equal to

$$c_{2n} = \langle \mathfrak{C}_{2n}, G \rangle.$$

By the definitions of GPV-inv., Vassiliev inv. and Theorem 17, we have Result .

Result 4

Non-trival F_n invariants of every order exist.
Further, $\{F_i(1 \leq i \leq n+1)\}$ is strictly stronger than $\{F_i(1 \leq i \leq n)\}$.

Thank you for your attention.