Dijkgraaf-Witten invariants of cusped hyperbolic 3-manifolds

Naoki Kimura

Waseda University

December 23, 2017

Introduction

M : a closed oriented 3-manifold

G : a finite group

$\alpha \in Z^3(BG, U(1))$: a normalized 3-cocycle

The Dijkgraaf-Witten invariant is defined as follows:

$$Z(M) = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M), G)} \langle \gamma^*[\alpha], [M] \rangle \quad \in \mathbb{C}.$$

$Z(M)$ is a topological invariant (homotopy invariant).

Introduction

The Dijkgraaf-Witten invariant is constructed by using a triangulation:

$$Z(M) = \frac{1}{|G|^a} \sum_{\varphi \in \mathrm{Col}(M)} \prod_{\text{tetrahedron}} \alpha(g, h, k)^{\pm 1}.$$

a : the number of the vertices of M

$g, h, k \in G$: colors of edges of a tetrahedron

We consider the Dijkgraaf-Witten invariant as "the Turaev-Viro type invariant".

G : a finite group

$$C^n(G, U(1)) = \begin{cases} U(1) & (n = 0) \\ \{\alpha : \overbrace{G \times \cdots \times G}^{n} \to U(1)\} & (n \geq 1) \end{cases}$$

$$\delta^n : C^n(G, U(1)) \to C^{n+1}(G, U(1))$$

$$(\delta^0 a)(g) = 1 \quad (a \in U(1),\ g \in G)$$

$$(\delta^n \alpha)(g_1, \ldots, g_{n+1}) = \alpha(g_2, \ldots, g_{n+1}) \times$$

$$\prod_{i=1}^{n} \alpha(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})^{(-1)^i} \times \alpha(g_1, \ldots, g_n)^{(-1)^{n+1}}$$

$$(\alpha \in C^n(G, U(1)),\ g_1, \ldots, g_{n+1} \in G,\ n \geq 1)$$

Group cohomology

$$\alpha \in C^n(G, U(1)) \text{ is normalized}$$

$$\Leftrightarrow \alpha(1, g_2, \ldots, g_n) = \alpha(g_1, 1, g_3, \ldots, g_n) = \cdots$$
$$= \alpha(g_1, \ldots, g_{n-1}, 1) = 1.$$
$$(g_1, \ldots, g_n \in G)$$

cocycle condition

$\alpha \in C^3(G, U(1))$,

$$\alpha \in Z^3(G, U(1))$$

$$\Leftrightarrow \alpha(h, k, l)\alpha(g, hk, l)\alpha(g, h, k) = \alpha(gh, k, l)\alpha(g, h, kl).$$
$$(g, h, k, l \in G)$$

Local order

Fix a triangulation of M.

Give an orientation to each edge such that

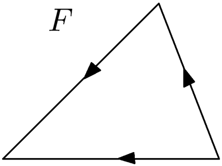
for any 2-face F, the orientations of the three edges of F

are not cyclic.

Then a total order on the set of the vertices of each

tetrahedron is induced by the choice of the orientations.

We call it a local order of M.



Coloring

Fix a triangulation of M with oriented edges and oriented faces.

A coloring of M is a map

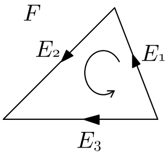
$\varphi : \{\text{oriented edges of } M\} \to G$ such that

for any oriented 2-face F,

$$\begin{aligned}\varphi(E_3)^{\epsilon_3}\varphi(E_2)^{\epsilon_2}\varphi(E_1)^{\epsilon_1} &= \varphi(E_2)^{\epsilon_2}\varphi(E_1)^{\epsilon_1}\varphi(E_3)^{\epsilon_3} \\ &= \varphi(E_1)^{\epsilon_1}\varphi(E_3)^{\epsilon_3}\varphi(E_2)^{\epsilon_2} = 1 \in G,\end{aligned}$$

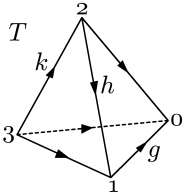
where E_1, E_2, and E_3 are the oriented edges of F and

$$\epsilon_i = \begin{cases} 1 & \text{the orientation of } E_i \text{ agrees with that of } \partial F \\ -1 & \text{otherwise.} \end{cases}$$



$\epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = -1.$

Definition of DW invariant



Correspond $\alpha(g,h,k)^{\epsilon}$ to the colored tetrahedron T, where

$$\epsilon = \begin{cases} 1 & \text{local order of } T \text{ agrees with the orientation of } M \\ -1 & \text{otherwise.} \end{cases}$$

The Dijkgraaf-Witten invariant $Z(M)$ is defined as follows:

$$Z(M) = \frac{1}{|G|^{a}} \sum_{\varphi \in \mathrm{Col}(M)} \prod_{\text{tetrahedron}} \alpha(g,h,k)^{\epsilon}.$$

a : the number of the vertices of M

Invariance

The Dijkgraaf-Witten invariant is actually a topological inavariant, i.e.

it is independent of

(1) a choice of a local order for a fixed triangulation of M

and

(2) a choice of a triangulation of M.

The invariance follows from the following theorem and the cocycle condition.

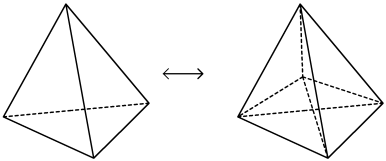
$$(\alpha(h,k,l)\alpha(g,hk,l)\alpha(g,h,k) = \alpha(gh,k,l)\alpha(g,h,kl))$$

Pachner moves

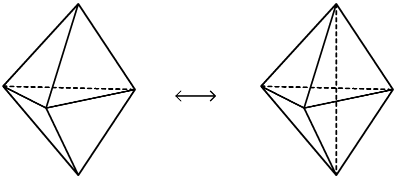
Theorem (Pachner)

Any two triangulations of a 3-manifold M can be transformed one to another by a finite sequence of the following two types of transformations.

(1,4)-Pachner move



(2,3)-Pachner move



Properties of DW invariant

M : a closed oriented 3-manifold

G : a finite group

$\alpha \in Z^3(G, U(1))$

(1) $Z(M)$ only depends on the cohomology class of α.

(2) $Z(-M) = \overline{Z(M)}$,

where $-M$ is the oriented 3-manifold with the opposite orientation to M.

Manifold with boundary case

The Dijkgraaf-Witten invariant is also defined for a compact oriented 3-manifold M with $\partial M \neq \emptyset$. However, $Z(M, \psi)$ depends on a locally ordered triangulation of ∂M and on a coloring ψ of ∂M.

cf. Dijkgraaf-Witten TQFT

$Z(M) : Z(\partial M) \to Z(\emptyset) = \mathbb{C}$: a linear map

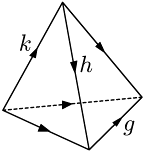
Hard to calculate the linear maps and to compare them.

We consider an extension of the Dijkgraaf-Witten invariant to 3-manifolds with boundary or cusped hyperbolic 3-manifolds using by an ideal triangulation as follows:

Definition (Extended Dijkgraaf-Witten invariant)

$$Z(M) = \sum_{\varphi \in \mathrm{Col}(M)} \prod_{\text{ideal tetrahedron}} \alpha(g, h, k)^{\pm 1}.$$

A locally ordered triangulation of ∂M and a coloring of ∂M are unnecessary for this definition.



Invariance

The extended Dijkgraaf-Witten invariant is actually
a topological inavariant, i.e.
it is independent of

(1) a choice of a local order for a fixed ideal triangulation of M

and

(2) a choice of an ideal triangulation of M.

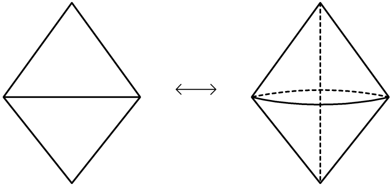
The invariance follows from the following theorem
and the cocycle condition.

Pachner moves (ideal triangulation)

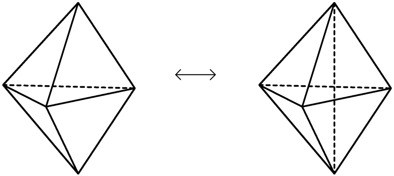
Theorem (Matveev)

Any two ideal triangulations of a 3-manifold M can be transformed one to another by a finite sequence of the following two types of transformations.

(0,2)-Pachner move

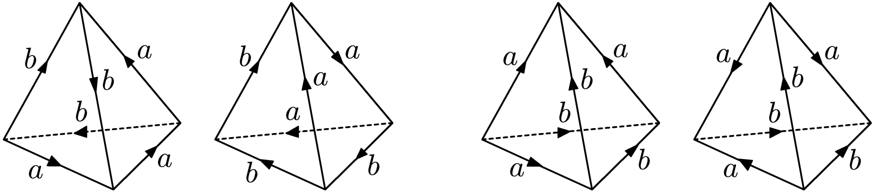


(2,3)-Pachner move



$m003$ and $m004$

$m003$ $\qquad$ $m004$ $(= S^3 \setminus 4_1)$



$\mathrm{Vol}(m003) = \mathrm{Vol}(m004) \approx 2.02988.$

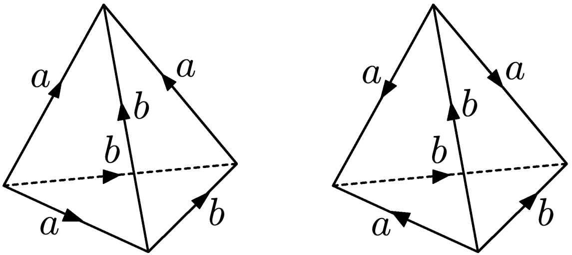
$$TV(m003) = \sum_{(a,a,b),(a,b,b)\in adm} w_a w_b \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix} \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix}$$

$$= TV(m004).$$

$H_1(m003;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5,\ \ H_1(m004;\mathbb{Z}) = \mathbb{Z}.$

Extended DW invariant of $m004$

$m004 = S^3 \setminus 4_1$



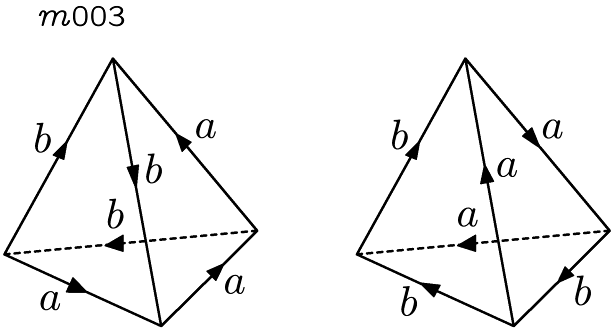
$a = ba, b = ab$

$\Rightarrow a = b = 1 \in G$

$m004$ has only a trivial coloring.

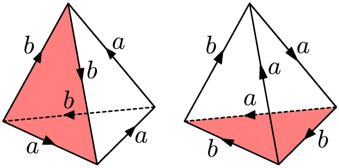
For any G and any α,

$Z(m004) = 1.$



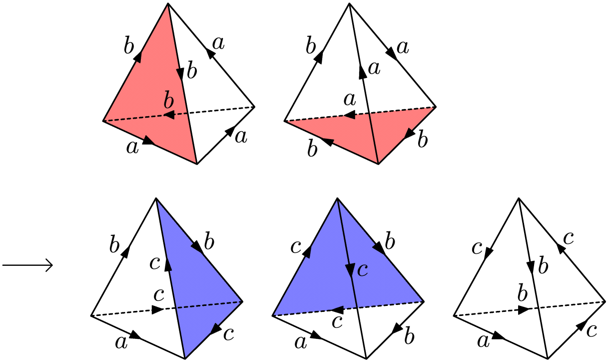
This ideal triangulation of $m003$ does not admit a local order.

In order to assign a local order, transform the ideal triangulation of $m003$ by (2,3)-Pachner moves.



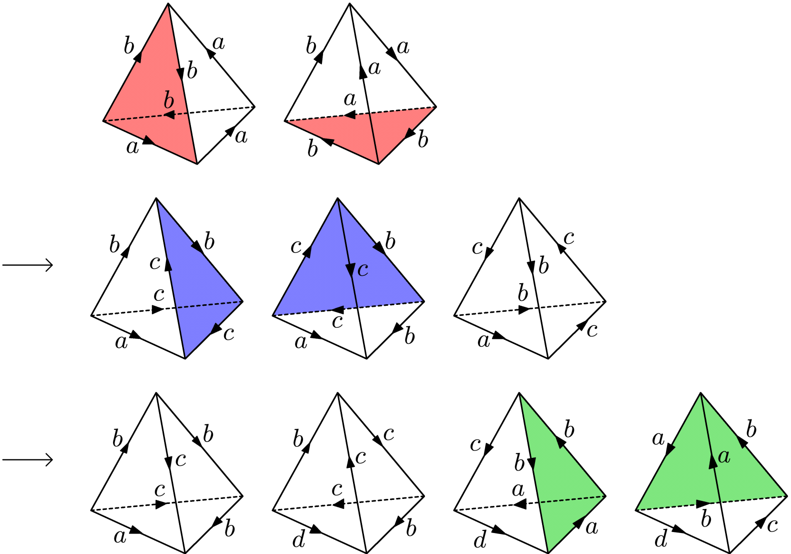
$m003$

In order to assign a local order, transform the ideal triangulation of $m003$ by (2,3)-Pachner moves.



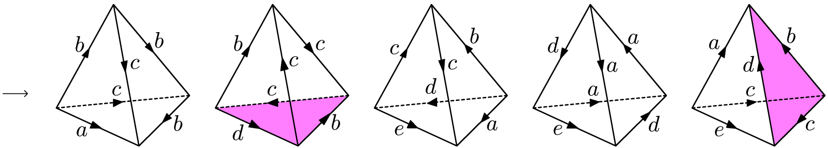
$m003$

In order to assign a local order, transform the ideal triangulation of $m003$ by (2,3)-Pachner moves.

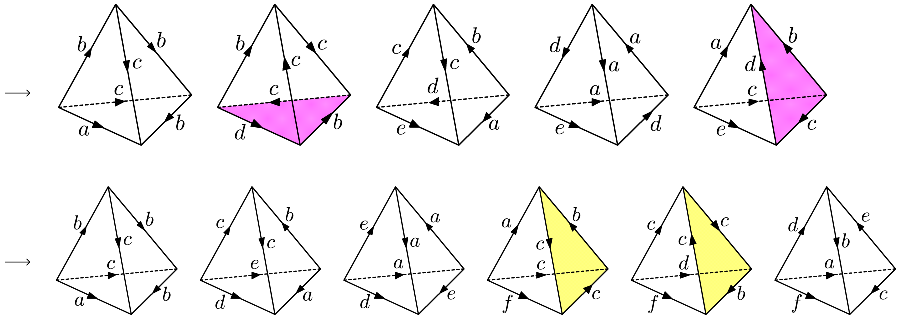


$m003$

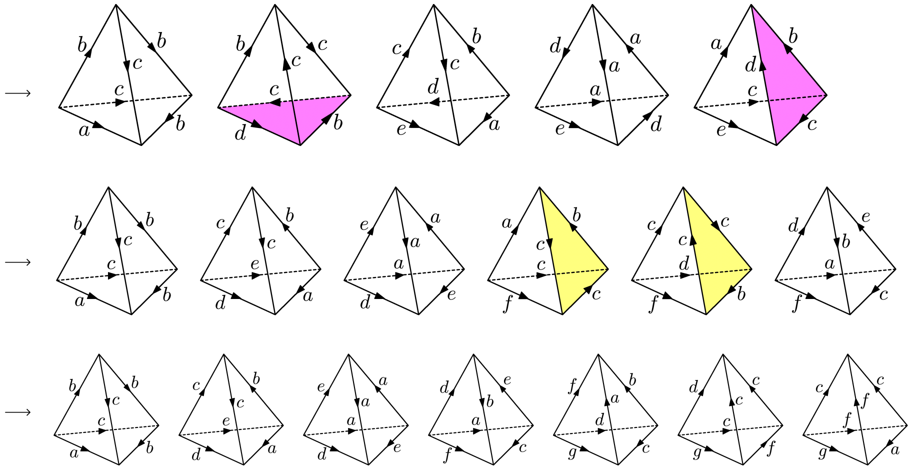
In order to assign a local order, transform the ideal triangulation of $m003$ by (2,3)-Pachner moves.



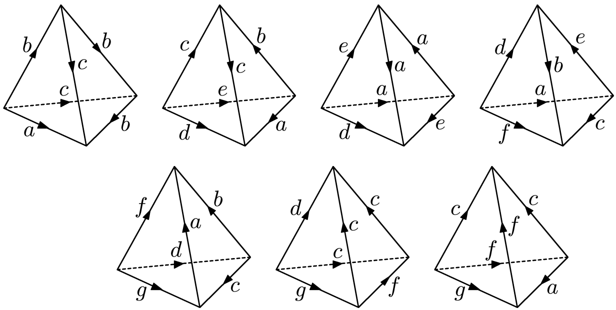
In order to assign a local order, transform the ideal triangulation of $m003$ by (2,3)-Pachner moves.



In order to assign a local order, transform the ideal triangulation of $m003$ by (2,3)-Pachner moves.



Extended DW invariant of $m003$



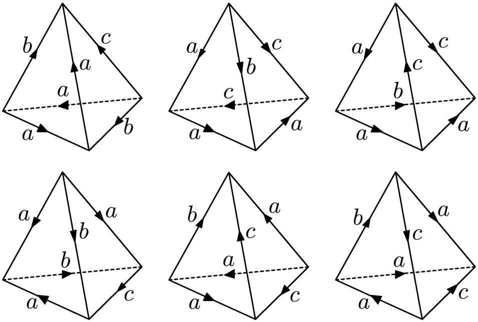
$a = b^3, c = b^2, d = b^4, e = b, f = 1, g = b^2,\ b^5 = 1.$

$$Z(m003) = \sum_{b^5=1} \alpha(b,b,b)^{-1}\alpha(b^2,b,b)\alpha(b^3,b^3,b^3)$$
$$\times\alpha(b,b,b^3)\alpha(b,b^2,b^2)\alpha(b^2,b^3,b^2).$$

$G = \mathbb{Z}_5$, α is a generator of $H^3(\mathbb{Z}_5, U(1)) \cong \mathbb{Z}_5$,

$Z(m003) = \frac{1}{2}(5+\sqrt{5}+i\sqrt{10+2\sqrt{5}})$. $\quad (Z(m004) = 1.)$

$m006$ and $m007$



$m006$

$m007$

$$\mathrm{Vol}(m006) = \mathrm{Vol}(m007) \approx 2.56897.$$

$$TV(m006) = \sum w_a w_b w_c \begin{vmatrix} a & b & c \\ a & b & a \end{vmatrix} \begin{vmatrix} a & b & c \\ a & c & a \end{vmatrix} \begin{vmatrix} a & b & c \\ a & c & a \end{vmatrix}$$

$$= TV(m007).$$

$H_1(m006;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5,\ H_1(m007;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_3.$

Extended DW invariants of $m006$ and $m007$

$$Z(m006) = \sum_{a^5=1} \alpha(a,a,a)^3 \alpha(a,a^2,a)\alpha(a^3,a^3,a^3).$$

$$Z(m007) = \sum_{a^3=1} \alpha(a,a,a)\alpha(a^{-1},a^{-1},a^{-1}).$$

$G = \mathbb{Z}_5$, α is a generator of $H^3(\mathbb{Z}_5, U(1)) \cong \mathbb{Z}_5$,

$Z(m006) = -\frac{\sqrt{5}}{2}, \quad Z(m007) = 1.$

$s778$ and $s788$

The previous two pairs of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and the same Turaev-Viro invariants are distinguished by their homology groups.

The following pair of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and the same homology groups have the distinct Dijkgraaf-Witten invariants.

$\mathrm{Vol}(s778) = \mathrm{Vol}(s788) \approx 5.33349.$

$H_1(s778; \mathbb{Z}) = H_1(s788; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_{12}.$

$Z(s778) \neq Z(s788).$

Extended DW invariants of $s778$ and $s788$

$$Z(s778) = \sum_{a^{12}=1} \alpha(a,a,a^2)\alpha(a^2,a,a)\alpha(a^2,a,a^2)$$
$$\times\alpha(a^3,a^2,a^3)\alpha(a^3,a^{10},a^3)\alpha(a^5,a^5,a^{10})$$
$$\times\alpha(a^{10},a^5,a^5)\alpha(a^{10},a^5,a^{10}).$$

$$Z(s788) = \sum_{a^{12}=1} \alpha(a^5,a,a^2)\alpha(a^6,a^2,a^3)\alpha(a^8,a,a^2)$$
$$\times\alpha(a^8,a,a^8)^{-1}\alpha(a^8,a^5,a^8)^{-1}\alpha(a^8,a^9,a^8)^{-1}$$
$$\times\alpha(a^9,a^5,a^3)^{-1}\alpha(a^9,a^8,a)^{-1}\alpha(a^9,a^9,a^5).$$

$G = \mathbb{Z}_{12}$, α is a generator of $H^3(\mathbb{Z}_{12}, U(1)) \cong \mathbb{Z}_{12}$,

$Z(s778) = -6, \quad Z(s788) = 3 - 2\sqrt{3}.$

References

[1] B.A.Burton, R.Budney, W.Pettersson, et al., *Regina: Software for low-dimensional topology*, http://regina-normal.github.io/, 1999-2016.

[2] M.Culler, N.M.Dunfield, M.Goerner, and J.R.Weeks, *SnapPy, a computer program for studying the geometry and topology of 3-manifolds*, http://snappy.computop.org/.

[3] R.Dijkgraaf and E.Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. 129 (1990), no. 2, 393-429.

[4] S.V.Matveev, *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics, vol. 9, Springer, 2003.

[5] V.G.Turaev and O.Y.Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology 31 (1992), 865-902.

[6] M.Wakui, *On Dijkgraaf-Witten invariant for 3-manifolds*, Osaka J. Math. 29 (1992), no. 4, 675-696.