# 3次元球面に埋め込まれたコンパクト曲面の全同位による分類について

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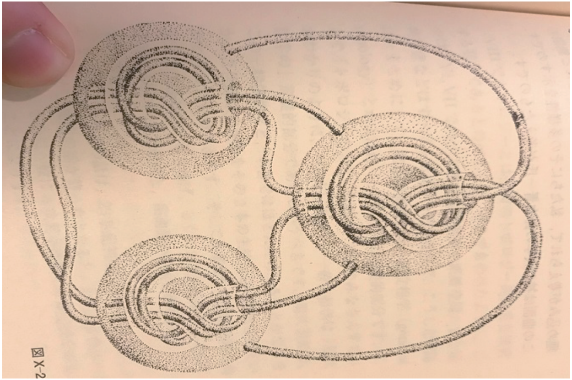
# §1. MOTIVATIONS

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- Can we represent them by using “diagrams” ?
- Are there “Reidemister moves” for their diagrams ?

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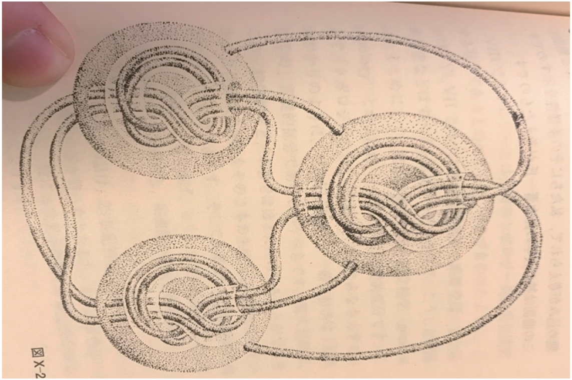
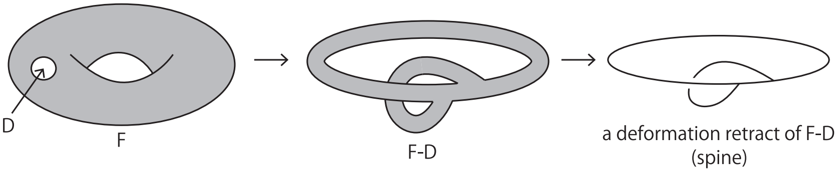


- I am interested in classification of closed surfaces embedded in $S^3$.
- Can we represent them by using “diagrams” ?
- Are there “Reidemister moves” for their diagrams ?

## Easy observation.

$F$: a closed surface embedded in $S^3$.

- If we remove an open disk $D$ on $F$,
  we have a compact surface with boundary $F - D$ .
- $F - D$ contains a graph as a deformation retract of $F - D$.




We might be able to represent the middle surface by using (trivalent) graph !

## Proposition

*$F$: a connected closed surface embedded in $S^3$.*
*$D$: a disk in $F$.*
*$D'$: a disk in $S^3$ s.t. $\partial D' = \partial D$, $D' \cap (F \setminus D) = \emptyset$.*

$\Rightarrow$ *$F$ and $D' \cup (F \setminus D)$ are ambient isotopic on $S^3$.*

## Remark

In the case where $F$ is not connected,
if $F$ is non-splittable, an analogous proposition holds.

Essentially, we should deal with non-splittable surfaces !

# § 2. A SPATIAL SURFACE.

In this talk,

1. A graph is finite and
2. Every vertex of a graph is valence-2 or valence-3.
3. An spatial graph is a graph embedded in $S^3$.

Definition

$F$: a compact 2-manifold in $S^3$.

$F$ is a spatial surface.

$\overset{\text{def}}{\Leftrightarrow}$ $\forall C$: a connected component of $F$, $\partial C \neq \emptyset$.

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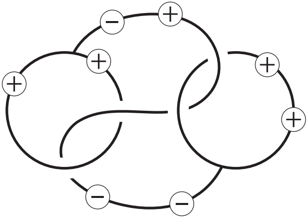
# Definition

$D$: a diagram of a spatial graph $G$.

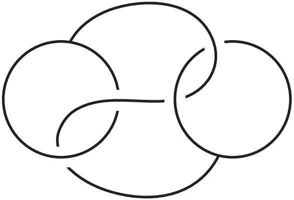
- A map $s : V_2(G) \to \{+1, -1\}$ is a sign for $D$.
- A pair $(D, s)$ is a signed diagram.

( $V_2(G)$ is the set consisting of valence-2 vertices of $G$.)

We regard the empty map $0 : V_2(G) \to \{+1, -1\}$ as a sign for $D$.

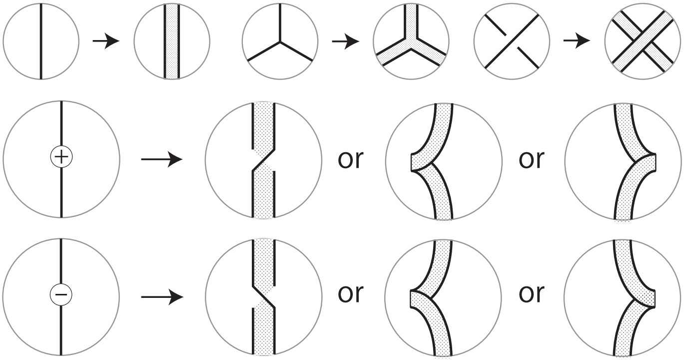


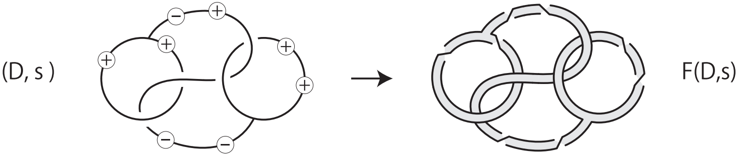

(D, s )



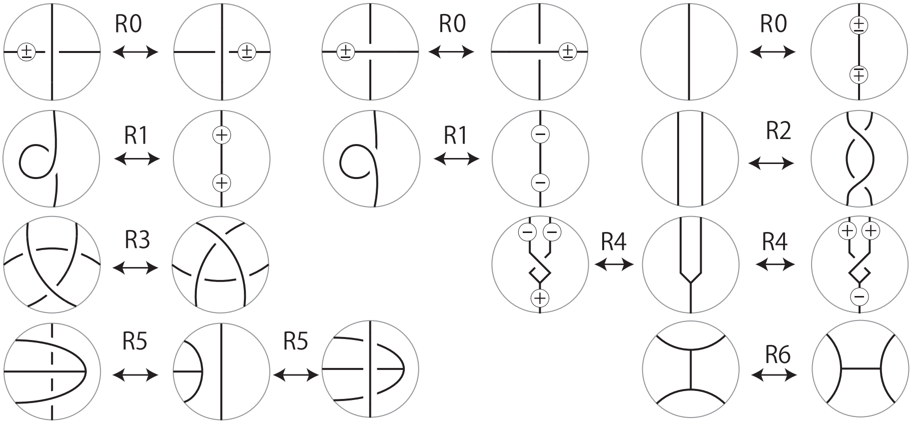
(D, 0)

# A spatial surface $F(D, s)$ obtained from $(D, s)$



# Reidemesiter moves for signed diagrams



## Remark

$(D, s)$, $(D', s')$ are related by Reidemeister moves.
$\Rightarrow F(D, s) \overset{\text{a.i.}}{\sim} F(D', s')$ (ambient isotopic).

## Remark

Every spatial surface can be represented by some signed diagram.

## Theorem (M)

*$(D, s)$, $(D', s')$: signed diagrams.*

$$F(D, s) \overset{\text{a.i.}}{\sim} F(D', s').$$

$\Leftrightarrow$ *$(D, s)$, $(D', s')$ are related by R0, R1, R2, R3, R4, R5, R6 on $\mathbb{R}^2$.*

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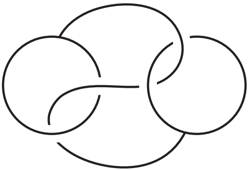
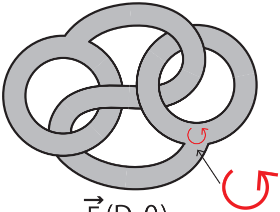
# § 3. AN ORIENTED SPATIAL SURFACE

## Notation

$D$: a diagram of spatial graphs.
$\vec{F}(D, 0)$: an oriented spatial surface defined as follows.




## Remark

Every oriented spatial surface can be represented by some signed diagram $(D, 0)$.

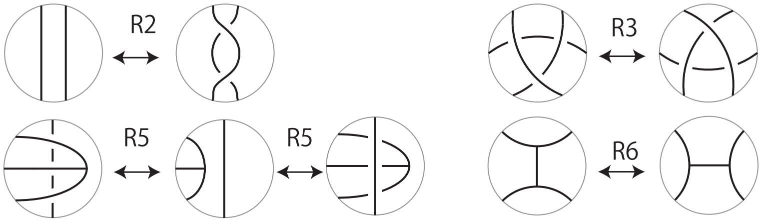
# Theorem (M)

*$D$, $D'$: diagrams of spatial trivalent graphs.*

$$\vec{F}(D,0) \overset{\text{a.i.}}{\sim} \vec{F}(D',0) \text{ (orientation preserving)}$$

$\Leftrightarrow$ $(D,0)$, $(D',0)$ *are related by*

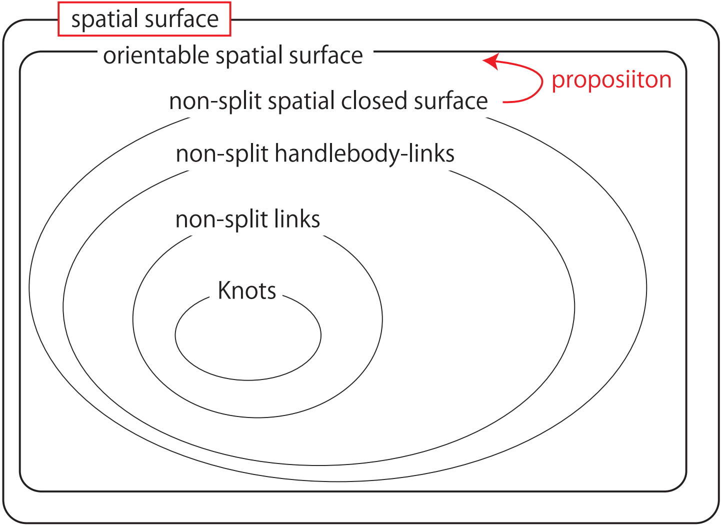
*R2, R3, R5, R6 on* $S^2$.



# §4. RELATION BETWEEN KNOTS, LINKS, HANDLEBODY-LINKS, SPATIAL SURFACES

## Observation.

- There is the injection from the set of link (knot) types to the set of handlebody-link types.
- There is the injection from the set of handlebody-link types to the set of spatial closed surface types.
(The boundary of a handlebody-link is a spatial orientable closed surface.)
- There is the injection from the set of non-split spatial closed surface types to the set of orientable spatial surface types.




All embeddings above are representable by signed diagrams!

**Very rough outline of Proof.**

## Theorem (再掲)

*$(D, s)$, $(D', s')$: signed diagrams.*

$F(D, s) \overset{\text{a.i.}}{\sim} F(D', s')$

$\Leftrightarrow$ *$(D, s)$, $(D', s')$ are related by R0, R1, R2, R3, R4, R5, R6 on $\mathbb{R}^2$.*

## Proof($\Rightarrow$).

$\{h_t\}_{t\in[0,1]}$: an ambient isotopy s.t. $h_1(F(D, s)) = F(D', s')$.

Step1.

We assign a sign $s_1$ for $h_1(D)$ s.t. $F(h_1(D), s_1) \overset{\text{a.i.}}{\sim} F(D', s')$.

Step2.

Prove $(D, s) \sim (h_1(D), s_1)$ (by R0, R1, R2, R3, R4, R5).

Step3.

Prove $(h_1(D), s_1) \sim (D', s')$ (by R0, R1, R2, R3, R4, R5, R6). □

**Very rough outline of Proof.**

## Theorem (再掲)

*$D$, $D'$: diagrams of spatial graphs.*

$\vec{F}(D,0) \overset{\text{a.i.}}{\sim} \vec{F}(D',0)$ *(orientation preserving).*

$\Leftrightarrow$ $(D,0)$, $(D',0)$ *are related by R2, R3, R5, R6 on* $S^2$*.*

## Proof ($\Rightarrow$).

- By Theorem 1, $(D,0) \sim (D',0)$ (by R0, R1, R2, R3, R4, R5, R6).
- A sequence of Reidemeister moves from $(D,0)$ to $(D',0)$ has the following condition:

  For each vertex of $D$, R4 occurs even times.

- We replace R0, R1, R4 with R2, R3, R5. □

**Future work.**

construct invariants for spatial surfaces.
(for example, using rack, quandle, ... etc.)

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