ハンドル体結び目のcutting数と constituentハンドル体結び目

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結び目の数学X
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Outline

§1 The cutting number and constituent handlebody-knots

§2 A G-family of quandles

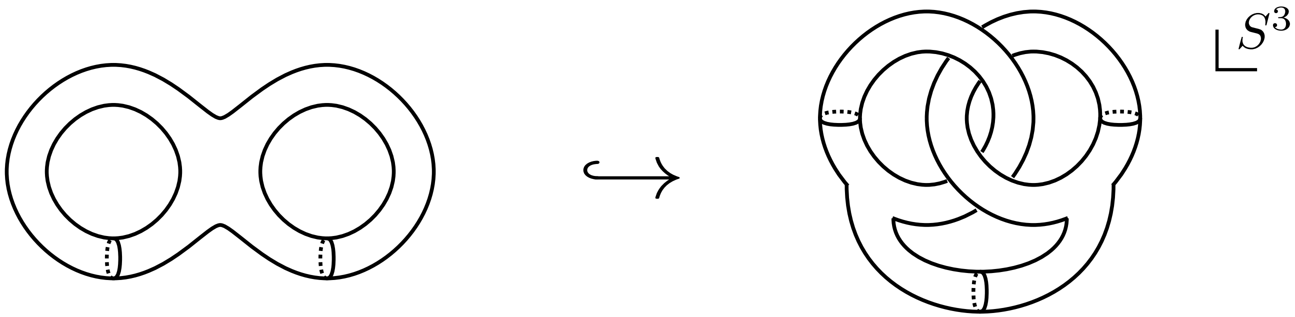
§3 Colorings

§4 Results and examples

§1 The cutting number and constituent handlebody-knots

<u>**Def**</u>

- **handlebody-knot** : handlebody $\hookrightarrow S^3$

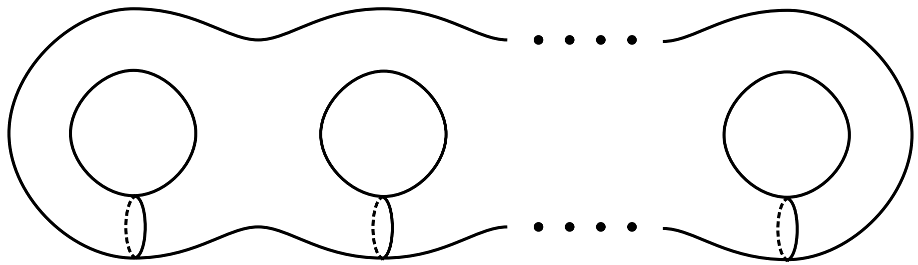


- H_1, H_2 : handlebody-knots

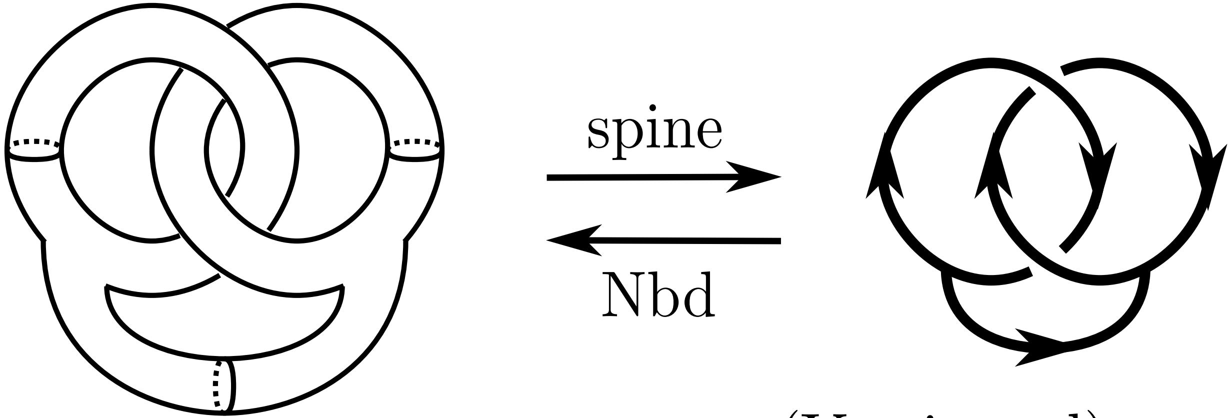
$H_1 \cong H_2 \overset{\text{def}}{\Longleftrightarrow} \exists f : S^3 \to S^3$ ori.-pres. homeo. ; $f(H_1) = H_2$

- O_g : **trivial** handlebody-knot of genus g

$\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{cl}(S^3 - O_g)$: handlebody of genus g

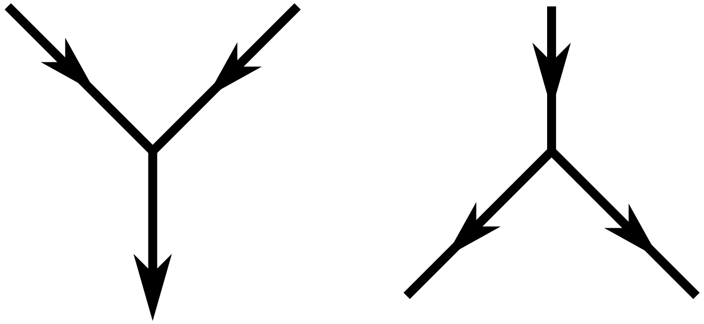


O_g

- 

handlebody-knot

(Y-oriented) spatial trivalent graph

- 

Y-orientation

Def

H : handlebody-knot of genus g

- H' : a **constituent hdbdy-knot** of H $(H' < H)$

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists D_i$: meridian disk of H $(1 \leq i \leq l)$ s.t. $H - \mathrm{Nbd}(\bigcup_{i=1}^{l} D_i) \cong H'$

- $\mathrm{Cut}(H) := g - \max\{\, g' \mid \exists O_{g'} < H \,\}$

$$= \min\left\{\, l \;\middle|\; \begin{array}{l} \exists D_i : \text{meridian disk of } H \ (1 \leq i \leq l) \\ \text{s.t. } H - \mathrm{Nbd}(\bigcup_{i=1}^{l} D_i) \cong O_{g-l} \end{array} \right\}$$

(allow genus 0)

: the **cutting number** of H

Rmk

$0 \leq \mathrm{Cut}(H) \leq g$ $\;(\mathrm{Cut}(H) = 0 \Longleftrightarrow H$: trivial$)$

§2 A G-family of quandles

Def

$(X, *)$: quandle

$\overset{\text{def}}{\Longleftrightarrow}$

$\forall x, y, z \in X,$

$\bullet x * x = x$

$\bullet * x : X \to X; a \mapsto a * x$: bijection

$\bullet (x * y) * z = (x * z) * (y * z)$

Ex

$\mathbb{Z}_k[t^{\pm 1}]$: **Alexander quandle**

with $x * y = tx + (1 - t)y$

Def

X : quandle

$$\mathrm{type}\, X := \min \left\{ n > 0 \;\middle|\; x *^n y := (\cdots((x \underbrace{*y) * y) * \cdots * y}_{n}) = x \ (\forall x, y \in X) \right\}$$

Def [Ishii-Iwakiri-Jang-Oshiro]

G : group,

$(X, \{*^g\}_{g \in G})$: **G-family of quandles**

$\overset{\text{def}}{\iff}$

$\forall g, h \in G, \forall x, y, z \in X,$

- $x *^g x = x$
- $x *^{gh} y = (x *^g y) *^h y, \;\; x *^e y = x$
- $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$

Ex

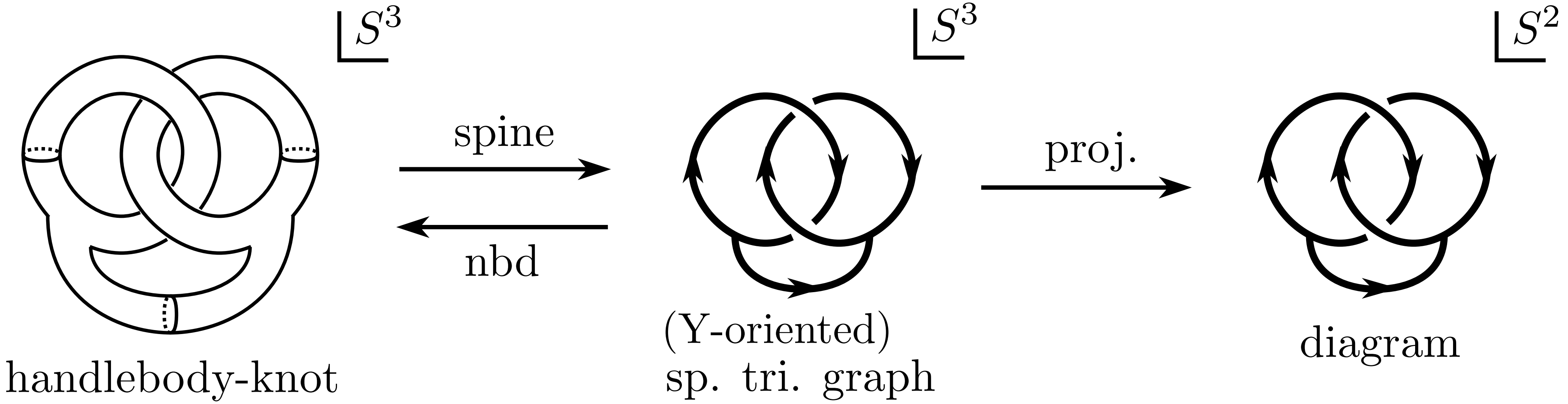
$(X, *)$: (Alexander) quandle, $k := \text{type}\, X$,

$\Longrightarrow (X, \{*^{[i]}\}_{[i] \in \mathbb{Z}_k})$: **$\mathbb{Z}_k$-family of (Alexander) quandles**

with $x *^{[n]} y := x *^n y = (\cdots((x \underbrace{* y) * y) * \cdots * y}_{n})$

§3 Colorings

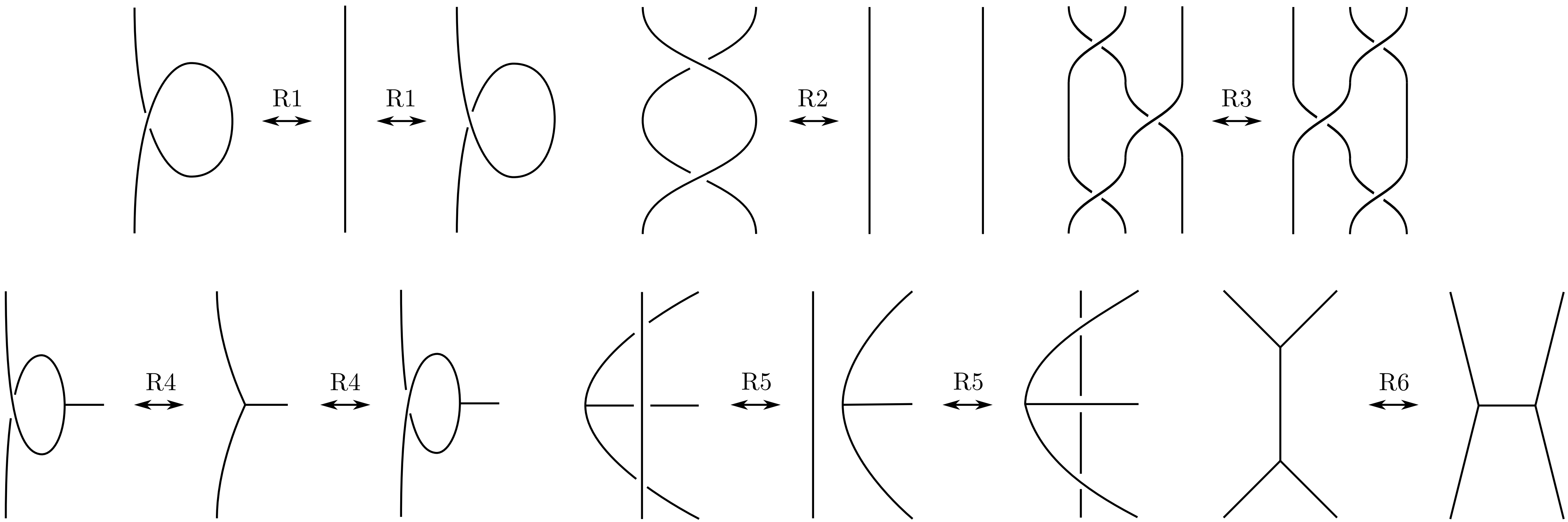
Def



<u>Thm</u> [Ishii]

D_1, D_2 : diagrams of handlebody-knots H_1, H_2,

$$H_1 \cong H_2 \Longleftrightarrow D_1 \overset{\text{(R1} \sim \text{R6 moves)}}{\longleftrightarrow} \cdots\cdots \longleftrightarrow D_2$$



(preserving Y-orientations)

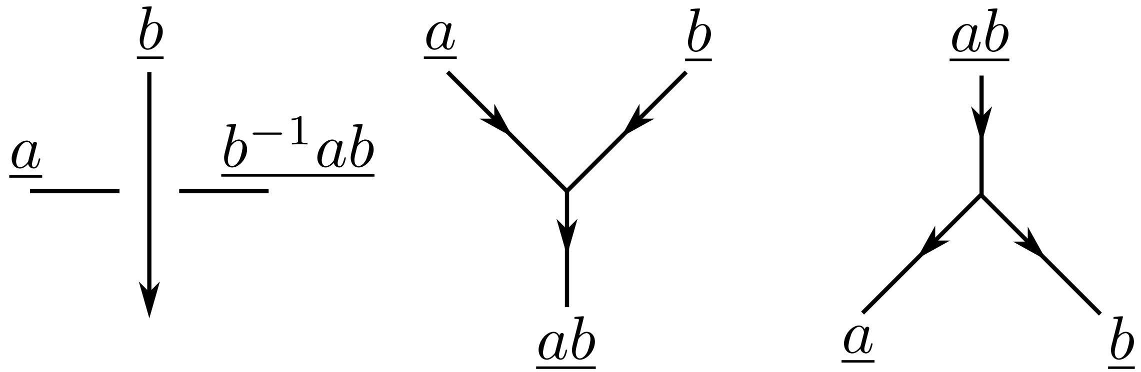
Def

D : diagram of a handlebody-knot H

G : group

• $\phi : \mathcal{A}(D) = \{$arc of $D\} \to G$: **G-flow** of D

$\stackrel{\text{def}}{\Longleftrightarrow}$



$a, b \in G$

$\Longleftrightarrow \phi \in \mathrm{Hom}(\pi_1(S^3 - H), G)$

• $\mathrm{Flow}(D; G) :=$ { G-flow of D }

• $\phi \in \mathrm{Flow}(D; G)$: **trivial G-flow** $\stackrel{\text{def}}{\Longleftrightarrow} \mathrm{Im}(\phi) = \{e\}$

Rmk

$D_1 \xleftrightarrow{\text{(one of R1} \sim \text{R6 moves)}} D_2$

$\Longrightarrow \forall \phi_1 \in \mathrm{Flow}(D_1; G),\ \ \exists! \phi_2 \in \mathrm{Flow}(D_2; G)$ s.t.

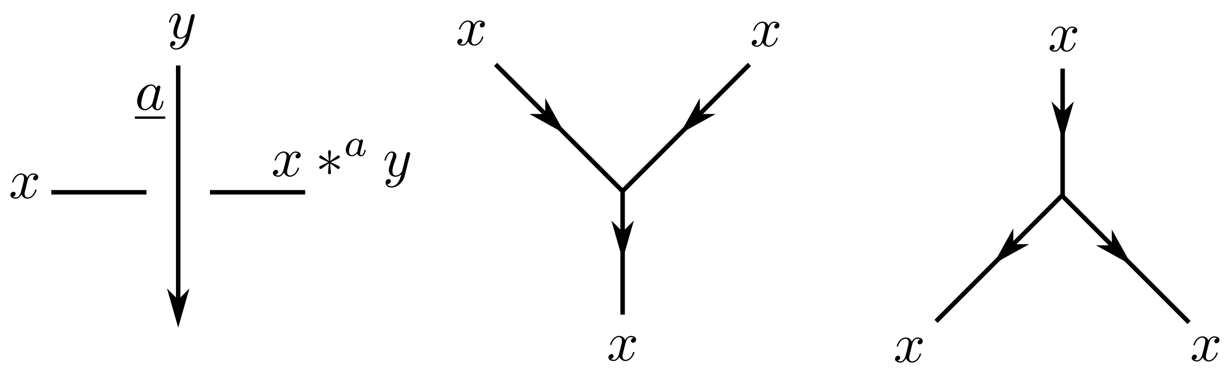
ϕ_2 coincides with ϕ_1 except near the point where the move applied.

Def

(D,ϕ) : G-flowed diag. of a hdbdy-knot H, X : G-family of quandles

- $C : \mathcal{A}(D,\phi) = \{$arc of $(D,\phi)\} \to X$: **X-coloring** of (D,ϕ)

$\overset{\text{def}}{\Longleftrightarrow}$



$a \in G$
$x, y \in X$

- $\mathrm{Col}_X(D,\phi) := \{X$-coloring of $(D,\phi)\}$
- $C \in \mathrm{Col}_X(D,\phi)$: **trivial coloring** $\overset{\text{def}}{\Longleftrightarrow}$ C : constant map

- $\phi \in \mathrm{Flow}(D;G)$: **trivial coloring G-flow**

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall Y$:G-family of quandle, $\forall C \in \mathrm{Col}_Y(D,\phi)$, C : trivial coloring

- $\mathrm{Flow}_{\text{trivial}}(D;G) := \{\phi \in \mathrm{Flow}(D;G) | \phi :$ trivial coloring G-flow$\}$

Rmk

- $(D_1,\phi_1) \xleftrightarrow{\text{(R1} \sim \text{R6 moves)}} (D_2,\phi_2) \Rightarrow \#\mathrm{Col}_X(D_1,\phi_1) = \#\mathrm{Col}_X(D_2,\phi_2)$
- X : G-family of Alex. qnd., field $\Rightarrow$ $\mathrm{Col}_X(D,\phi)$: vec. sp. over X

§4 Results and examples

Theorem [M]
H, H' : handlebody-knots of genus g, g' $(g' < g)$
D : diagram of H
(D', ϕ') : G-flowed diagram of H'
X : G-family of Alexander quandles, field

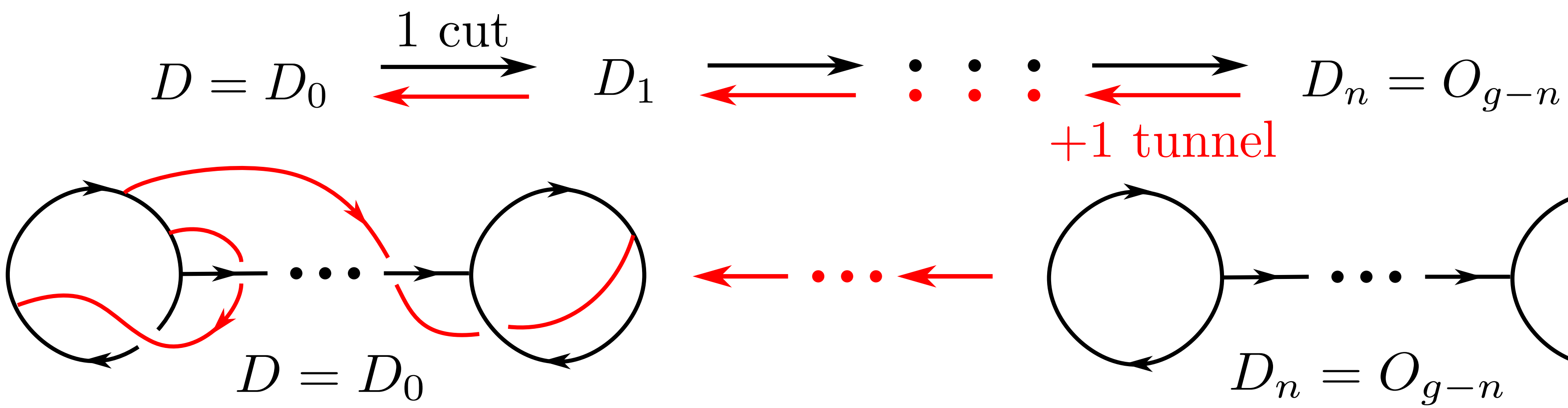
(1) $H' < H$
$\Rightarrow \exists \phi \in \mathrm{Flow}(D; G)$ s.t. $\dim \mathrm{Col}_X(D', \phi') - \dim \mathrm{Col}_X(D, \phi) \leq g - g'$

(2) $g - \log_{|G|} \#\mathrm{Flow}_{\text{trivial}}(D; G) \leq \mathrm{Cut}(H)$

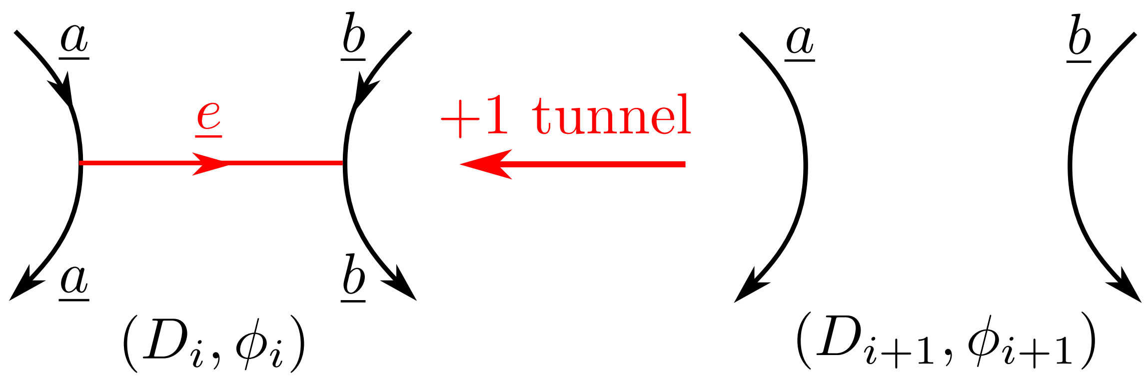
$$\left(\mathrm{Flow}_{\text{trivial}}(D; G) = \left\{ \phi \in \mathrm{Flow}(D; G) \;\middle|\; \begin{array}{l} \forall Y : G\text{-fam. of qnd.}, \\ \forall C \in \mathrm{Col}_Y(D, \phi), \\ C : \text{trivial coloring} \end{array} \right\} \right)$$

Proof (2)

$n := \mathrm{Cut}(H)$



$$\#\mathrm{Flow}_{\mathrm{trivial}}(D_n; G) = \#\mathrm{Flow}(D_n; G) = |G|^{g-n}$$

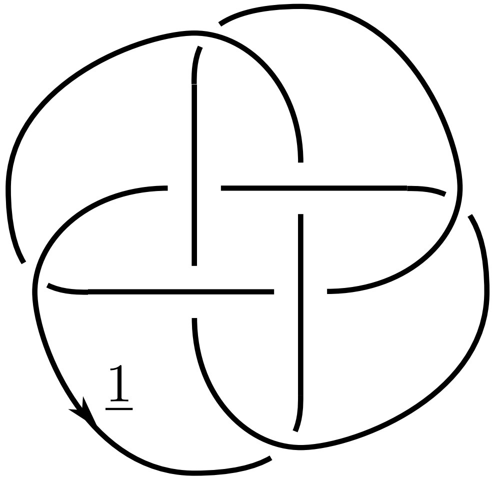


$$\phi_{i+1} \in \mathrm{Flow}_{\mathrm{trivial}}(D_{i+1}; G) \Rightarrow \phi_i \in \mathrm{Flow}_{\mathrm{trivial}}(D_i; G)$$
$$\therefore \#\mathrm{Flow}_{\mathrm{trivial}}(D; G) \geq \#\mathrm{Flow}_{\mathrm{trivial}}(D_n; G) = |G|^{g-n}$$

Ex1

$X := \mathbb{Z}_3[t^{\pm 1}]/(t+1)$: $\mathbb{Z}_2$-family of Alexander quandles, field

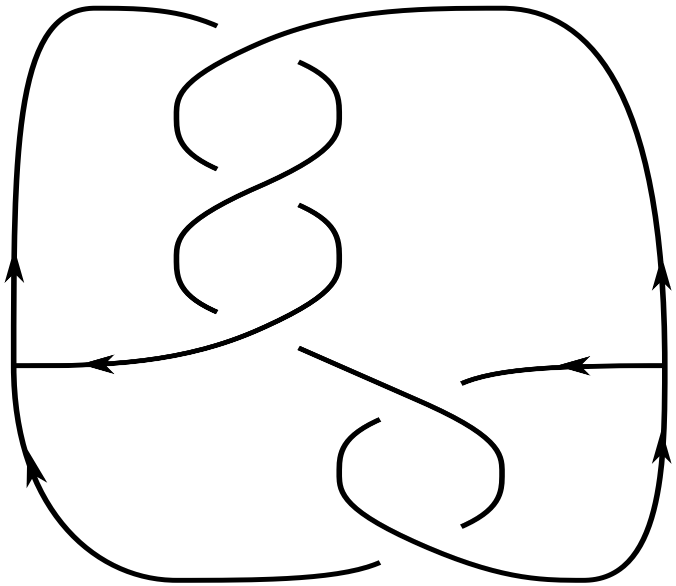
$K(= 8_{18})$:



(D', ϕ')

ϕ' : $\mathbb{Z}_2$-flow of D'

$H(= 5_1)$:



D

$\dim \mathrm{Col}_X(D', \phi') = 3$

$\dim \mathrm{Col}_X(D, \forall\phi) = 1$

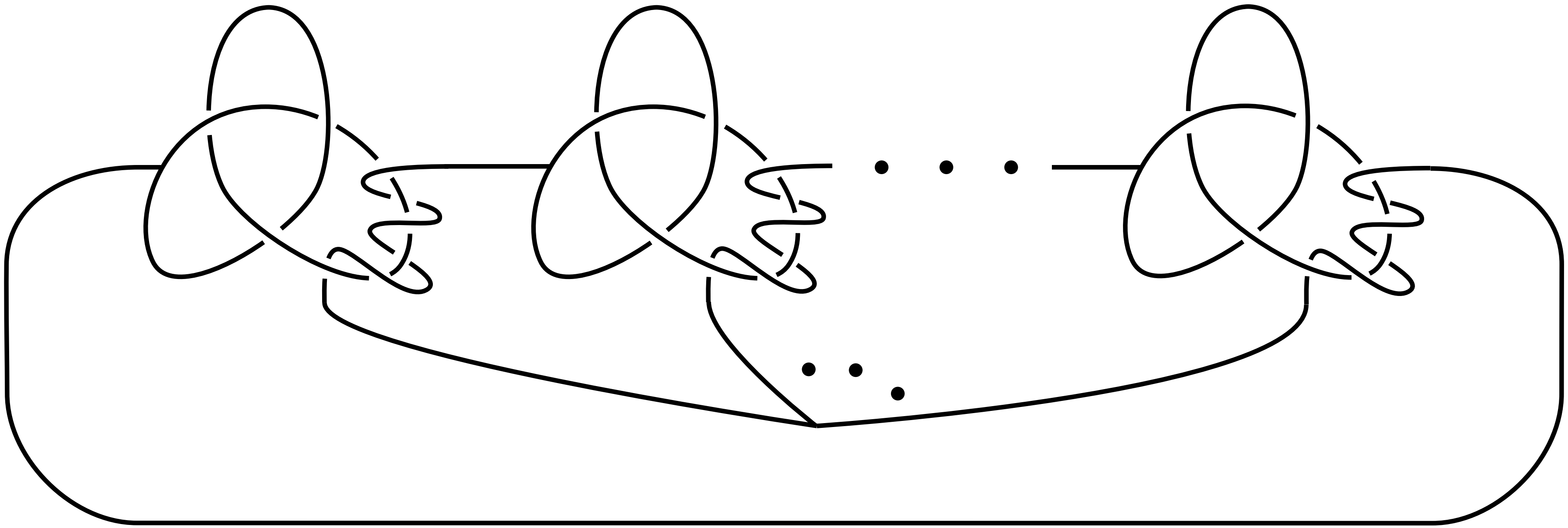
$\forall \phi \in \mathbf{Flow}(D, \mathbb{Z}_2), \quad \dim \mathrm{Col}_X(D', \phi') - \dim \mathrm{Col}_X(D, \phi) = 2$

$\therefore K \not\leq H$ ($\because$ Theorem (1))

Ex2

$X := \mathbb{Z}_2[t^{\pm 1}]/(t^2 + t + 1)$: $\mathbb{Z}_3$-family of Alexander quandles

H_g :



D_g

$\forall \phi \in \mathbf{Flow}(D_g; \mathbb{Z}_3)$:non-trivial flow, $\exists C \in \mathbf{Col}_X(D_g, \phi)$:non-trivial col.
(i.e. $\mathbf{Flow}_{\text{trivial}}(D_g; \mathbb{Z}_3) = \{\phi$: trivial flow$\}$)

$\therefore g \leq \mathbf{Cut}(H_g)$ ($\because$ Theorem (2))

$\therefore \mathbf{Cut}(H_g) = g$