Property of the interior polynomial from the HOMFLY polynomial

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東京工業大学理学院数学系博士課程後期 1 年

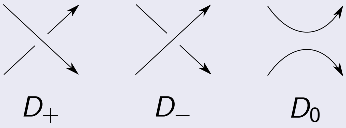
2017 年 12 月 24 日

1. Computing the HOMFLY polynomial using the combinatorics
   - HOMFLY polynomial
   - Interior polynomial
   - Root polytope and Ehrhart polynomial
   - Signed version of interior polynomial and Ehrhart polynomial

2. Properties of the interior polynomial
   - Mirror image
   - Flyping and mutation

Definition 1 (HOMFLY polynomial)

There is a function $P : \{oriented\ links\ in\ S^3\} \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ defined uniquely by
(i) $P(unknot) = 1$,
(ii) $v^{-1}P_{D_+} - vP_{D_-} = zP_{D_0}$, where D_+, D_-, D_0 are an oriented skein triple.



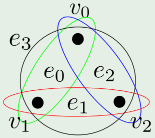
Definition 2 (top of the HOMFLY polynomial)

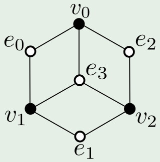
$\mathrm{Top}_D(v) =$ the coefficient of $z^{c(D)-s(D)+1}$ in the HOMFLY polynomial of D.

Interior polynomial

- $\mathscr{H} = (V, E)$: hypergraph.
- $I_{\mathscr{H}}(x)$: interior polynomial (T. Kálmán 2013).
- It generalizes the evaluation $x^{|V|-1}T_G(1/x, 1)$ of the classical Tutte polynomial $T_G(x, y)$ of the graph $G = (V, E)$.
- We regard the interior polynomial as an invariant of bipartite graph $G = (V, E, \mathcal{E})$ with color classes E and V (T. Kálmán and A. Postnikov 2016).

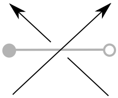
Example





$$I_G(x) = 1x^0 + 3x^1 + 3x^2.$$

For any plane bipartite graph G, Let L_G be the alternating link obtained from G by replacing each edge by a crossing.



Obviously, L_G is a special alternating diagram.

Theorem 3 (T. Kálmán, H. Murakami and A. Postnikov, 2016)

$G = (V, E, \mathcal{E})$: a connected plane bipartite graph.

$$\mathrm{Top}_{L_G}(v) = v^{|\mathcal{E}|-(|V|+|E|)+1} I_G(v^2),$$

where $I_G(x)$ is the interior polynomial of G.

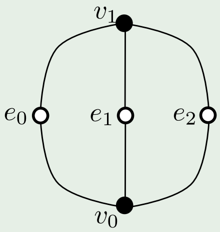
$G = (V, E, \mathcal{E})$: a bipartite graph

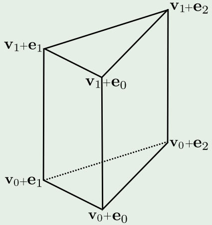
Definition 4

For $v \in V$ and $e \in E$, let $\mathbf{v}$ and $\mathbf{e}$ denote the standard generators of $\mathbb{R}^V \oplus \mathbb{R}^E$. Then the root polytope of G is defined to be

$$Q_G = \mathrm{Conv}\{\mathbf{v} + \mathbf{e} \mid ve \text{ is an edge of } G\}.$$

Example





$d = \dim Q_G = |V| + |E| - 2.$

Q_G : the root polytope of a bipartite graph G.

Definition 5 (Ehrhart polynoial)

$$\varepsilon_{Q_G}(s) := |s \cdot Q_G \cap \mathbb{Z}^V \oplus \mathbb{Z}^E|.$$

Definition 6 (Ehrhart series)

$$\mathrm{Ehr}_{Q_G}(x) = \sum_{s \in \mathbb{Z}_{\geq 0}} \varepsilon_{Q_G}(s) x^s.$$

Theorem 7 (T. Kálmán and A. Postnikov, 2016)

$G = (V, E, \mathcal{E})$: connected bipartite graph.
I_G : the interior polynomial of G.

$$\frac{I_G(x)}{(1-x)^{d+1}} = \mathrm{Ehr}_{Q_G}(x).$$

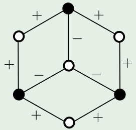
$G = (V, E, \mathcal{E}_- \cup \mathcal{E}_+)$: a connected signed bipartite graph.

Definition 8 (signed interior polynomial)

$$I_G^+(x) = \sum_{\mathcal{S} \subseteq \mathcal{E}_-} (-1)^{|\mathcal{S}|} I_{G \setminus \mathcal{S}}(x),$$

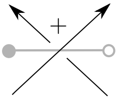
where $G \setminus \mathcal{S}$ is bipartite graph obtained from G by deleting $\forall e \in \mathcal{S}$ and by forgetting sign.

Example

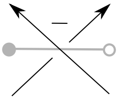


$$I_G^+ = 1x^3.$$

For any signed plane bipartite graph G, Let L_G be the oriented link obtained from G by replacing each edge to a crossing.



positive edge



negative edge

Obviously, L_G is a special diagram.

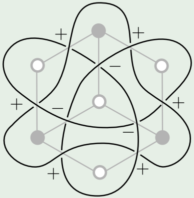
Theorem 9 (K.)

$G = (V, E, \mathcal{E}_+ \cup \mathcal{E}_-)$: *plane signed bipartite graph.*

$$\mathrm{Top}_{L_G}(v) = v^{|\mathcal{E}_+|-|\mathcal{E}_-|-(|V|+|E|)+1} I_G^+\left(v^2\right),$$

where $I_G^+(x)$ *is the interior polynomial of* G.

Example



$$I_G^+ = 1x^3.$$

$$P_{L_G}(v,z) = \begin{array}{ccc} & +1v^3z^3 & \\ & +4v^3z & -1v^5z \\ -1vz^{-1} & +3v^3z^{-1} & -2v^5z^{-1}. \end{array}$$

Proposition 10 (Murasugi and Pryzytycki, 1989)

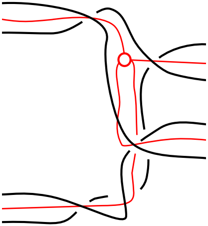
$D_1 * D_2$: a link diagram obtained by Murasugi-sum. Then

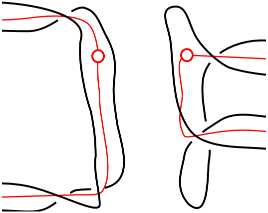
$$\mathrm{Top}_{D_1 * D_2}(v) = \mathrm{Top}_{D_1}(v)\,\mathrm{Top}_{D_2}(v).$$

Proposition 11

$G_1 * G_2$: a signed bipartite graph obtained by identifying one vertex. Then

$$I^{+}_{G_1 * G_2}(x) = I^{+}_{G_1}(x) I^{+}_{G_2}(x).$$





Theorem 12 (K.)

D : oriented link diagram.
$G = (V, E, \mathcal{E}_+ \cup \mathcal{E}_-)$: the Seifert graph of D. Then

$$\mathrm{Top}_D(v) = v^{|\mathcal{E}_+|-|\mathcal{E}_-|-(|V|+|E|)+1} I_G^+ \left(v^2\right).$$

$G = (V, E, \mathcal{E}_+ \cup \mathcal{E}_-)$: a signed bipartite graph.

Definition 13 (the signed Ehrhart series)

$$\mathrm{Ehr}_G^+(x) = \sum_{\mathcal{S} \subseteq \mathcal{E}_-(G)} (-1)^{|\mathcal{S}|} \, \mathrm{Ehr}_{Q_{G \setminus \mathcal{S}}}(x) \, .$$

Theorem 14 (K.)

$I_G^+(x)$: the signed interior polynomial of G. Then

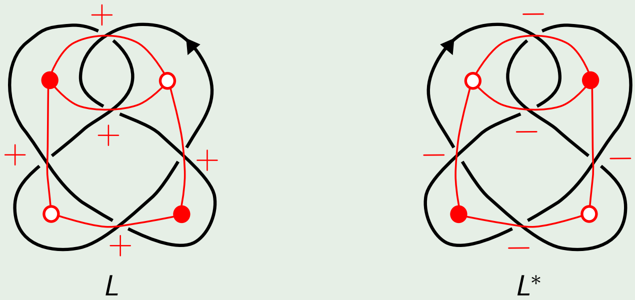
$$\frac{I_G^+(x)}{(1-x)^{d+1}} = \mathrm{Ehr}_G^+(x).$$

Theorem 15

L^* : mirror image of L. Then

$$P_{L^*}(v,z) = P_L(-v^{-1}, z).$$

Example



Theorem 16 (Ehrhart reciprocity)

P : rational convex polytope

$$\mathrm{Ehr}_P(1/x) = (-1)^{\dim P+1}\,\mathrm{Ehr}_{\mathrm{int}\,P}(x).$$

$G = (V, E, \mathcal{E} = \mathcal{E}_+)$: bipartite graph with only positive edge.
Q_G : the root polytope of G (forgetting sign).

$$\mathrm{Ehr}_{Q_G}(1/x) = (-1)^{d+1}\,\mathrm{Ehr}_{\mathrm{int}\,Q_G}(x).$$

Lemma 17

$$(-1)^d\,\mathrm{Ehr}_{\mathrm{int}\,Q_G}(x) = \sum_{\mathcal{S}\subset\mathcal{E}}(-1)^{|\mathcal{S}|-1}\,\mathrm{Ehr}_{Q_\mathcal{S}}(x),$$

where $Q_\mathcal{S}$ is the root polytope of the bipartite graph whose edges consist of $\mathcal{S}$.

Therefore,

$$\mathrm{Ehr}_{Q_G}(1/x) = \sum_{\mathcal{S}\subset\mathcal{E}} (-1)^{|\mathcal{S}|}\,\mathrm{Ehr}_{Q_\mathcal{S}}(x).$$

By definition of the signed Ehrhart series,

$$\begin{aligned}(-1)^{|\mathcal{E}|}\,\mathrm{Ehr}_{Q_G}(1/x) &= \sum_{\mathcal{S}\subset\mathcal{E}} (-1)^{|\mathcal{E}|-|\mathcal{S}|}\,\mathrm{Ehr}_{Q_\mathcal{S}}(x) \\ &= \mathrm{Ehr}^{+}_{Q_{-G}}(x),\end{aligned}$$

where Q_{-G} is the root polytope of the bipartite graph obtained from G by changing sign.

By using Theorem 14,

$$(-1)^{|\mathcal{E}|}\frac{I_G^+(1/x)}{(1-1/x)^{d+1}}=\frac{I_{-G}^+(x)}{(1-x)^{d+1}}.$$

We get

$$(-1)^{|\mathcal{E}|+d+1}x^{d+1}I_G^+(1/x)=I_{-G}^+(x).$$

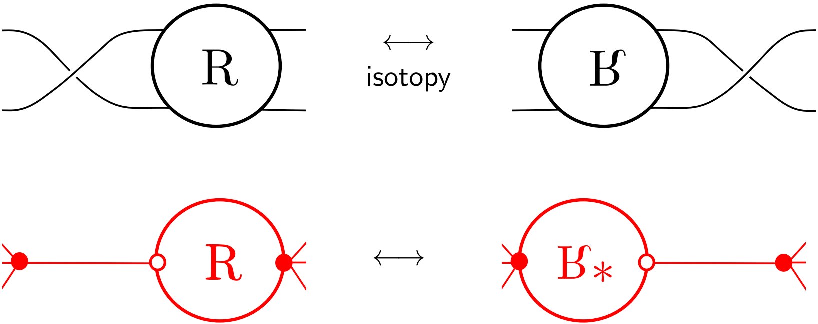
And by using induction on $|\mathcal{E}_-|$, we prove the following theorem.

Theorem 18 (K.)

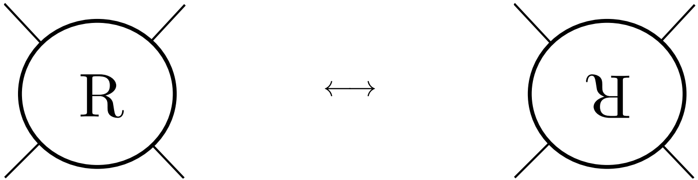
$G=(V,E,\mathcal{E}_+\cup\mathcal{E}_-)$: signed bipartite graph.
$-G$: the signed bipartite graph obtained from G by changing sign.
Then

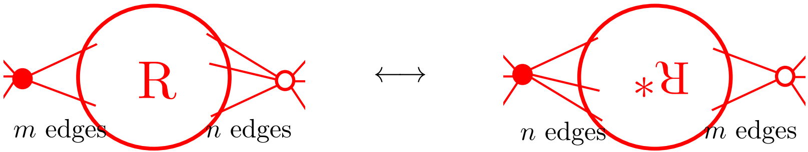
$$(-1)^{|\mathcal{E}_+|+|\mathcal{E}_-|+|E|+|V|-1}x^{|E|+|V|-1}I_G^+(1/x)=I_{-G}^+(x).$$

Flyping



Mutation





Theorem 19 (K.)

Flyping and Mutation of bipartite graph doesn't change the interior polynomial.

We use the folloeing theorem in the proof of Theorem 19.

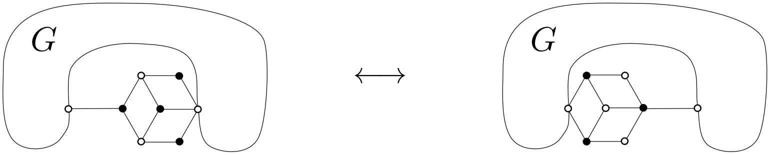
Theorem 20 (K.)

G : bipartite graph containning a cycle $\epsilon_1, \delta_1, \epsilon_2, \delta_2, \cdots, \epsilon_n, \delta_n$

$$I_G(x) = \sum_{\phi \neq \mathcal{S} \subset \{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}} (-1)^{|\mathcal{S}|-1} I_{G \setminus \mathcal{S}}(x).$$

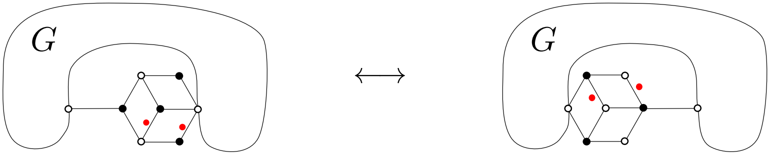
Proof of Theorem 19

By induction on the nullity in R.



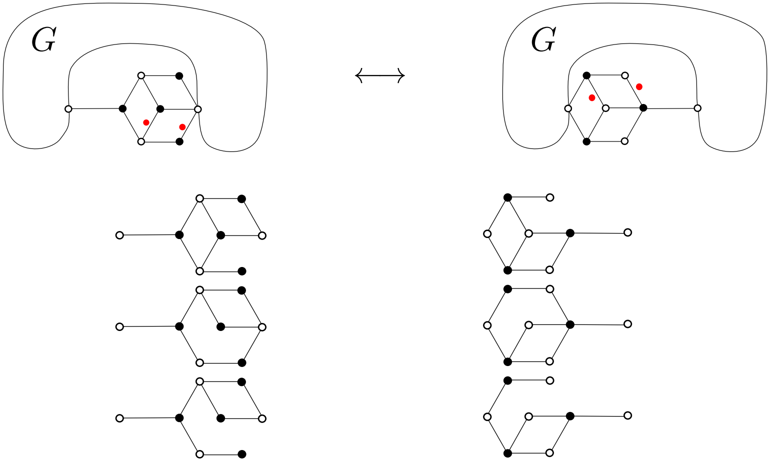
Proof of Theorem 19

By induction on the nullity in R.



Proof of Theorem 19

By induction on the nullity in R.



Proof of Theorem 19

By induction on the nullity in R.