# A Characterization of Alternating Link Exteriors

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# Motivation

### Question: Ralph Fox

*"What is an alternating knot?"*

i.e. Give an intrinsic characterization of alternating links
which are defined diagrammatically.

### Answers: Greene, Howie

Characterizations in terms of "spanning surfaces" of alternating links.

### Problem

Can we characterize the **EXTERIORS** of alternating links?

# Aitchison Complex and Dehn Complex

For a link represented by a connected diagram and its exterior,

## Aitchison Complex

A certain cubical decomposition of the exterior
obtained from the diagram
(by Aitchison, Agol, Adams, Thurston, Yokota).
We call the cubed complex the **Aitchison complex**.

## Dehn Complex

A squared complex obtained from the diagram,
which forms a spine of the exterior.

# Goal

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Give a characterization of alternating link exteriors.

## Tool

We introduce a "**Signed Colored**" complex (SC-complex), which is a cubed complex consisting of

- Cells with "signed" vertices and "colored" edges,
- Gluing information of the cells.

## Method

Describe the Aitchison complex and the Dehn complex by using the SC-complexes.

# Plan of Talk

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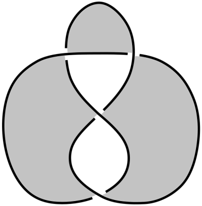
# Answer: Greene

$\boldsymbol{L}$: a link,
$\boldsymbol{\Sigma}$: a surface spanning $\boldsymbol{L} \Leftrightarrow \partial\boldsymbol{\Sigma} = \boldsymbol{L}$.

e.g. Black/white surfaces
associated to a checkerboard coloring
of a diagram of $\boldsymbol{L}$.



## Theorem (Greene, 2017)

*A link $\boldsymbol{L} \subset \boldsymbol{S}^3$ is alternating*
$\Leftrightarrow \exists\{\boldsymbol{\Sigma}, \boldsymbol{\Sigma}'\}$: *a pair of connected surfaces spanning $\boldsymbol{L}$*
*s.t. $\{\boldsymbol{\Sigma}, \boldsymbol{\Sigma}'\}$: positive/negative definite.*

i.e. $\langle\ ,\ \rangle : \boldsymbol{H}_1(\boldsymbol{\Sigma}) \times \boldsymbol{H}_1(\boldsymbol{\Sigma}) \to \mathbb{Z}$: the Gordon-Litherland pairing
is positive/negative definite.

# Answer: Howie

## Theorem (Howie, 2017)

*A non-trivial knot* $K \subset S^3$ *is alternating*
$\Leftrightarrow \exists\{\Sigma, \Sigma'\}$: *a pair of connected surfaces spanning* $K$
*s.t.* $\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial\Sigma, \partial\Sigma') = 2.$

## Theorem (Howie, 2017)

$L \subset S^3$: *a non-trivial non-split link with marked meridians on boundary components of its exterior.*
$L$ *is alternating*
$\Leftrightarrow \exists(\Sigma, \Sigma')$: *a pair of connected surfaces spanning* $L$
*s.t.* $\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial\Sigma, \partial\Sigma') = 2$
*and* $i(\partial\Sigma, \partial\Sigma') = |\sum_{j=1}^{m} i_a(\sigma_j, \sigma'_j)|.$

# Answer: Howie

## Fact

$\{\boldsymbol{\Sigma}, \boldsymbol{\Sigma}'\}$: *the pair satisfying the conditions in the theorems,*
$\boldsymbol{L} = \partial \boldsymbol{\Sigma}$: *the alternating link/knot,* $\boldsymbol{\Gamma}$: *a diagram of* $\boldsymbol{L}$.
*Then,* $\{\boldsymbol{\Sigma}, \boldsymbol{\Sigma}'\} \simeq \{\boldsymbol{\Sigma}_B, \boldsymbol{\Sigma}_W\}$.
($\boldsymbol{\Sigma}_B, \boldsymbol{\Sigma}_W$: *black/white checkerboard surfaces associated to* $\boldsymbol{\Gamma}$.)

# Plan of Talk

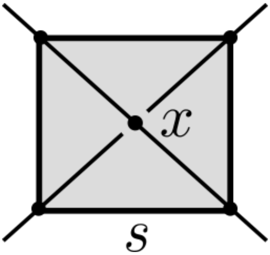
# Settings

$\Gamma \subset S^2$: a connected alternating link diagram,
$L \subset S^3$: the alternating link represented by $\Gamma$,
$E(L) = S^3 \setminus \mathrm{int}\, N(L)$: the exterior of $L$ ($N(L)$: a tubular nbd).
$P_+, P_- \in S^3 \setminus S^2$: points regarded to lie above and below $S^2$,
$S^3 \setminus \{P_+, P_-\} \cong S^2 \times \mathbb{R}$.

Assume the following:
$\Gamma$ is regarded as a 4-valent graph in $S^2 \times \{0\}$,
$|L \pitchfork (S^2 \times \{0\})| = 2n$ ($n$: the crossing number of $\Gamma$).

# Squares at Crossings

For each vertex $x$ of $\Gamma$,
$s \subset S^2 = S^2 \times \{0\}$: a square
which forms a relative regular nbd
of $x$ in $(S^2, \Gamma)$
s.t. the four vertices of $s$
lie in the four germs of edges around $x$.



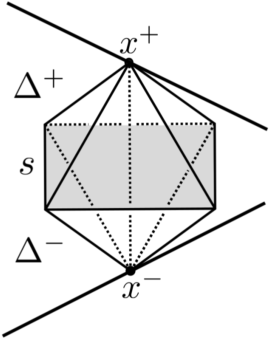
# Pyramids at Crossing Arcs

$x^{\pm} \subset L$: the points which lie above and below the vertex $x$ of $\Gamma$.

$\Delta^{\pm} := x^{\pm} * s \subset S^3$: pyramids.

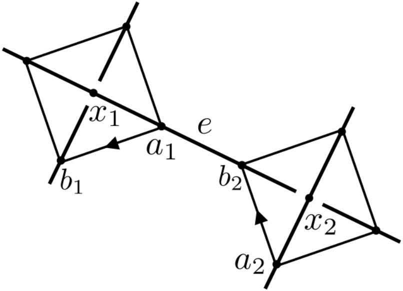
Assume
$\Delta^{\pm} \cap L = \{x^{\pm}\}$, $\Delta^{+} \cap \Delta^{-} = s$.

Let $\{\Delta_1^{\pm}, \ldots, \Delta_n^{\pm}\}$
be the set of $2n$ pyramids in $S^3$
located around the crossing arcs of $L$.



# Relative Isotopies

$e \subset \Gamma$: an edge, $x_1, x_2 \subset \Gamma$: the endpoints of $e$,
$\tilde{e} \subset L$: the arc corresponding to $e$ joining $x_1^+$, $x_2^-$.
$w = \tilde{e} \cap S^2$: the "middle point" of $\tilde{e}$,
$wP_+$, $wP_-$: the vertical line segments.
We have the following isotopies in $(S^3, L)$:
$\triangle x_1^+ a_1 b_1 \sim \triangle w P_- P_+ \sim \triangle x_2^- a_2 b_2$.



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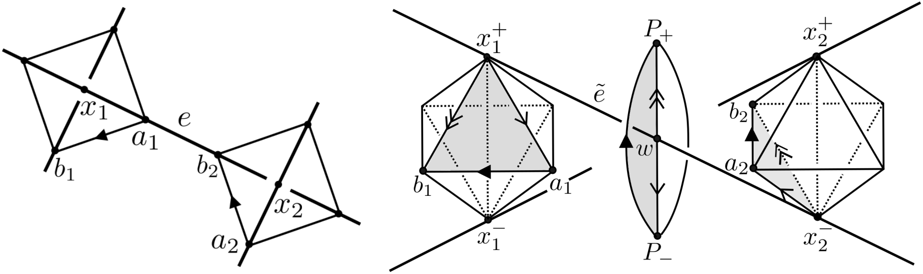

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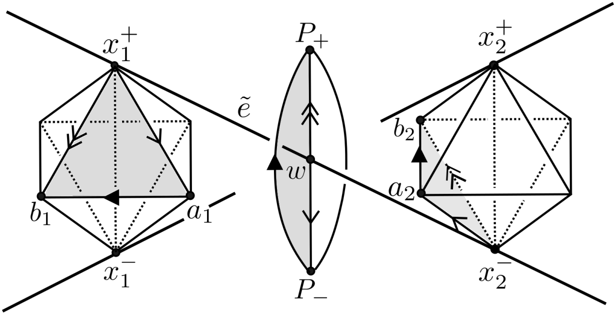
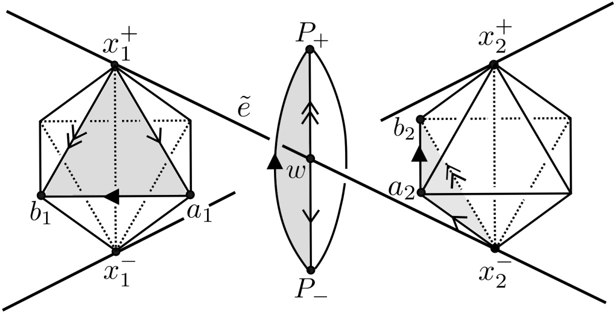


# Cubes and Gluing Information

$\boldsymbol{\Delta}_i^{\pm}$: the pyramids,
$\downarrow$ chopping off small nbds of $x_i^{\pm}$
$O_i^{\pm}$: the cubes.
The isotopies
$\rightsquigarrow$ homeomorphisms between the faces of $\{\boldsymbol{\Delta}_i^{+} \cup \boldsymbol{\Delta}_i^{-}\}_i$
$\rightsquigarrow$ a gluing information for the cubes $\{O_i^{\pm}\}_i$.



# Obtain Aitchison Complex

$\mathcal{A}(\Gamma)$: the cubed complex $\{O_i^{\pm}\}_i$
with the gluing information of the faces,

$\mathcal{D}(\Gamma)$: the subcomplex of $\mathcal{A}(\Gamma)$
consisting of the squares $\{s_i = O_i^+ \cap O_i^-\}_i$.

## Fact

- *$\mathcal{A}(\Gamma)$ gives a cubical decomposition of $E(L)$.*
- *There is a deformation retraction $r\colon E(L) \to \mathcal{D}(\Gamma)$ and $\mathcal{A}(\Gamma)$ is identified with the mapping cylinder of $r|_{\partial E(L)}$.*
- *$\mathcal{A}(\Gamma)$ and $\mathcal{D}(\Gamma)$ have non-positively curvature $\Leftrightarrow$ $\Gamma$ is prime.*

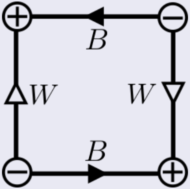
$\mathcal{A}(\Gamma)$ is the **Aitchison complex** of $\Gamma$,
$\mathcal{D}(\Gamma)$ is the **Dehn complex** of $\Gamma$.

# Plan of Talk

# Signed Colored Squares

## Definition

A **signed colored square** (**SC-square**) is the square $s := [0,1]^2$ with the following information:
(1) The vertices $(0,0)$, $(1,1)$ have the sign $-$, and the vertices $(0,1)$, $(1,0)$ have the sign $+$.
(2) The horizontal edges have the color $B$ (Black), and the vertical edges have the color $W$ (White).

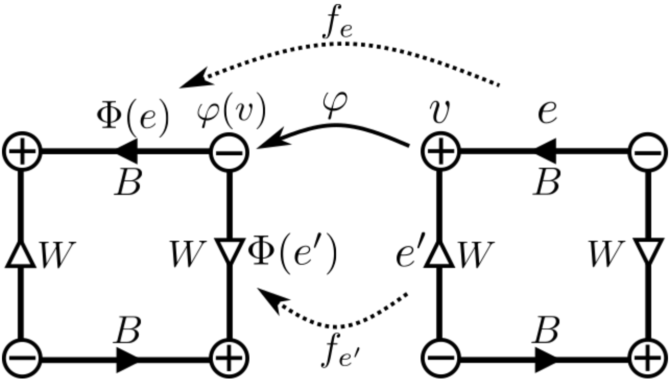



Assume that each edge is oriented from the $-$ vertex to the $+$ vertex.

$S = \{s_1, \ldots, s_n\}$: the set of $n$ copies of the SC-square.

# Gluing Information for SC-squared Complex

$V_{\pm}(S)$: the $\pm$ vertex set, $E(S)$: the edge set.
$\varphi$: $V_{+}(S) \to V_{-}(S)$: a bijection
$\rightsquigarrow$ $\Phi$: $E(S) \to E(S)$: a "color-preserving" bijection
$\rightsquigarrow$ $\{f_e : e \to \Phi(e)\}_{e \in E(S)}$: a family of glueing homeomorphisms.



# Definition of SC-Squared Complex

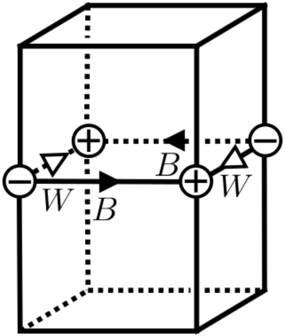
## Definition (SC-Squared Complex)

For a set $\boldsymbol{S}$ of SC-squares and a bijection $\varphi\colon V_+(S) \to V_-(S)$, the **signed colored squared complex** is a squared complex obtained by gluing SC-squares in $\boldsymbol{S}$ together according to the information determined by $\varphi$.
We denote it by $\mathcal{C}^2(S, \varphi)$.

# Associated Signed Colored Cubes

For each SC-square $s$,
$s \times [0,1]$: an "**upper SC-cube**,"
$s \times [-1,0]$: a "**lower SC-cube**."

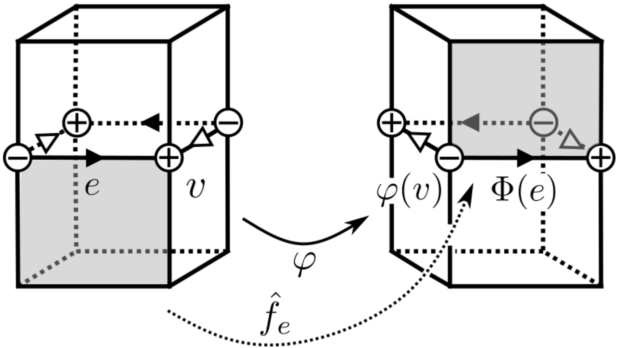
For each edge $e \subset s$,
$e \times [0,1]$, $e \times [-1,0]$:
a **black face** (if $e$: a black edge).
a **white face** (if $e$: a white edge).




For the set $S = \{s_1, \ldots, s_n\}$
of the SC-squares in $\mathcal{C}^2(S, \varphi)$,
$C_+ = \{s_i \times [0,1]\}_{i=1}^{n}$: the set of the upper SC-cubes,
$C_- = \{s_i \times [-1,0]\}_{i=1}^{n}$: the set of the lower SC-cubes.

# Gluing Information for SC-Cubed Complex

$F(C_{\pm})$: the face set.
$\varphi\colon V_+(S) \to V_-(S)$: the bijection
$\rightsquigarrow \Phi\colon E(S) \to E(S)$: the color-preserving bijection
$\rightsquigarrow \hat{\Phi}\colon F(C_-) \to F(C_+)$: a color-preserving bijection
by setting $\hat{\Phi}(e \times [-1, 0]) = \Phi(e) \times [0, 1]$.

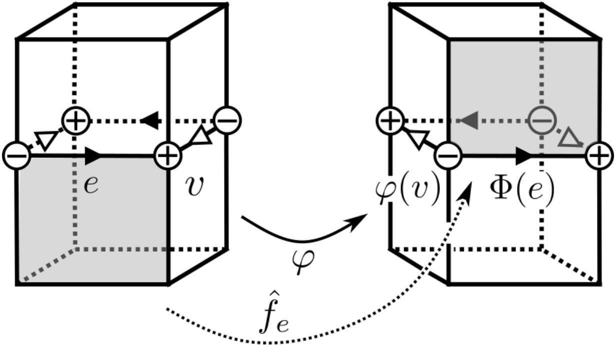


# Gluing Information for SC-Cubed Complex

$\{f_e : e \to \Phi(e)\}_{e \in E(S)}$: the glueing information of edges

$\rightsquigarrow \left\{ \hat{f}_e : e \times [-1, 0] \to \Phi(e) \times [0, 1] \right\}_{e \in E(S)}$

: a family of glueing homeomorphisms
defined by $\hat{f}_e(x, t) = (f_e(x), -t)$ $(x \in e, t \in [-1, 0])$.



# Definition of SC-Cubed Complex

## Definition (SC-Cubed Complex)

For a set $\boldsymbol{S}$ of SC-squares and a bijection $\varphi\colon V_+(S) \to V_-(S)$, the **signed colored cubed complex** is a 3-dim cubed complex obtained by gluing SC-cubes associated with $\boldsymbol{S}$ together according to the information determined by $\varphi$.
We denote it by $\mathcal{C}^3(S,\varphi)$.

Note that $\mathcal{C}^2(S,\varphi)$ is a subcomplex of $\mathcal{C}^3(S,\varphi)$, and there is a deformation retraction of $\mathcal{C}^3(S,\varphi)$ onto $\mathcal{C}^2(S,\varphi)$.

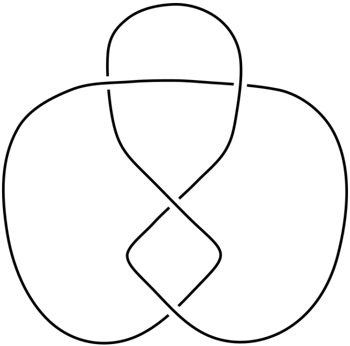
# Dehn Complex and Aitchison Complex are SC

$\Gamma \subset S^2$: a connected alternating link diagram with $n$ crossings.

Give $\Gamma$ a checkerboard coloring.

$S = \{s_1, \ldots, s_n\}$
: relative regular nbds of vertices of $\Gamma$
$\rightsquigarrow$ SC-squares.

Arcs $\Gamma \setminus \bigcup_{i=1}^{n} \mathrm{int}(s_i)$
$\rightsquigarrow$ a bijection $\varphi\colon V_+(S) \to V_-(S)$.



**Proposition**

*$\mathcal{C}^2(S, \varphi) \cong \mathcal{D}(\Gamma)$ and $\mathcal{C}^3(S, \varphi) \cong \mathcal{A}(\Gamma)$.*

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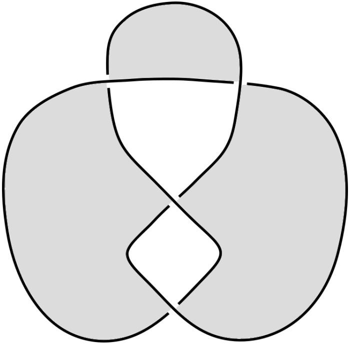

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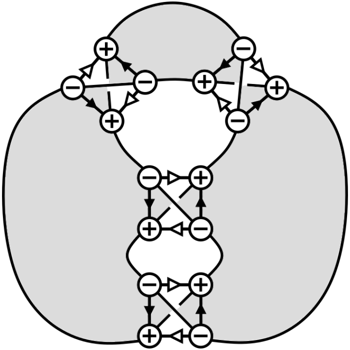

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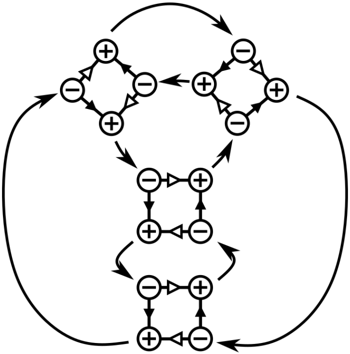


## Proposition

*$\mathcal{C}^2(S,\varphi) \cong \mathcal{D}(\Gamma)$ and $\mathcal{C}^3(S,\varphi) \cong \mathcal{A}(\Gamma)$.*

# Plan of Talk

# Main Theorem

$S = \{s_1, \ldots, s_n\}$: a set of SC-squares,
$\varphi\colon V_+(S) \to V_-(S)$: a bijection,
$\Phi\colon E(S) \to E(S)$: induced by $\varphi$.

## Theorem

*The SC-cubed complex $\mathcal{C}^3(S, \varphi)$ is isomorphic to the Aitchison complex $\mathcal{A}(\Gamma)$ of a connected alternating diagram $\Gamma \subset S^2$*
$\Leftrightarrow$ *the map $\Phi$ satisfies*

$$|E(S)/\langle\Phi\rangle| = |S| + 2,$$

*where $E(S)/\langle\Phi\rangle$ denotes the quotient space of the cyclic group action on $E(S)$ induced by the bijection $\Phi$.*

# Proof: Only If Part

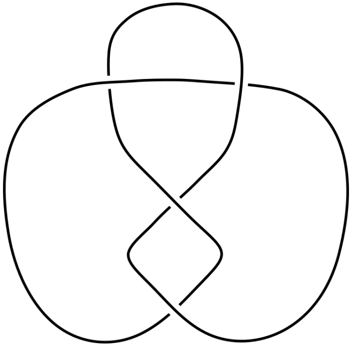
Suppose that $\mathcal{C}^3(S, \varphi) \cong \mathcal{A}(\Gamma)$
of a connected alternating diagram $\Gamma \subset S^2$,
where $(S, \varphi)$ is constructed from $\Gamma$.

Observe that
$E(S)/\langle\Phi\rangle \xleftrightarrow{1:1} \{\text{regions}\}$.

Consider the cell decomposition of
the projection plane $S^2$ obtained from $\Gamma$.

$2 = \chi(S^2) = |S| - 2|S| + |E(S)/\langle\Phi\rangle|$.

Hence, $|E(S)/\langle\Phi\rangle| = |S| + 2$.



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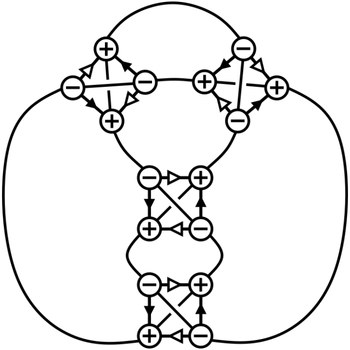

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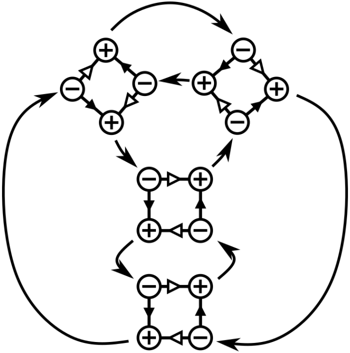


# If Part: Construct Diagram on $S^2$

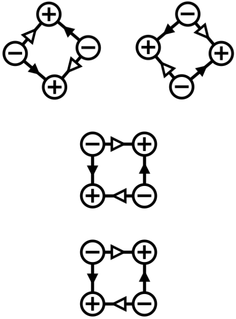
Suppose $|E(S)/\langle\Phi\rangle| = |S| + 2$.
Construct a connected alternating diagram $\Gamma \subset S^2$
s.t. $\mathcal{C}^3(S,\varphi) \cong \mathcal{A}(\Gamma)$.

(0) The set $S = \{s_i\}_{i=1}^n$ of SC-squares

(1) Attach $\gamma = \langle v, \varphi(v)\rangle$
for each $v \in V_+(S)$.

(2) Attach overpasses and underpasses
to SC-square.

(3) For each orbit in $E(S)/\langle\Phi\rangle$,
attach 2-cells along simple 1-cycles.



$M$: the resulting 2-dim cell complex.

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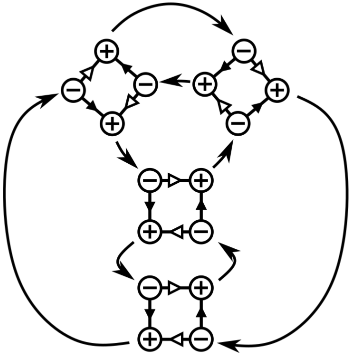

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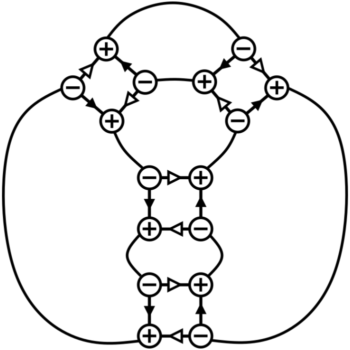

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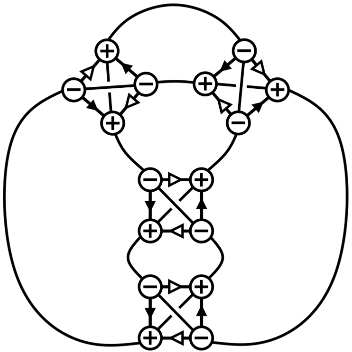


$M$: the resulting 2-dim cell complex.

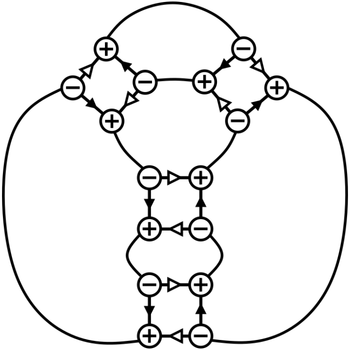
# If Part: Confirm $M$ is $S^2$

Observe that $M$ is an orientable 2-manifold.
Compute the Euler characteristic $\chi(M)$:

(0) $\mathbf{4}n$, with $n = |S|$.
(∵ Each SC-square has **4** vertices.)

(1) $\mathbf{6}n$.
(∵ $\mathbf{4}n$ edges of the SC-squares and $\mathbf{2}n$ "connecting" edges.)

(2) $n + |E(S)/\langle\Phi\rangle|$.
(∵ $n$ SC-squares and $|E(S)/\langle\Phi\rangle|$ 2-cells.)



$\chi(M) = 4n - 6n + (n + |E(S)/\langle\Phi\rangle|) = -n + |E(S)/\langle\Phi\rangle| = 2.$
Hence, $M \cong S^2$.

# Summary

- $S = \{s_1, \ldots, s_n\}$: a set of SC-squares,
  $\varphi\colon V_+(S) \to V_-(S)$: a bijection
  $\rightsquigarrow \mathcal{C}^3(S, \varphi)$: SC-cubed complex.

- $\mathcal{C}^3(S, \varphi) \cong \mathcal{A}(\Gamma)$: Aitchison complex of a connected alternating diagram $\Gamma$
  $\Leftrightarrow |E(S)/\langle\Phi\rangle| = |S| + 2$.

# Characterization

## Corollary

*A compact 3-manifold* $M$ *is homeomorphic to the exterior of an alternating link represented by a connected alternating diagram*
$\Leftrightarrow$ $M$ *is homeomorphic to the underlying space of an SC-cubed complex* $\mathcal{C}^3(S,\varphi)$ *s.t.* $|E(S)/\langle\Phi\rangle| = |S| + 2$.

Thank you very much.